Classification Problems on Toric Manifolds via Cohomology

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Manchester July 8, 2008

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1. Variety classification and equivariant cohomology

Def. A toric variety X of $\dim_{\mathbb{C}} n$ is a normal algebraic variety of $\dim_{\mathbb{C}} n$ with $T = (\mathbb{C}^*)^n$-action having a dense orbit.

We note $\mathrm{Hom}(\mathbb{C}^*, T) = \mathbb{Z}^n$. $\quad \big(g \to (g^{a_1}, \dots, g^{a_n})\big)$

Def. A fan Δ of $\dim_{\mathbb{R}} n$ is a collection of rational polyhedral cones in $\mathrm{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R} = \mathbb{R}^n$ satisfying

(1) each face of a cone in Δ is a also a cone in Δ;

(2) the intersection of two cones in Δ is a face of each.

Fundamental theorem on toric varieties

$$\text{category of toric varieties} \overset{\text{equivalent}}{\longleftrightarrow} \text{category of fans}$$

$$X \qquad\qquad\qquad \Delta(X)$$

In particular,

$$X \cong_{wT} X' \quad \Longleftrightarrow \quad \Delta(X) \cong \Delta(X')$$

• $X \cong_{wT} X'$ means that $\exists f\colon X \to X'$ (isomorphism) together with $\rho\colon T \to T$ (group iso) s.t.

$$f(tx) = \rho(t)f(x) \quad (t \in T,\ x \in X).$$

• $\Delta(X) \cong \Delta(X')$ menas that $\exists g \in \mathrm{GL}(n;\mathbb{Z})$ s.t. $g(\Delta(X)) = \Delta(X')$.

toric <u>manifold</u> = compact smooth toric variety

Fact. $X \cong X'$ as varieties $\Longleftrightarrow X \cong_{wT} X'$.

Proof. $\mathrm{Aut}(X)$ = the group of automorphisms of X (an algebraic group with T as a maximal torus).
An isomorphism $f\colon X \to X'$ induces

$$f^*\colon\ \mathrm{Aut}(X') \to \mathrm{Aut}(X)$$

by $f^*(h) = f^{-1}hf$. Since $f^*(T)$ is a maximal torus of $\mathrm{Aut}(X)$, $\exists \sigma \in \mathrm{Aut}(X)$ s.t.

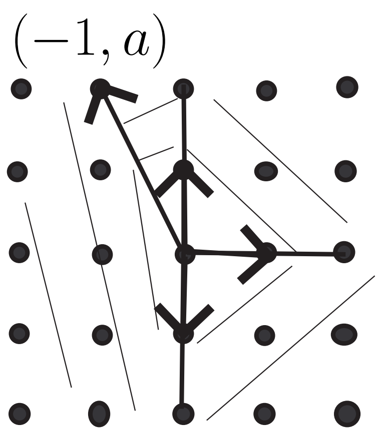
$$\sigma T \sigma^{-1} = f^*(T) = f^{-1}Tf, \quad \text{so} \quad (f\sigma)T(f\sigma)^{-1} = T$$

Therefore $f\sigma\colon X \to X'$ is a weakly equiv. iso. □

Fact implies

Classification of toric manifolds as varieties
= Classification of fans up to isomorphism

Exam. $a \in \mathbb{Z}$
$H_a = P(\gamma^a \oplus \mathbb{C}) \to \mathbb{C}P^1$ (Hirzebruch surface)
$H_a \cong H_{a'}$ as varieties $\Longleftrightarrow |a| = |a'|$.



$(a = 2)$

Equivariant cohomology

We have a fibration

$$X \longrightarrow ET \times_T X \xrightarrow{\pi} BT$$

and

$$H_T^*(X) := H^*(ET \times_T X)$$

is not only a ring but also an algebra over $H^*(BT)$ via π^*.

Def. $H_T^*(X) \cong H_T^*(X')$ (weakly isomorphic)
$\Longleftrightarrow$
$\exists \psi\colon H_T^*(X) \to H_T^*(X')$ (ring iso) together with $\rho\colon T \to T$ (group iso) such that

$$\psi(u\xi) = \rho^*(u)\psi(\xi) \quad \text{for } u \in H^*(BT),\ \xi \in H_T^*(X)$$

Thm. *$X \cong X'$ as varieties $\Longleftrightarrow H_T^*(X) \cong H_T^*(X')$.*

Sketch of Proof

($\Longrightarrow$) follows from Fact mentioned before. In fact, $\exists \varphi\colon X' \to X$ (iso.) and $\exists \rho\colon T \to T$ (group iso) s.t.

$$\varphi(tx) = \rho(t)\varphi(x) \quad (t \in T,\ x \in X').$$

So $\psi = (\rho, \varphi)^*$.

($\Longleftarrow$) follows from the following three observations.

X_i $(i = 1, \ldots, m)$: invariant divisors

$$X_i \qquad \overset{P.D.}{\longleftrightarrow} \qquad \tau_i \in H^2_T(X)$$

Lem. *As a ring*

$$H^*_T(X) = \mathbb{Z}[\tau_1, \ldots, \tau_m] \Big/ \Big(\prod_{i \in I} \tau_i \mid \bigcap_{i \in I} X_i = \emptyset \Big)$$

i.e., $H^*_T(X)$ is the face ring of a simplicial complex

$$K_X := \{ I \subset \{1, \ldots, m\} \mid \bigcap_{i \in I} X_i \neq \emptyset \}$$

Recall $H^*_T(X)$ is an algebra over $H^*(BT)$ via π^* where $X \longrightarrow ET \times_T X \overset{\pi}{\longrightarrow} BT$.

Lem. $\exists_1$ $v_i \in H_2(BT)$ $(i = 1, \ldots, m)$ *s.t.*

$$\pi^*(u) = \sum_{i=1}^{m} \langle u, v_i \rangle \tau_i \quad \textit{for } \forall u \in H^2(BT)$$

Note $H_2(BT) = [B\mathbb{C}^*, BT] = \mathrm{Hom}(\mathbb{C}^*, T)$.
Span a cone in $H_2(BT) \otimes \mathbb{R} = \mathbb{R}^n$ by $\{v_i\}_{i \in I}$ whenever $I \in K_X$. This produces the fan of X.

Lem. *An algebra iso ψ: $H^*_T(X) \to H^*_T(X')$ maps $\{\tau_i\}$ in $H^2_T(X)$ to $\{\tau'_i\}$ in $H^2_T(X')$ bijectively up to sign.*

2. Diffeomorphism classification and cohomology

$H_T^*(X)$ determines X as a variety. So it is natural to ask how much information $H^*(X)$ has.

Cohomological Rigidity Problem (CRP) for Toric Manifolds. Let X, X' be toric manifolds.

$H^*(X) \cong H^*(X')$ as graded rings
$\Longrightarrow$
$X \cong X'$ diffeo (or homeo) ?

Exam. Recall $H_a = P(\gamma^a \oplus \mathbb{C}) \xrightarrow{\mathbb{C}P^1} \mathbb{C}P^1$ $(a \in \mathbb{Z})$.

- $H_a \cong H_{a'}$ as varieties $\Longleftrightarrow |a| = |a'|$.
- $H_a \cong H_{a'}$ diffeo $\Longleftrightarrow a \equiv a' \pmod 2$
  $\Longleftrightarrow H^*(H_a) \cong H^*(H_{a'})$

CRP is affirmative for Hirzebruch surfaces H_a's.

Exam (with Choi and Suh)**.**
CRP is affirmative for $P(E \oplus \mathbb{C})$'s where E is a sum of line bundles over $\mathbb{C}P^{d_1}$.

$$P(E \oplus \mathbb{C}) \xrightarrow{\mathbb{C}P^{d_2}} \mathbb{C}P^{d_1}$$

Generalized Bott (or Dobrinskaya) tower of height n

$$X_n \xrightarrow{\mathbb{C}P^{d_n}} X_{n-1} \xrightarrow{\mathbb{C}P^{d_{n-1}}} \cdots \xrightarrow{\mathbb{C}P^{d_3}} X_2 \xrightarrow{\mathbb{C}P^{d_2}} X_1 \xrightarrow{\mathbb{C}P^{d_1}} \{*\}$$

where $X_{k+1} = P(E_k \oplus \mathbb{C}) \to X_k$ and E_k is a sum of line bundles. We call X_n a generalized Bott manifold.

The tower is called a Bott tower when $d_i = 1$ for $\forall i$.

Thm (with Panov, with Choi and Suh).

Let X be a toric manifold. Then

$H^*(X) \cong H^*(\prod_{i=1}^n \mathbb{C}P^{d_i})$ *as graded rings*

$\Longrightarrow$

$X \cong \prod_{i=1}^n \mathbb{C}P^{d_i}$ *(diffeo)*

Sketch of Proof

(1) If $H^*(X) \cong H^*(X')$ where X' is a generalized Bott manifold, then $K_X \cong K_{X'}$ (the underlying simplicial complexes of the fans); so X is also a generalized Bott manifold.

(2) By (1) we may assume that X is a generalized Bott manifold.

$$H^*(X) \cong H^*(\prod_{i=1}^n \mathbb{C}P^{d_i}) \Longrightarrow c(E_k \oplus \mathbb{C}) = 1$$
$$\Longrightarrow E_k \oplus \mathbb{C} \text{ is trivial } \Longrightarrow X \cong \prod_{i=1}^n \mathbb{C}P^{d_i} \qquad \square$$

Another affirmative partial solution to CRP is

Exam (with Choi and Suh)**.** CRP is affirmative for Bott manifolds of $\dim_{\mathbb{C}} 3$.

Sketch of Proof. Let X, X' be Bott manifolds of $\dim_{\mathbb{C}} 3$. Then one can show any isomorphism $\psi\colon H^*(X) \to H^*(X')$ preserves their Pontrjagin classes (and Stiefel-Whitney classes as well). So the classification results on 6-manifolds by Wall (spin case) and Jupp (non-spin case) imply our theorem. $\square$

This leads us to ask

Question. Let X, X' be toric mfds. If $\psi\colon H^*(X) \to H^*(X')$ is an isomorphism, then $\psi(p(X)) = p(X')$?

c.f.

Petrie's conjecture (1972). Let M be a homotopy $\mathbb{C}P^n$ and $f\colon M \to \mathbb{C}P^n$ be a homotopy equivalence. If M supports a non-trivial smooth S^1-action, then $f^*(p(\mathbb{C}P^n)) = p(M)$, i.e. $p(M) = (1 + x^2)^{n+1}$ where $x \in H^2(M)$ is a generator.

Surgery Theory $\Longrightarrow$
$\exists\ \infty$ many homotopy $\mathbb{C}P^n$ when $n \geq 3$ and they are distinguished by their Pontrjagin classes up to finite ambiguity.

3. Real toric manifolds

Toric manifold X admits a "complex conjugation" and its fixed point set $X(\mathbb{R})$ is called a <u>real</u> toric manifold.

$$(\mathbb{Z}_2)^n \subset (\mathbb{R}^*)^n \curvearrowright X(\mathbb{R})$$

Exam. $X = \mathbb{C}P^n$, $X(\mathbb{R}) = \mathbb{R}P^n$.

(Similarity to the complex case)

- $H^*(X(\mathbb{R});\mathbb{Z}/2) \cong H^{2*}(X;\mathbb{Z}) \otimes \mathbb{Z}/2$.

(Non-similarity to the complex case)

- $\pi_1(X) = \{1\}$, but $\pi_1(X(\mathbb{R})) \neq \{1\}$ and $X(\mathbb{R})$'s provide many examples of aspherical manifolds.

Cohomological Rigidity Problem (CRP) for <u>Real</u> Toric Manifolds.

Let M, M' be real toric manifolds.

$H^*(M;\mathbb{Z}/2) \cong H^*(M';\mathbb{Z}/2)$ as graded rings

$\Longrightarrow$

$M \cong M'$ diffeo (or homeo) ?

Exam. Real Bott tower of height n

$$M_n \xrightarrow{\mathbb{R}P^1} M_{n-1} \xrightarrow{\mathbb{R}P^1} \cdots \xrightarrow{\mathbb{R}P^1} M_2 \xrightarrow{\mathbb{R}P^1} M_1 \xrightarrow{\mathbb{R}P^1} \{*\}$$

Here $M_{k+1} = P(L_k \oplus \mathbb{R}) \to M_k$ and L_k is a real line bundle. We call M_n a real Bott manifold.

$$\text{Choices of } L_k \longleftrightarrow H^1(M_k; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^k.$$

An upper triangular $(0,1)$ matrix $A = (A^i_j)$ is associated to the real Bott tower, and

$$M_n = \mathbb{R}^n/\Gamma(A) \ (= M(A))$$

where $\Gamma(A)$ is generated by $s_1, \ldots, s_n$ and

$$s_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, u_i + \frac{1}{2}, (-1)^{A^i_{i+1}} u_{i+1}, \ldots, (-1)^{A^i_n} u_n)$$

$$1 \to \langle s_1^2, \ldots, s_n^2 \rangle = \mathbb{Z}^n \to \Gamma(A) \to (\mathbb{Z}_2)^n \to 1$$

The real Bott manifold $M_n = M(A)$ admits a riemannian flat metric invariant under the $(\mathbb{Z}_2)^n$-action.

(Conversely a real toric manifold which admits an invariant riemannian flat metric is a real Bott manifold.)

Thm (with Kamishima)**.**
$H^*(M(A);\mathbb{Z}/2) \cong H^*(M(B);\mathbb{Z}/2)$ *as graded rings*
$\Longrightarrow$
$M(A) \cong M(B)$ *(diffeo)*

Sketch of Proof. Remember $M(A) = \mathbb{R}^n/\Gamma(A)$ and $\Gamma(A) = \pi_1(M(A))$ is generated by $s_1, \ldots, s_n$

$$s_i(u_1, \ldots, u_n) = (u_1, \ldots, u_{i-1}, u_i + \frac{1}{2}, (-1)^{A^i_{i+1}} u_{i+1}, \ldots, (-1)^{A^i_n} u_n)$$

$$1 \to \langle s_1^2, \ldots, s_n^2 \rangle = \mathbb{Z}^n \to \Gamma(A) \to (\mathbb{Z}_2)^n \to 1$$

(1) $H^*(M(A);\mathbb{Z}/2) = \mathbb{Z}/2[x_1, \ldots, x_n]/(x_j^2 = x_j \sum_{i=1}^{j-1} A^i_j x_i \mid 1 \leq j \leq n)$

(2) Let $\psi\colon H^*(M(A);\mathbb{Z}/2) \to H^*(M(B);\mathbb{Z}/2)$ iso.
ψ restricted to H^1 induces a matrix $P \in \mathrm{GL}(n;\mathbb{Z}/2)$.
P satisfies some conditions which involve A and B.

(3) Define $\rho\colon \Gamma(B) \to \Gamma(A)$ by

$$\rho(t_r) = s_1^{P^r_1} s_2^{P^r_2} \ldots s_n^{P^r_n} \quad (r = 1, \ldots, n).$$

Well-defined and homomorphism by (3).

(4) $\rho\colon \Gamma(B) \to \Gamma(A)$ is injective and
$\rho(\Gamma(B)) \subset \Gamma(A)$ odd index.
$\overset{\text{group ext}}{\Longrightarrow} \Gamma(B) \cong \Gamma(A) \overset{\text{Bieberbach}}{\Longrightarrow} M(B) \cong M(A)$ $\square$

Exam. $D_n = \#$ of diffeomorphism classes in real Bott manifolds of dim n.

$$D_2 = 2, \quad D_3 = 4, \quad D_4 = 12, \quad D_5 = 54(?),$$
$$D_n > 2^{(n-2)(n-3)/2} \text{ for any } n.$$

(cf. $\#$ of real Bott towers of height n is $2^{n(n-1)/2}$)

Using the previous theorem, one can prove

Thm. *Decomposition of a real Bott manifold into a product of indecomposable real Bott manifolds is unique (up to a permutation of factors).*

In particular, if $S^1 \times M \cong S^1 \times M'$ where M, M' are real Bott manifolds, then $M \cong M'$ (i.e. cancellation property holds).

Not true for general riemannian flat manifolds.

We may ask

Question. Is a decomposition of a (real) toric manifold into a product of indecomposable (real) toric manifolds unique?

Thm. *Any isomorphism*

$$\psi\colon H^*(M(A);\mathbb{Z}/2) \to H^*(M(B);\mathbb{Z}/2)$$

is induced from a diffeomorphism $M(B) \to M(A)$.

<u>Sketch of Proof.</u> Remember

$$H^*(M(A);\mathbb{Z}/2) = \mathbb{Z}/2[x_1,\ldots,x_n]/(x_j^2 = x_j \sum_{i=1}^{j-1} A_j^i x_i \mid 1 \le j \le n)$$

Set $\alpha_j = \sum_{i=1}^{j-1} A_j^i x_i$ $(1 \le j \le n)$. Then $x_j^2 = \alpha_j x_j$.

(1) • If $x^2 = \alpha x$ for $0 \neq x \neq \alpha \in H^1(M(A);\mathbb{Z}/2)$, then $\alpha = \alpha_j$ for some j.

• The solutions of $x^2 = \alpha x$ are a vector space $V(\alpha)$ generated by $x_j, x_j + \alpha_j$ with $\alpha_j = \alpha$.

(2) $\psi(V(\alpha_j)) = V(\beta_{\sigma(j)})$ with some permutation σ, so ψ decomposes into a composition of three types of isomorphisms:

[1] $\{x_1,\ldots,x_n\} \to \{y_1,\ldots,y_n\}$ (may permute)
[2] $x_j \to y_j + \beta_j$ and $x_j + \alpha_j \to y_j$ for some j
[3] linear transformation $V(\alpha_j) \to V(\beta_j)$.

(3) Each of [1], [2], [3] above is induced by an (affine) diffeomorphism. $\square$

4. Addendum

We may ask the previous problems for a more general family of manifolds.

Def. A quasitoric manifold M is a closed smooth manifold of dim $2n$ with smooth $T^n = (S^1)^n$-action s.t.

(1) the action is *locally standard*, i.e., locally same as $\mathbb{C}^n$ with the T-action defined by
$$(z_1, \ldots, z_n) \to (t_1 z_1, \ldots, t_n z_n).$$
(Note $\mathbb{C}^n/T^n = (\mathbb{R}_{\geq 0})^n$)

(2) M/T^n is a simple convex polytope of dim n.

Exam. $\mathbb{C}P^2 \# \mathbb{C}P^2$ is quasitoric but not toric.

$$\{\text{toric manifolds}\} \overset{(?)}{\subsetneq} \{\text{quasitoric manifolds}\}$$

Remark (by S. Kuroki). Let $\gamma \to S^2$ be a canonical line bundle and $M = S(\gamma \oplus \mathbb{C} \oplus \mathbb{R})$, $M' = S^2 \times S^4$. Then M and M' have locally standard T^3-actions (so that the orbits spaces are manifolds with corners) and $H^*(M) \cong H^*(M')$, but they are not diffeomorphic.

Reference (except for real Bott manifold)

M. Masuda and D. Y. Suh,
Classification problems of toric manifolds via topology,

Toric Topology, Contemp. Math. 460, pp. 273–286.

(Expected publication date is **July 11, 2008**)