Realisation of cycles by aspherical manifolds

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N. Steenrod's problem on realisation of cycles.

Suppose X is a compact polyhedron, $z \in H_n(X;\mathbb{Z})$ is a homology class.

Do there exist an oriented closed manifold N^n and a continuous mapping $f : N^n \to X$ such that $f_*[N^n] = z$?

Theorem (R. Thom, 1954). For a compact polyhedron X and a homology class $z \in H_n(X;\mathbb{Z})$ there is a nonzero integer $k = k(n)$ such that the class kz is realisable in sense of Steenrod.

For $n \leq 6$ all homology classes are realisable.

For $n \geq 7$ there are non-realisable classes.

R. Thom's approach

For a compact polyhedron X and a positive integer n there is an embedding $i : X \hookrightarrow Q^q$ such that

- Q^q is an oriented closed manifold;
- $i_* : \pi_j(X) \to \pi_j(Q^q)$ is an isomorphism for $j \leq n$.

For q sufficiently large the realisation of cycles of X in sense of N. Steenrod is equivalent to the realisation of cycles of Q^q by oriented submanifolds.

By R. Thom's transversality theorem a homology class $z \in H_n(X; \mathbb{Z})$ is realisable by a submanifold $\Leftrightarrow$ there is a mapping $g : Q^q \to MSO(q-n)$ such that $g^*\iota = Dz$.

There is a mapping

$$MSO(k) \to K(\mathbb{Z}, k) \times K(\mathbb{Z}, k+4) \times K(\mathbb{Z}, k+8)^2 \times \ldots$$

that induces an isomorphism of rational homology groups.

Explicit realisation of homology classes

Problem 1. Given a singular simplicial cycle $\xi \in C_n(X;\mathbb{Z})$ construct explicitly a manifold N^n and a continuous mapping $f : N^n \to X$ realising $k[\xi]$ for a nonzero integer k.

Problem 2. Describe a class $\mathcal{M}_n$ of oriented closed n-dimensional manifolds such that every n-dimensional homology class of every compact polyhedron can be realised with a multiplicity by an image of a manifold belonging to $\mathcal{M}_n$.

Example. No multiple of the fundamental class of a torus T^n can be realised by an image of a sphere S^n.

Manifold of isospectral symmetric tridiagonal matrices

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \dots & 0 \\ 0 & b_2 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n+1} \end{pmatrix}, \qquad a_i, b_i \in \mathbb{R}$$

M^n is the manifold of all matrices L with a fixed spectrum $\lambda_1 < \lambda_2 < \ldots < \lambda_{n+1}$.

Toda flow

$$B(L) = \begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ -b_1 & 0 & b_2 & \dots & 0 \\ 0 & -b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\frac{dL}{dt} = [B(L), L]$$

Main theorem

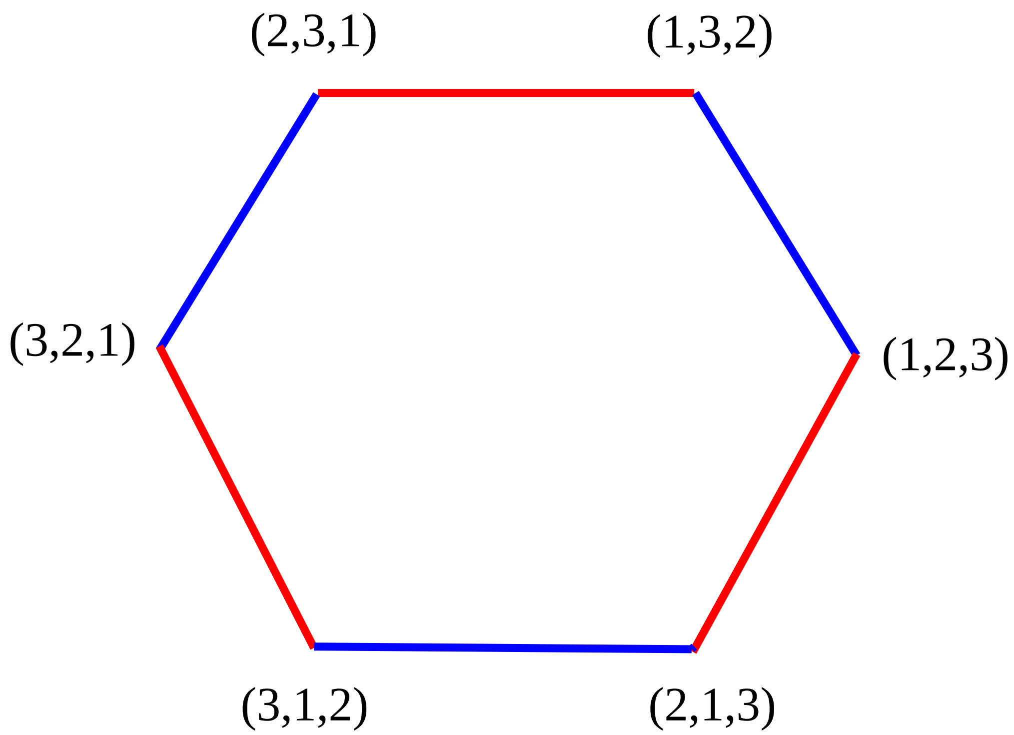
Theorem. Every n-dimensional integral homology class of every compact polyhedron can be realised with some multiplicity by a continuous image of a finite-fold covering of the manifold M^n.

Theorem (C. Tomei, 1984). M^n is aspherical, that is, $\pi_j(M^n) = 0$ for $j > 1$.

Corollary. Every integral homology class of every connected compact polyhedron can be realised by a continuous image of an aspherical manifold.

Permutahedra.

The permutahedron Π^n is the convex hull of the points obtained by permutations of coordinates of the point $(1, 2, \ldots, n+1) \in \mathbb{R}^{n+1}$.



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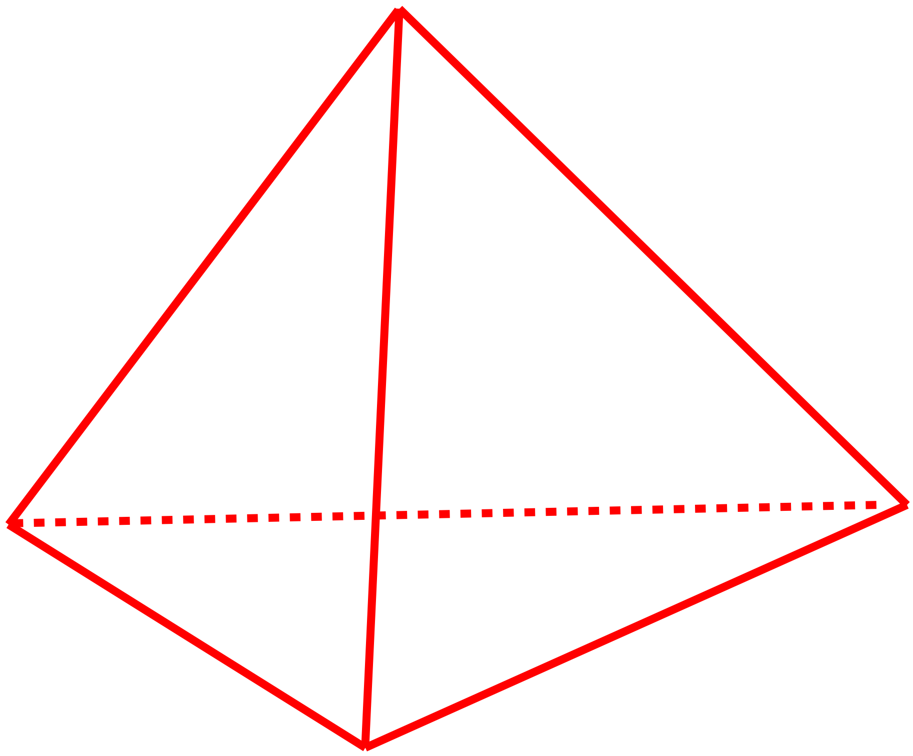

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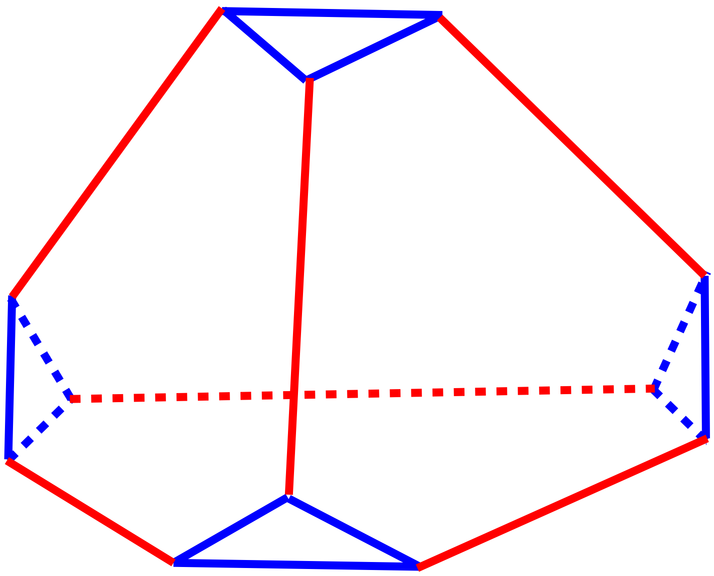

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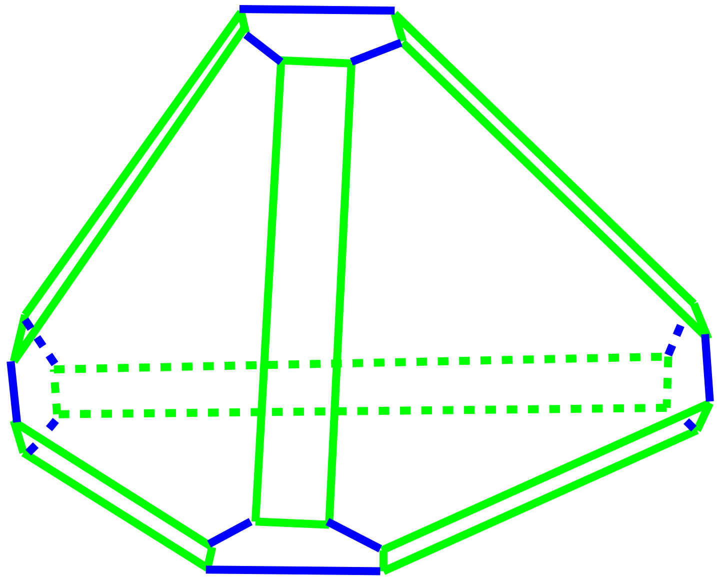

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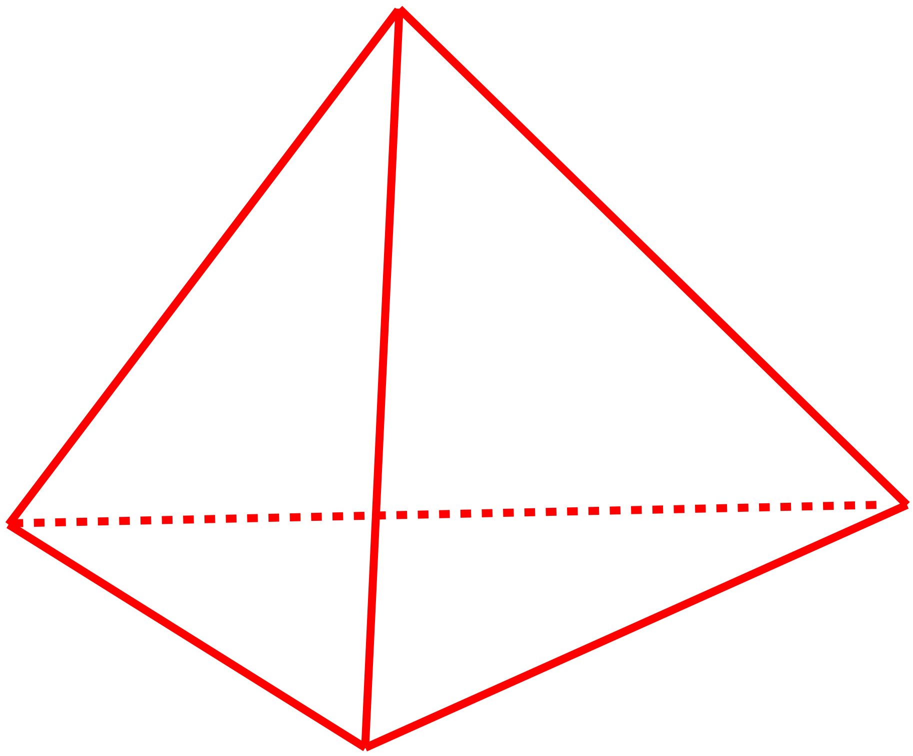
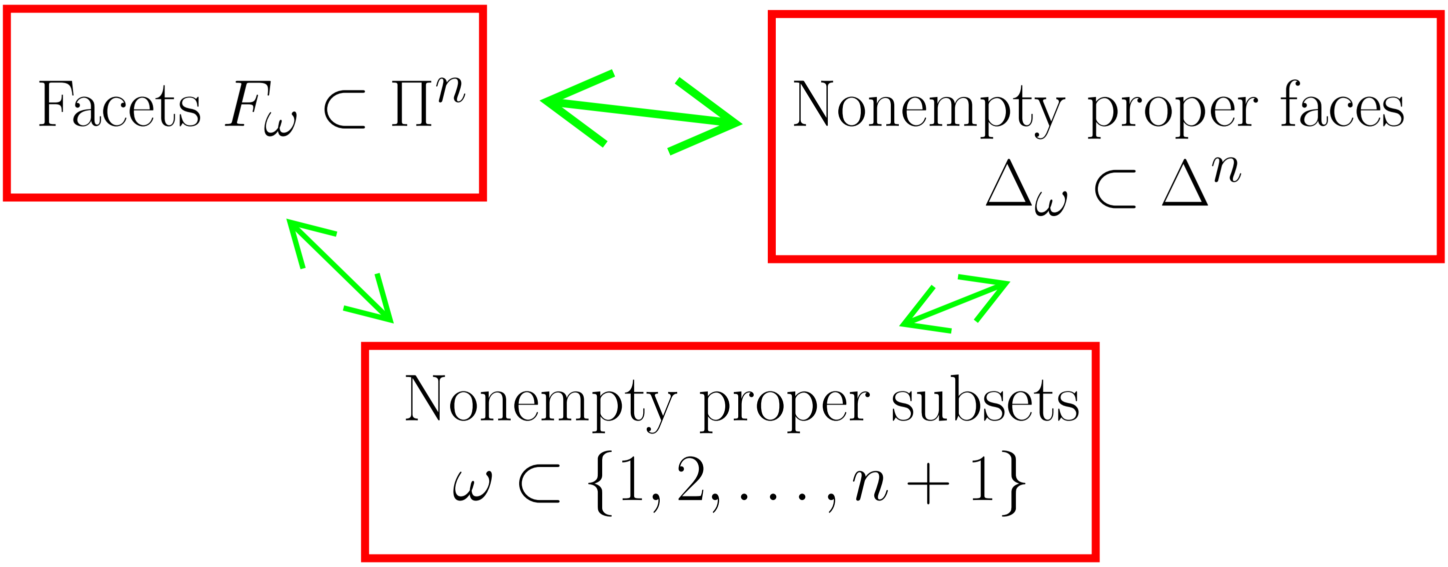


Facets of the permutahedron



$$F_\omega : \qquad \sum_{i \in \omega} x_i = \frac{|\omega|(2n + 3 - |\omega|)}{2}$$

$$\dim \Delta_\omega = |\omega| - 1$$

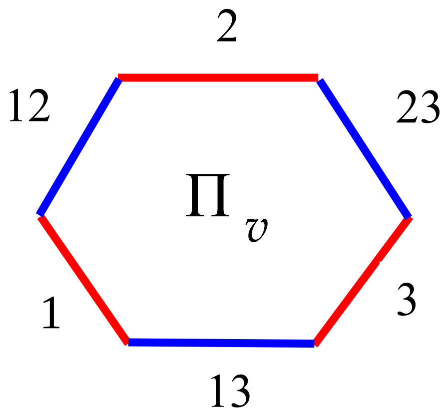
$$F_{\omega_1} \cap F_{\omega_2} \neq \emptyset \quad \Leftrightarrow \quad \omega_1 \subset \omega_2 \text{ or } \omega_2 \subset \omega_1$$

Gluing manifolds from permutahedra

Suppose V is a finite set and $\Phi_\omega : V \to V$ are involutions without fixed points such that

- $F_{\omega_1} \circ F_{\omega_2} = F_{\omega_2} \circ F_{\omega_1}$ whenever $\omega_1 \subset \omega_2$;
- there is a mapping $p : V \to \mathbb{Z}_2^n$ such that $p\left(\Phi_\omega(v)\right) = p(v) + e_{|\omega|}$, where $(e_1, \dots, e_n)$ is the basis of $\mathbb{Z}_2^n$.

Take a permutahedron Π_v^n for each $v \in V$ and glue together the permutahedra Π_v^n and $\Pi_{\Phi_\omega(v)}^n$ along their facets F_ω for every v and ω.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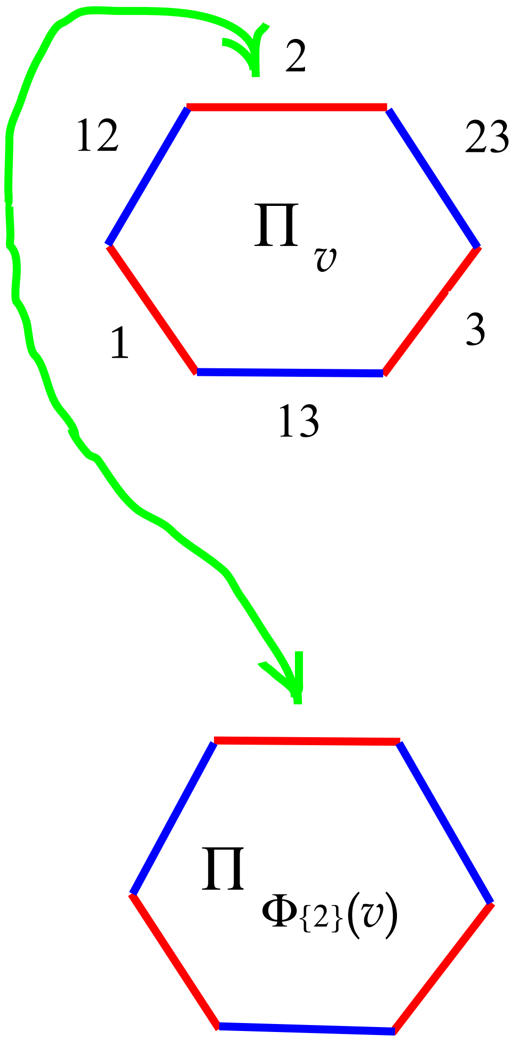
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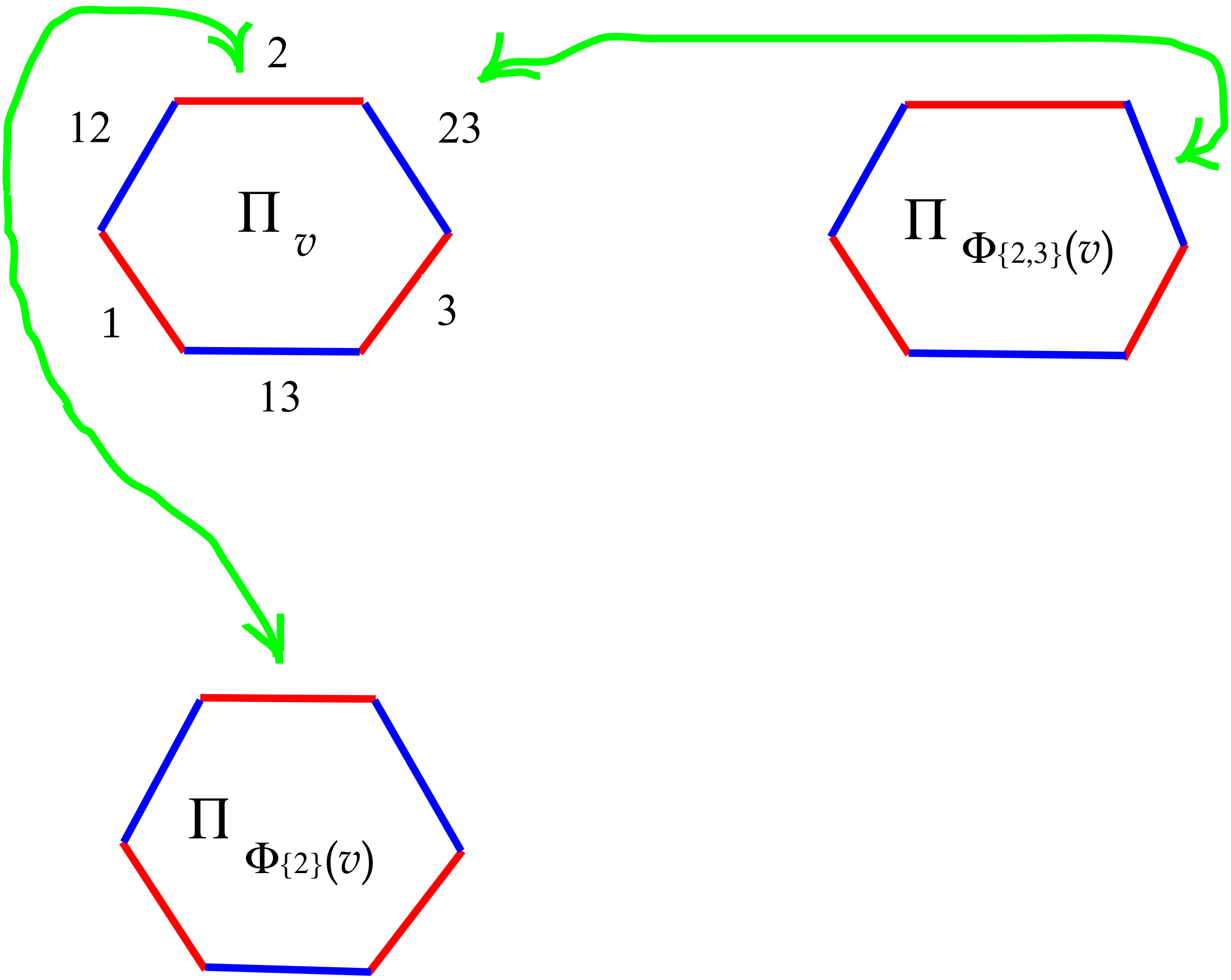
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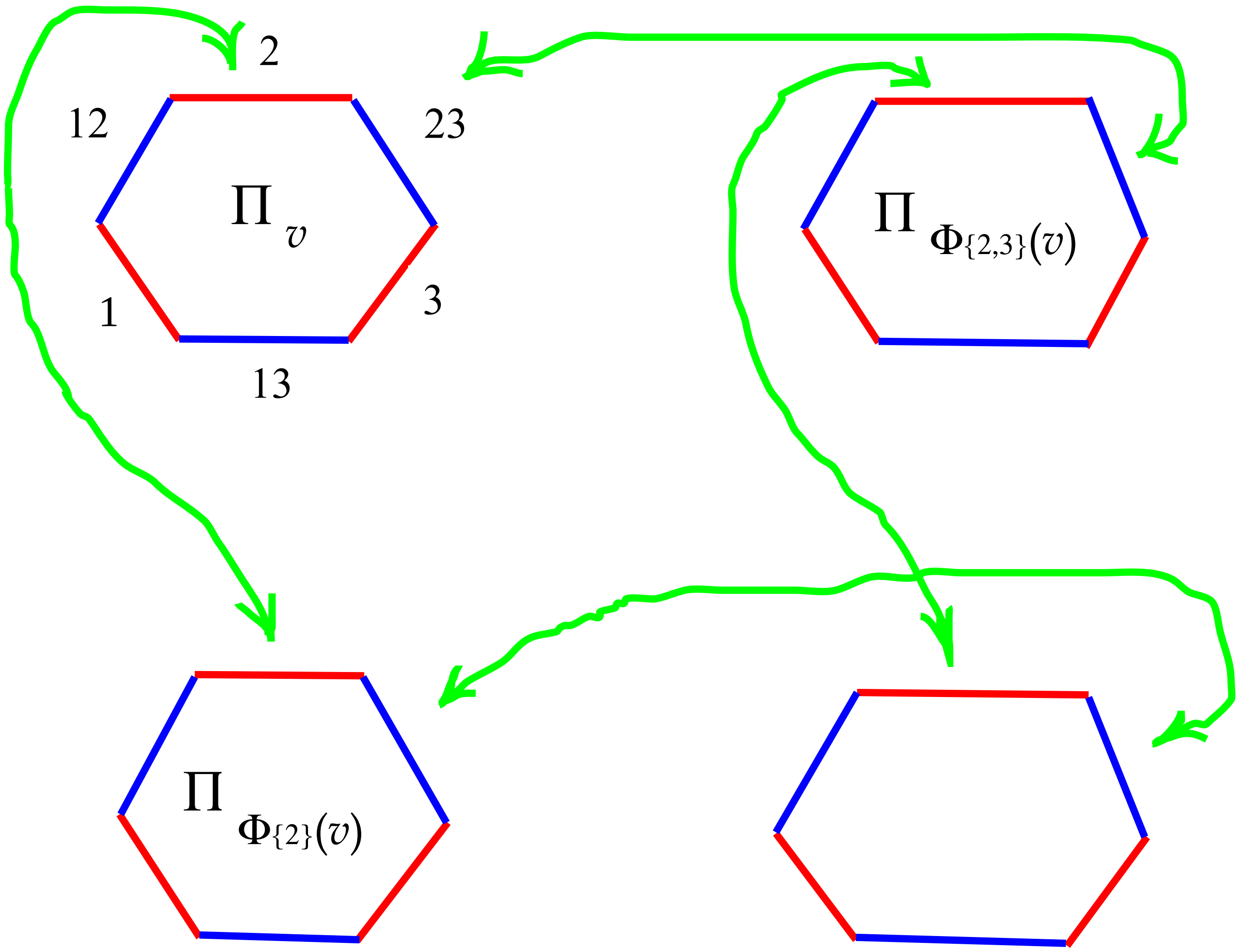
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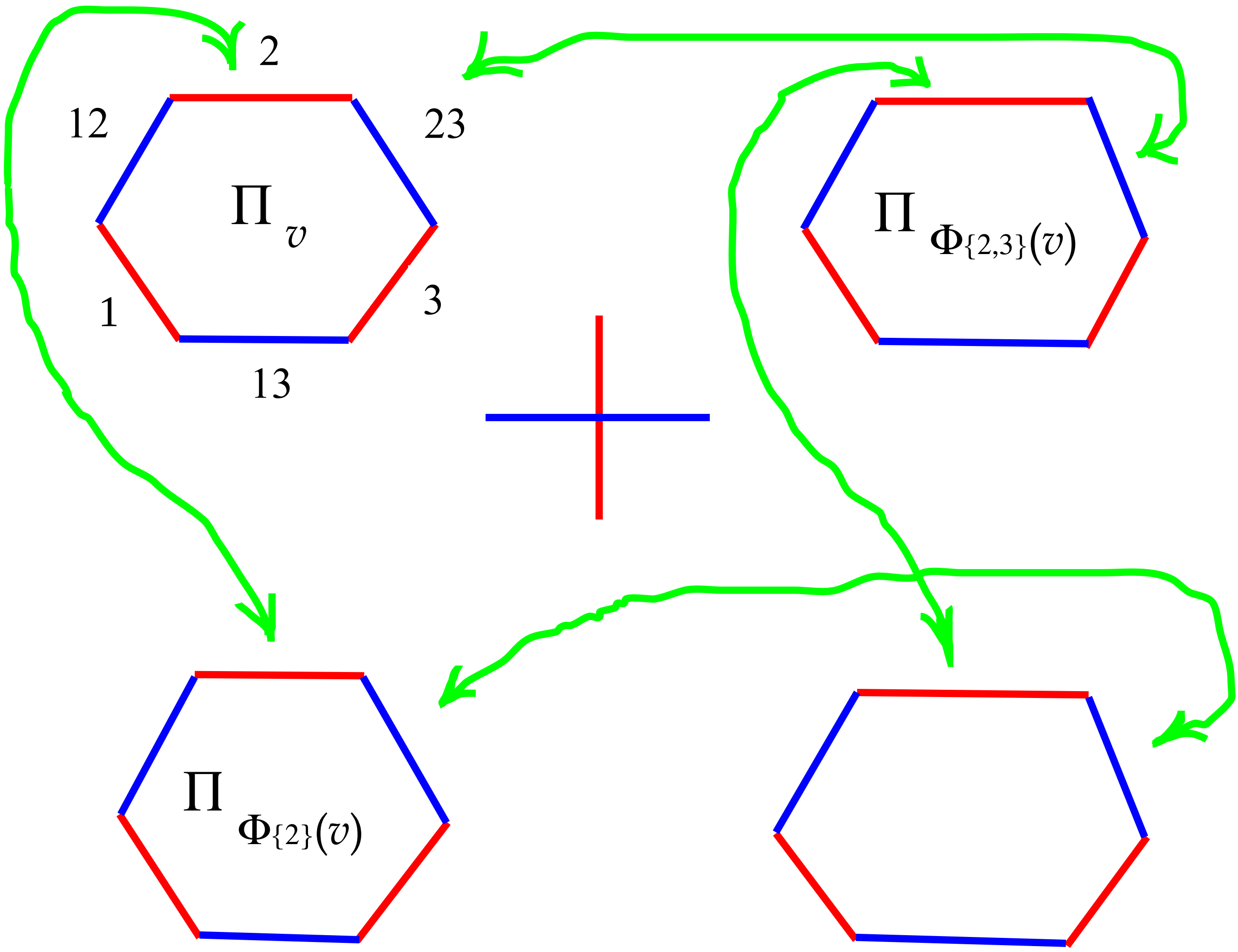
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Gluing manifolds from permutahedra

Suppose V is a finite set and $\Phi_\omega : V \to V$ are involutions without fixed points such that

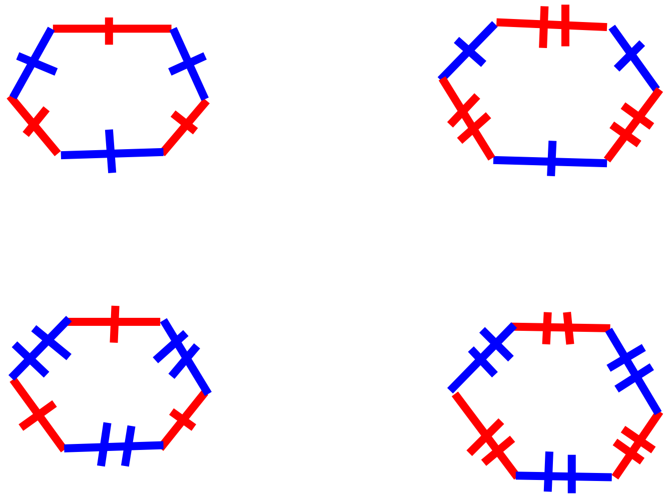
- $F_{\omega_1} \circ F_{\omega_2} = F_{\omega_2} \circ F_{\omega_1}$ whenever $\omega_1 \subset \omega_2$;
- there is a mapping $p : V \to \mathbb{Z}_2^n$ such that $p\left(\Phi_\omega(v)\right) = p(v) + e_{|\omega|}$, where $(e_1, \ldots, e_n)$ is the basis of $\mathbb{Z}_2^n$.

Take a permutahedron Π_v^n for each $v \in V$ and glue together the permutahedra Π_v^n and $\Pi_{\Phi_\omega(v)}^n$ along their facets F_ω for every v and ω.

The manifold obtained is denoted by $M^n(V, \{\Phi_\omega\})$. It is canonically smoothable.

Main example

Suppose $V = \mathbb{Z}_2^n$, $\Phi_\omega(g) = g + e_{|\omega|}$.



Then $M^n(V, \{\Phi_\omega\})$ is diffeomorphic to the manifold M^n of isospectral symmetric tridiagonal real $(n+1) \times (n+1)$ matrices. (C. Tomei, 1984)

For arbitrary V and Φ_ω the manifold $M^n(V, \{\Phi_\omega\})$ is a finite-fold covering of M^n.

Let $M^n_{\varepsilon_1,\dots,\varepsilon_n}$, $\varepsilon_i = \pm 1$, be the set of all matrices

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \dots & 0 \\ 0 & b_2 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n+1} \end{pmatrix}, \qquad a_i, b_i \in \mathbb{R}$$

with fixed spectrum $\lambda_1 < \dots < \lambda_{n+1}$ such that $\varepsilon_i b_i > 0$ for all i. (J. Moser.)

$M^n_{\varepsilon_1,\dots,\varepsilon_n}$ is the integral manifold of the Toda flow. Since the Toda flow is integrable, it follows from Liouville's theorem that $M^n_{\varepsilon_1,\dots,\varepsilon_n} \approx \mathbb{R}^n$.

The closure $\overline{M^n_{\varepsilon_1,\dots,\varepsilon_n}}$ is a permutahedron. The facet F_ω consists of those matrices L for which $b_{|\omega|} = 0$, the first block of L has eigenvalues λ_i, $i \in \omega$, and the second block has eigenvalues λ_i, $i \notin \omega$.

Pseudomanifolds

An n-dimensional *pseudomanifold* is a simplicial complex such that

- every simplex is contained in an n-dimensional simplex;
- every $(n-1)$-dimensional simplex is contained in exactly two n-dimensional simplices.

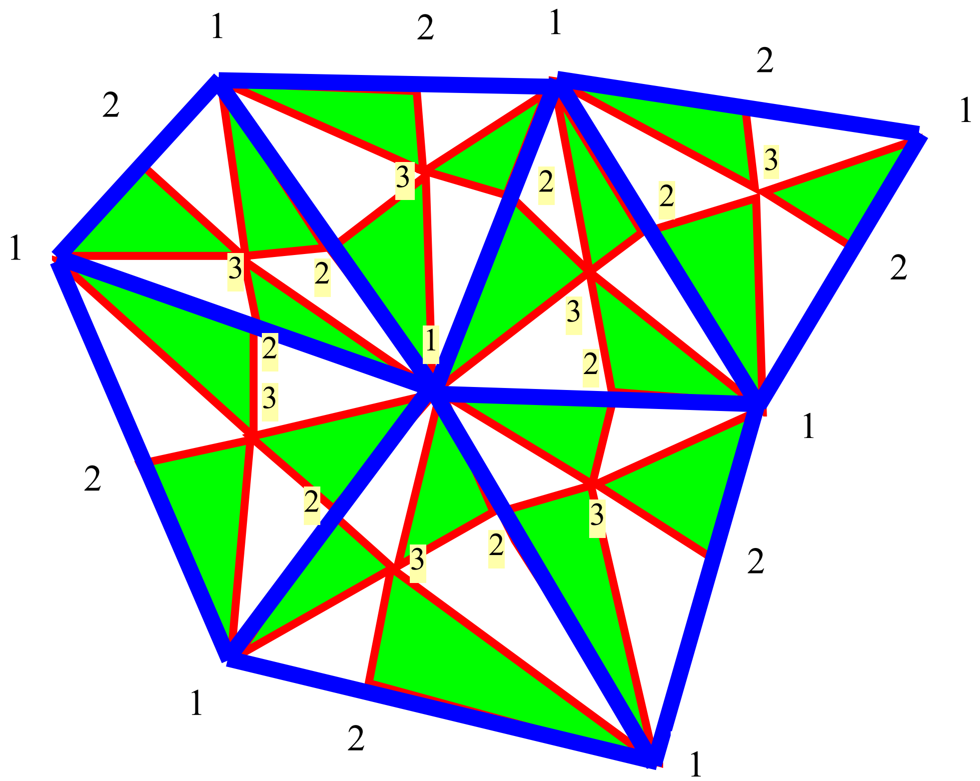
Example. ΣN^{n-1}.

Obviously, any homology class $z \in H_n(X;\mathbb{Z})$ can be represented by an image of a compact oriented pseudomanifold Z under a continuous mapping $h : Z \to X$.

Therefore the problem of realisation with multiplicity of an arbitrary homology class reduces to the problem of realisation with multiplicity of the fundamental class of a pseudomanifold Z.

Colourings of simplices of the barycentric subdivision Z'

- A regular colouring of vertices into $n+1$ colours. The barycenter of a k-dimensional simplex is coloured in colour $k+1$.
- A checkerboard colouring of n-dimensional simplices.



Sets $\mathcal{P}_\omega$.

Suppose U is the set of n-dimensional simplices of Z.

Let $\mathcal{P}_\omega$ be the set of all involutions $\Lambda : U \to U$ such that

- Λ inverses the checkerboard colouring;
- for every $\sigma \in U$ the simplices σ and $\Lambda(\sigma)$ have a common face of type Δ_ω.

$$V = U \times \prod_\omega \mathcal{P}_\omega \times \mathbb{Z}_2^n,$$

$$\Phi_\omega\left(\sigma, \left(\Lambda_\gamma\right)_\gamma, g\right) = \left(\Lambda_\omega(\sigma), \left(\widetilde{\Lambda}_\gamma\right)_\gamma, g + e_{|\omega|}\right),$$

$$\widetilde{\Lambda}_\gamma = \begin{cases} \Lambda_\omega \circ \Lambda_\gamma \circ \Lambda_\omega & \text{if } \gamma \subset \omega, \\ \Lambda_\gamma & \text{if } \gamma \not\subset \omega. \end{cases}$$

Take the manifold $M^n(V, \{\Phi_\omega\})$ and map the permutahedron corresponding to the triple $(\sigma, (\Lambda_\omega)_\omega, g)$ onto the simplex σ so that the facet F_ω is mapped onto the face Δ_ω.

Thus we obtain a well-defined mapping

$$f : M^n(V, \{\Phi_\omega\}) \to Z$$

that realises the fundamental class $[Z]$ with multiplicity

$$2^n \prod_\omega |\mathcal{P}_\omega|.$$