MATRIX FUNCTIONS WITH SYMMETRIES

(Recent) collaborations with U. Prells (Swansea), and L. Rodman (Williamsburg).

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Plan of the talk.

- Spectra of general matrix functions.
- Proper values for self-adjoint matrix functions.
- Sign characteristic - geometrically.
- Sign characteristic via realization.
- Matrix polynomials.
- Linearization and sign characteristics.
- (Strictly) isospectral families preserving structure.
- Palindromic polynomials and unitary structure.

Meromorphic functions

Given n^2 functions $w_{jk}(\lambda)$ meromorphic on a suitable domain, form

$$W(\lambda) := \left[\ w_{jk}(\lambda)\ \right]_{j,k=1}^{n}.$$

Assume regularity (throughout), i.e. $\det W(\lambda) \neq 0$.

An **eigenvalue** is a $\lambda_0 \in \mathbb{C}$ such that $\det W(\lambda_0) = 0$.

Spectrum of W, $\sigma(W)$, is set of all eigenvalues.

NB: A point can be both an ev and a pole.

Smith form

"Extended" Smith form:

Theorem

$\exists$ $n \times n$ analytic matrix fns $E(\lambda)$, $F(\lambda)$ invertible near λ_0 and integers $\nu_1, \ldots, \nu_n$ such that

$$W(\lambda) = E(\lambda) \begin{bmatrix} (\lambda - \lambda_0)^{\nu_1} & & \\ & \ddots & \\ & & (\lambda - \lambda_0)^{\nu_n} \end{bmatrix} F(\lambda)$$

holds near λ_0.

Partial multiplicites $\nu_1 \geq \cdots \geq \nu_n$ may be +ve, zero, or -ve.

Partial multiplicity of ev of $W(\lambda) \Leftrightarrow \nu_j > 0$.
Partial multiplicity of pole of $W(\lambda) \Leftrightarrow \nu_j < 0$.

Self-adjoint meromorphic functions

Definitions:

1. $W(\lambda)$ is self-adjoint (on the real line) if $W\left((\bar{\lambda})^*\right) = W(\lambda)$, $\lambda \notin \sigma(W^{-1})$.

2. Let $\lambda_0 \in \sigma(W) \cap \mathbb{R}$. A real-valued fn $\mu(\lambda)$ defined on a real n'hood Ω of λ_0 is a proper value of $W(\lambda)$ if $\exists$ analytic vector fn. $\phi(\lambda)$ s.t. $W(\lambda)\phi(\lambda) = \mu(\lambda)\phi(\lambda)$ on Ω.

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Theorem

Let $W(\lambda)$ be $n \times n$, self-adjoint, and meromorphic in a real n'hood Ω of λ_0. Then $\exists$ proper values $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ and an analytic matrix-valued fn $U(\lambda)$ defined on Ω where $U(\lambda)U(\lambda)^* = I$ and

$$M(\lambda) = U(\lambda)^{-1} \mathrm{diag}\left[\mu_1(\lambda), \ldots, \mu_n(\lambda)\right] U(\lambda).$$

Sign characteristic - geometric version

Self-adjoint, meromorphic $W(\lambda)$ near λ_0, as above, with proper values $\mu_1(\lambda), \ldots, \mu_n(\lambda)$. Define

$$\text{sgn}\left[\mu_j(\lambda)(\lambda - \lambda_0)^{-\nu_j}\right]_{\lambda=\lambda_0}$$

to be a member of the sign characteristic of $W(\lambda)$ at λ_0 associated with μ_j.

The sign characteristic is a collection of $+1$'s and -1's corresponding to **all** *n* proper values.

Toward linear algebra

Consider rational functions $W(\lambda)$ with a Laurent expansion at ∞:

$$W(\lambda) = \sum_{j=-\infty}^{q} W_j \lambda^j.$$

The "polynomial part" is $U(\lambda) = \sum_{j=0}^{q} W_j \lambda^j$. Then there are realizations:

$$W(\lambda) - U(\lambda) = C(\lambda I - A)^{-1} B,$$

(and A may be large). Furthermore, there are minimal relizations (minimizing the size of the "main matrix" A). Also,

$$\begin{array}{lcl} \text{Minimality} & \leftrightarrow & \text{controllability of } (A, B) \\ & & \text{plus observability of } (C, A). \end{array}$$

Notation:

If $X \in \mathbb{C}^{n \times \ell n}$, $Y \in \mathbb{C}^{\ell n \times n}$ and $M \in \mathbb{C}^{\ell n \times \ell n}$, we define $\ell n \times \ell n$ matrices by:

$$[X, M]_{\downarrow}^{\ell} := \begin{bmatrix} X \\ XM \\ \vdots \\ XM^{\ell-1} \end{bmatrix}, \qquad [M, Y]_{\rightarrow}^{\ell} := \begin{bmatrix} Y & MY & \cdots & M^{\ell-1} \end{bmatrix}.$$

When $[X, M]_{\downarrow}^{\ell}$ is nonsingular, the pair (X, M) is known as a **null-kernel** pair and, when $[M, Y]_{\rightarrow}^{\ell}$ is nonsingular, (M, Y) is known as a **full-range** pair.

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Definitions: (A, B) is **controllable** if (A, B) is a full-range pair. (C, A) is **observable** if (C, A) is a null-kernel pair.

Poles of $W(\lambda) \leftrightarrow$ evs of A (of a min. realn.).

Notions of "partial multiplicities" at poles $W(\lambda)$ are consistent with those of the Jordan form for A.

Zeros of $W(\lambda) \leftrightarrow$ evs of main matrix A of a min. realization for $W(\lambda)^{-1}$.

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MATRIX POLYNOMIALS

Generically,

$$L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j, \qquad \det A_\ell \neq 0.$$

The **resolvent** $L(\lambda)^{-1}$ has a min. realization:

$$L(\lambda)^{-1} = C(\lambda I - A)^{-1} B.$$

Matrix polynomials

In particular (when $\ell = 4$, etc.), can take $C = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$,

$$A = C_L = \begin{bmatrix} 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ -A_4^{-1}A_0 & -A_4^{-1}A_1 & -A_4^{-1}A_2 & -A_4^{-1}A_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_4^{-1} \end{bmatrix}.$$

call (C, C_L, B) a standard triple. Then all realizations of the resolvent are formed by standard triples

$$(CT^{-1}, TC_LT^{-1}, TB),$$

among which there are Jordan triples

$$(X, J, Y),$$

with J in Jordan form and X, Y displaying the right and left eigenvectors (in chains).

Self-adjoint Matrix Polynomials

Consider $L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j$ as before but now

$$A_j^* = A_j, \qquad j = 0, 1, \ldots, \ell.$$

The block-Hankel matrix

$$H := \begin{bmatrix} A_1 & A_2 & \cdots & A_\ell \\ A_2 & & . & 0 \\ \vdots & . & . & \vdots \\ A_\ell & 0 & \cdots & 0 \end{bmatrix}$$

has properties: $H^* = H$ and $(HC_L)^* = HC_L$, or, $HC_L = C_L^* H$. In other words, C_L is self-adjoint in the indefinite scalar product defined by H.

Linearization of Self-adjoint Matrix Polynomials

$I\lambda - C_L$ is a linearization of $L(\lambda)$:
So is $H(I\lambda - C_L) = H\lambda - B$ where

$$B := \begin{bmatrix} -A_0 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & A_3 & \cdots & A_\ell \\ 0 & A_3 & & . & 0 \\ \vdots & \vdots & . & . & \vdots \\ 0 & A_\ell & 0 & \cdots & 0 \end{bmatrix},$$

Thus $H\lambda - B$ is also self-adjoint. Call it the primary linearization of $L(\lambda)$.

Simultaneous reduction of 2 forms

Canonical forms for H and C under strict equivalence and congruence (both over $\mathbb{C}$) differ in only one major respect:

Congruence includes a SIGN CHARACTERISTIC.
Strict equivalence does not.

(See, for example, recent Lancaster/Rodman paper in SIAM Review. Note echoes of Kronecker and Weierstrass.)

Theorem

This sign characteristic is the same set of signs (ϵ, say) appearing in the theory of matrix functions - when applied to $L(\lambda)$.

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We call polynomial systems isospectral if all ev multiplicity data agree.

They are strictly isospectral if they are isospectral AND their sign characteristics agree.

Jordan triples

Simultaneous reduction by congruence:

$$T^*(HC_L)T = P_\epsilon J, \qquad T^*HT = P_\epsilon,$$

or,

$$T^{-1}C_LT = J, \qquad T^{-1}C_LT = J,$$

with

$$J = \begin{bmatrix} J_c & 0 & 0 & 0 \\ 0 & J_{R1} & 0 & 0 \\ 0 & 0 & J_{R2} & 0 \\ 0 & 0 & 0 & \bar{J}_c \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

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Jordan triples (canonical standard triples):

NO SYMMETRY: (X, J, Y).

HERMITIAN: (X, J, PX^*).

Isospectral families: preservation of structure

Theorem

Given $L(\lambda)$ (and hence C_L), let X be any matrix in $\mathbb{C}^{\ell \times \ell n}$ for which $[X, C_L]_{\downarrow}^{\ell}$ is nonsingular (a null-kernel pair) and define

$$Y = \left([X,\ C_L]_{\downarrow}^{\ell}\right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ A_\ell^{-1} \end{bmatrix}.$$

Then $[C_L,\ Y]_{\rightarrow}^{\ell}$ is nonsingular (a full-range pair) and, with

$$E := \left([C_L,\ Y]_{\rightarrow}^{\ell}\right)^{-1} A^{-1}, \quad F := \left([X,\ C_L]_{\downarrow}^{\ell}\right)^{-1},$$

the pencil

$$\lambda A' - B' := E(\lambda A - B)F$$

is the primary linearization of a polynomial $L'(\lambda)$ which is isospectral with $L(\lambda)$. Furthermore, all systems isospectral with $L(\lambda)$ can be generated in this way (i.e. by choice of the matrix X).

Strictly isospectral families: preservation of structure

Theorem

Let $L(\lambda)$ be an Hermitian matrix polynomial and $[X, C_L]_{\downarrow}^{\ell}$ be nonsingular. Then with F as above

$$\lambda A' - B' := F^*(\lambda A - B)F$$

is the primary linearization of a Hermitian polynomial $L'(\lambda)$ which is strictly isospectral with $L(\lambda)$.

Palindromic polynomials

The *reverse* polynomial of $L(\lambda)$ is

$$\mathrm{rev}L(\lambda) := \sum_{j=0}^{\ell} A_{\ell-j}\lambda^j$$

.

Definition: $L(\lambda)$ is $\mu -$ *palindromic* $(\mu = \pm 1)$, if

$$A^*_{\ell-j} = \mu A_j \quad \text{for} \quad j = 0, 1, \ldots, \ell$$

i.e.

$$(\mathrm{rev}L(\lambda))^* = \mu L(\lambda).$$

We take case $\mu = 1$ only. The anti-palindromic case, $\mu = -1$ is analogous.

Observation: A palindromic polynomial of degree one has the form $\lambda M + M^*$ for some square matrix M.

With H, B, C_L as above, define $S_j = AC_L^j$, $j = 0, 1, 2, \ldots$. (Then $S_0 = A = H,\ S_1 = B = AC_L$.) Also, let

$$R_{\ell,n} := \begin{bmatrix} 0 & \cdots & 0 & I_n \\ 0 & & I_n & 0 \\ \vdots & & & \vdots \\ I_n & 0 & \cdots & 0 \end{bmatrix}.$$

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Let $\mathcal{P} = \{(c_1, \ldots, c_\ell) \in \mathbb{C}^\ell \,:\, c_j = c^*_{\ell-j+1},\ j = 1, 2, \ldots, \ell\}$,

and denote the set of all zeros of $c(\lambda) := \sum_{j=1}^{\ell} c_j \lambda^{j-1}$ by $\sigma(c)$.

Define $A_c = R_{\ell n} \sum_{j=1}^{\ell} c_j S_{j-1}$ for each $c \in \mathcal{P}$.

Theorem

Let $L(\lambda)$ be palindromic, $c \in \mathcal{P}$ and $\sigma(c) \cap \sigma(L) = \phi$. Then $\lambda A_c + A_c^*$ is a palindromic linearization of $L(\lambda)$.

The primary linearization

Definition: The primary linearization for a palindromic matrix polynomial $L(\lambda)$ is the palindromic pencil $\lambda A_c + A_c^*$ where

$$A_c = \begin{cases} R_{\ell,n}(S_{k-1} + S_k) & \text{when } \ell = 2k \text{ and } -1 \notin \sigma(L), \\ R_{\ell,n} S_k & \text{when } \ell = 2k+1. \end{cases}$$

(Recall $S_j = AC_L^j$.)

Isospectral families: preservation of structure

Theorem

Let $L(\lambda)$ be palindromic and $[X, C_L]^{\ell}_{\downarrow}$ be nonsingular. Let

$$E := \left([C_L,\ Y]^{\ell}_{\rightarrow}\right)^{-1} A^{-1}, \quad F := \left([X,\ C_L]^{\ell}_{\downarrow}\right)^{-1},$$

(as above) and let $\lambda A_c + A_c^*$ be the primary linearization of $L(\lambda)$. Then

$$\lambda A' - B' := RER(\lambda A_c + A_c^*)F$$

is the primary linearization of an isospectral palindromic polynomial (so that $B' = -A'$).
Furthermore, all palindromic polynomials isospectral with $L(\lambda)$ can be generated in this way (i.e. by choice of the matrix X).

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Furthermore, all palindromic polynomials isospectral with $L(\lambda)$ can be generated in this way (i.e. by choice of the matrix X).

What about the signs? This strict-equivalence is inadequate.

Proposition

The spectrum of a palindromic pencil is symmetric w.r.t. the unit circle in the sense that the ev satisfy either $|\lambda_i| = 1$, or they occur in pairs $\lambda_j \neq \lambda_k$ with $\lambda_j \bar{\lambda}_k = 1$.

This suggests unitary properties for palindromic polynomials.

Can we find an H in which C_L is H-unitary?

i.e. for which $C_L^* H C_L = H$?

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To this end, define

$$\mathcal{Q} = \{z = (z_0, z_1, \ldots, z_\ell) \in \mathbb{C}^{\ell+1} \; : \; -z_i = \overline{z_{\ell-i}} \quad \text{for} \quad i = 0, 1, \ldots, \ell\}.$$

and, whenever $z \in \mathcal{Q}$ define , $q_z(\lambda) := \sum_{i=0}^{\ell} z_i \lambda^i$.

Candidates for indefinite scalar products

Theorem

Let $L(\lambda)$ be palindromic and matrices $S_0, \ldots, S_\ell$ be defined as above. Then all matrices of the form $H_z = R\sum_{j=0}^{\ell} z_j S_j$ with $z \in \mathcal{Q}$, are Hermitian and satisfy the equation $H_z C_L = (C_L^{-1})^* H_z$.

Furthermore, when $z \in \mathcal{Q}$ and $\det A_0 \neq 0$, H_z is nonsingular if and only if no zero of $q_z(\lambda)$ is an eigenvalue of $L(\lambda)$.

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Furthermore, when $z \in \mathcal{Q}$ and $\det A_0 \neq 0$, H_z is nonsingular if and only if no zero of $q_z(\lambda)$ is an eigenvalue of $L(\lambda)$.

When $\ell = 2k$ we choose

$$H = iRS_k$$

and when $\ell = 2k+1$ it is assumed that $1 \notin \sigma(L)$ and

$$H = R(S_k - S_{k+1}).$$

Palindromic Jordan triples

Recall the general Jordan triple: (X, J, Y).
For Hermitian polynomials we have

$$(X, J, PX^*), \quad \text{or} \quad (Y^*P, J, Y)$$

if we “start” with left eigenvectors, Y.

Canonical triple for palindromics?

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Canonical triple for palindromics?

First re-cast Jordan blocks:

$$J = J_1 \oplus J_2 \oplus J_3,$$

where ev of J_1 and J_2 are all those inside, and on the unit circle, resp..

Then J_3 has same structure as J_1, but ev are images of those of J_1; they are outside the unit circle.

Palindromic Jordan triples

Let

$$P_\epsilon = \begin{bmatrix} 0 & 0 & P_1 \\ 0 & P_2 & 0 \\ P_1 & 0 & 0 \end{bmatrix},$$

where P_1, P_2 have simple block structure determined by that of J_1 and J_2, and P_2 depends on ϵ:

$$P_\epsilon^2 = I \quad \text{and} \quad P_\epsilon^* = P_\epsilon.$$

We construct a canonical K from J by bilinear maneouvres, and then:

Theorem

If C_L is H-unitary there is a nonsingular matrix S reducing H and C_L simultaneously to the forms:

$$(S^*)^{-1} H S^{-1} = P_\epsilon, \qquad S C_L S^{-1} = K.$$

It follows that $K^* P_\epsilon K = P_\epsilon$.

Unitary triples

Definition: A standard triple $(X,\ T,\ Y)$ for a palindromic polynomial $L(\lambda)$ is a unitary triple if $T = K$ and

$$X = \begin{cases} iY^*(K^*)^{k-1}P_\epsilon & \text{when } \ell = 2k, \\ Y^*(K^*)^k(I - K^*)P_\epsilon & \text{when } \ell = 2k+1. \end{cases}$$

APPLICATIONS:

Consider the difference equation:

$$A_0 x_i + A_1 x_{i+1} + \cdots + A_\ell x_{i+\ell} = 0, \quad i = 0, 1, \ldots .$$

It is **bounded** if all solutions are bounded.

Theorem

Let $L(\lambda) := \sum_{j=0}^{l} \lambda^j A_j$ have palindromic symmetry with A_ℓ nonsingular. Then this difference equation is "bounded" if and only if $L(\lambda)$ has simple structure with respect to the unit circle.

(All ev on unit circle and only linear elementary divisors.)

Also, a criterion for stably bounded systems in the case of even degree.

REFERENCES:

For material on rational functions and systems theory see:

Gohberg, I., Lancaster, P., and Rodman, L., *Invariant Subspaces of Matrices with Applications*, John Wiley, 1986, SIAM, 2006.

(And other GLR works.)

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For our work on isospectral families and palindromics see:

Lancaster, P., and Prells, U., *Isospectral families of high-order systems*, Z.Angew.Math.Mech., **87**, 2007, 219-234.

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