Perturbations of Jordan Matrices

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This is an account of the preprint of the same name by Mildred Hager and myself, of November 2006.

Pseudospectra

The pseudospectral regions are defined by

$$\mathrm{Spec}_\varepsilon(A) = \{z : \|(zI - A)^{-1}\| > \varepsilon^{-1}\}.$$

and satisfy

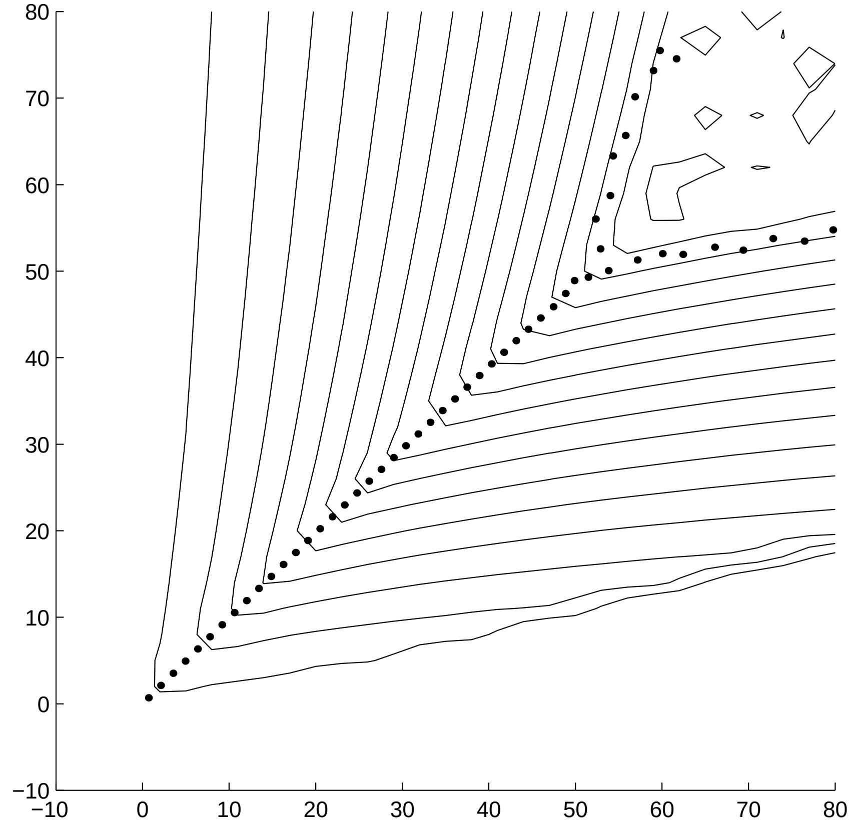
$$\mathrm{Spec}(A) \subseteq \mathrm{Spec}_\varepsilon(A).$$

The NSA harmonic oscillator

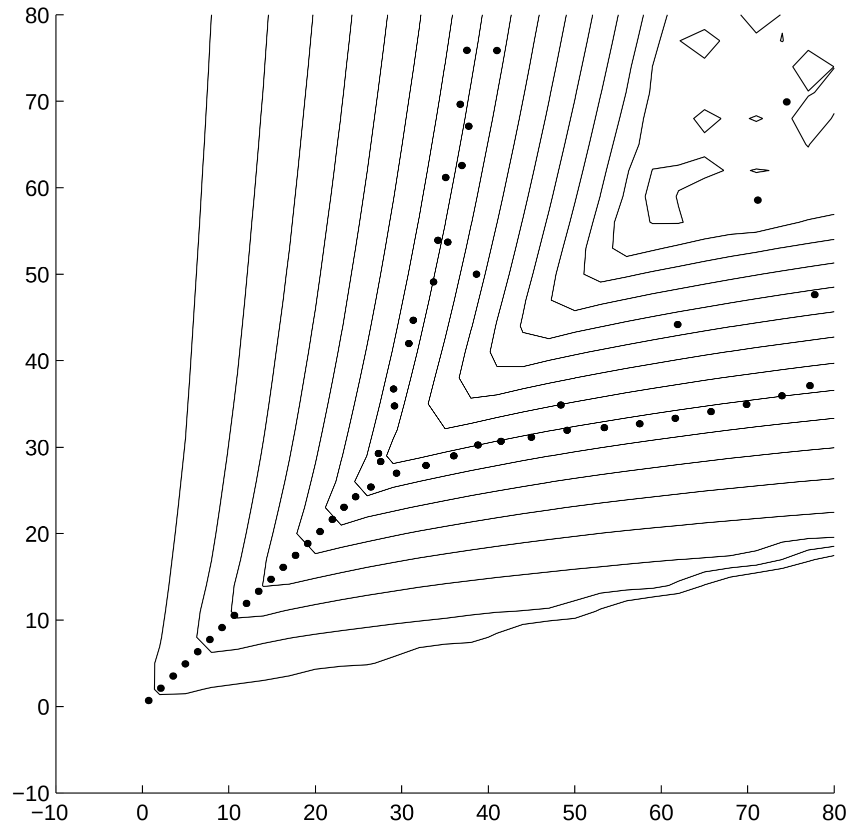
$$(Hf)(x) := -f''(x) + c^2x^2f(x)$$

acting in $L^2(\mathbf{R})$ has eigenvalues $\lambda_n := c(2n+1)$ where $n = 0, 1, \ldots.$

If c is complex then the norms of the spectral projections P_n increase at an exponential rate as $n \to \infty$.(EBD and Kuijlaars)



The contours correspond to $\varepsilon = 10^{-n}$ where $n = 0, 1, 2, ...$



With a perturbation of norm 10^{-6} the splitting of the eigenvalues along the pseudospectral contour is not due to model error or processor rounding errors.

The Jordan block

$$J_4 := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Using the explicit formula for $(zI - J_n)^{-1}$ one immediately obtains

$$\|(zI - J_n)^{-1}\|_1 = \frac{|z|^{-n} - 1}{1 - |z|}$$

so the norm is exponentially large inside the unit circle.

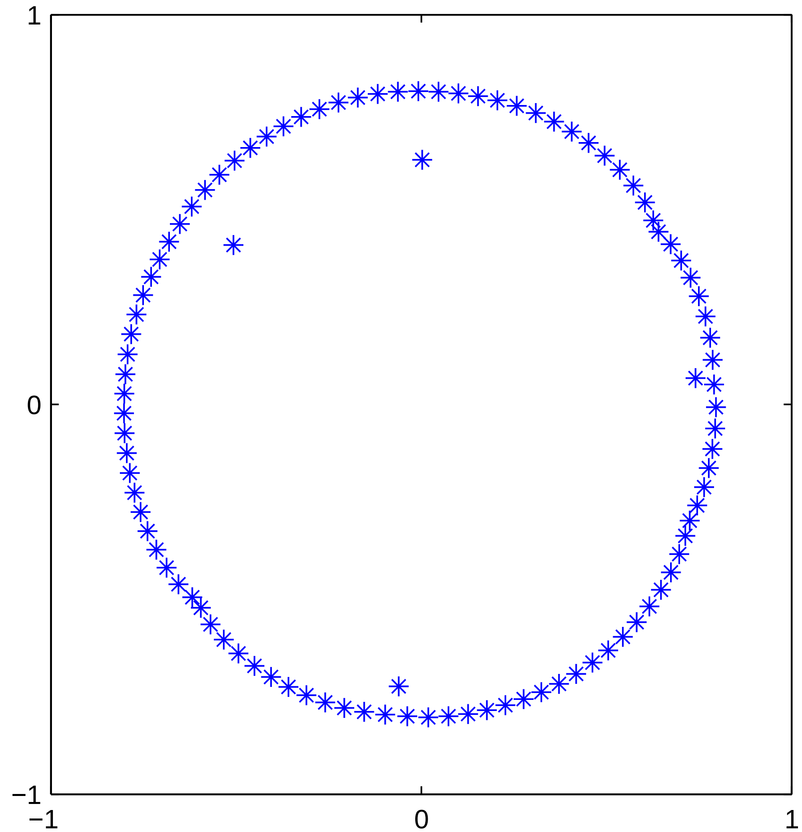
Perturbations of The Jordan block

If $\|B\| \leq 1$ and $0 < c < 1$ then

$$\mathrm{Spec}(J_n + c^n B) \subseteq \{z : |z| \leq c\}.$$

If B is chosen randomly one might expect the spectrum to be randomly distributed within this ball.

Mildred Hager showed that this was not correct. I got involved in looking with her in some detail at this problem.



The result of adding a small random perturbation to the Jordan matrix is to move most of the eigenvalues to the Lidskii circle, but a few are left at random positions inside the circle.

Theorem 1 *Let $M = J + c^n K$ where J is the standard $n \times n$ Jordan matrix, $0 < c < 1$ and K is a random matrix with independent Gaussian entries.*

Then for any $\varepsilon > 0$ with probability that converges to 1 as $n \to \infty$, the proportion of the eigenvalues that lie in any annulus

$$\{z : c - \varepsilon < |z| < c + \varepsilon\}$$

converges to 1.

The remaining eigenvalues lie inside the annulus.

Proof: Reduce the problem to finding the solutions of an equation of the form

$$w^n = f(w), \qquad\qquad w = z/c.$$

The analysis of the spectrum involves using theorems such as the following, and proving that the bounds hold with high probability.

Proposition 2 (The Poisson-Jensen formula) *Let f be a holomorphic function that does not vanish anywhere on the boundary of $D(0,R)$, where $0 < R < \infty$. Let M be the number of zeros of f in $D(0,Re^{-\sigma})$ for some positive constant σ. Then*

$$M \leq \frac{1}{\sigma}\left(-\ln \frac{|f(0)|}{\|f\|_{L^\infty(D(0,R))}}\right). \tag{1}$$

A simpler Example

Consider $A = J_n + c^n K$ where

$$K = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

and C is a fixed $k \times k$ matrix, for example

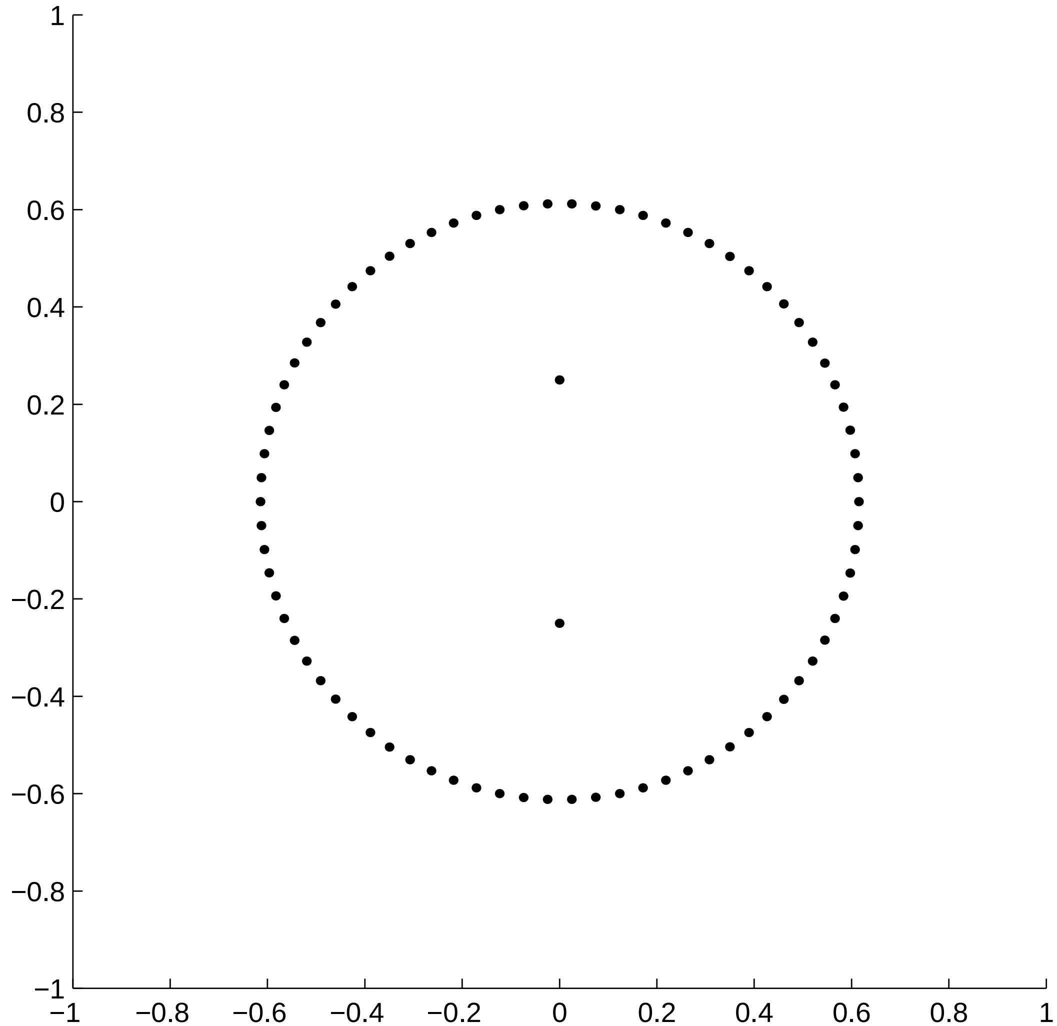
$$C = \begin{pmatrix} 8 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & -2 & 3 \end{pmatrix}.$$

THEOREM If $0 < c < \infty$ then $z \in \mathrm{Spec}(J_n + c^n K)$ if and only if

$$(z/c)^n = p(z)$$

where p is a fixed (i.e. n-independent) polynomial of degree $2k$. There is a large family of solutions for which $|z/c|$ is close to 1. If $|z/c| < 1$ then there are other solutions close to the zeros of $p(z)$.

The resulting spectrum is shown in the next figure.

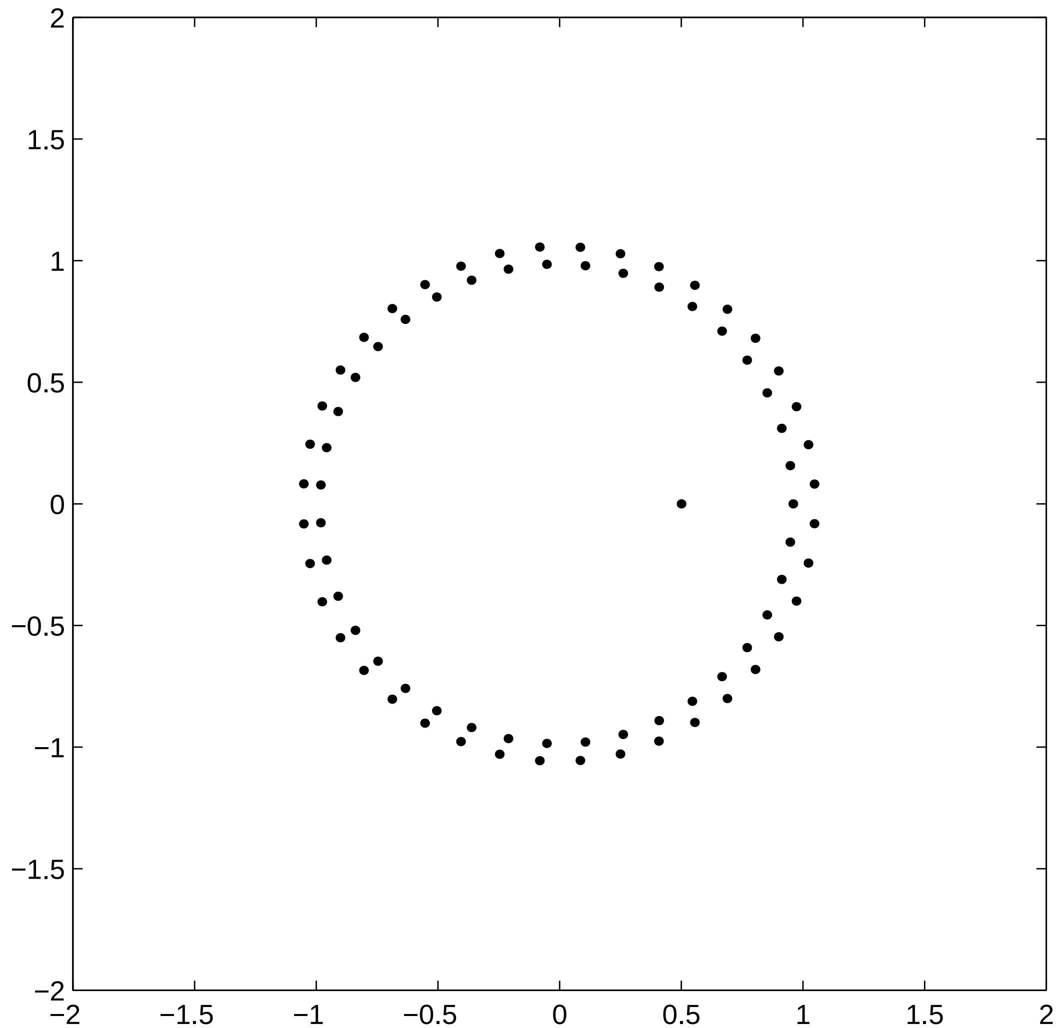


$A = J_n + c^n K$ where $n = 80$ and $c = 0.6$

More complex problems may lead to equations of the type

$$z^{2n} + p(z)z^n + q(z) = 0$$

or polynomial equations of higher order. The zeros of such equations are as shown in the following figure.


2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2