Perturbation theory for Lagrangian subspaces of symplectic matrices

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joint work with Volker Mehrmann, André Ran, and Leiba Rodman

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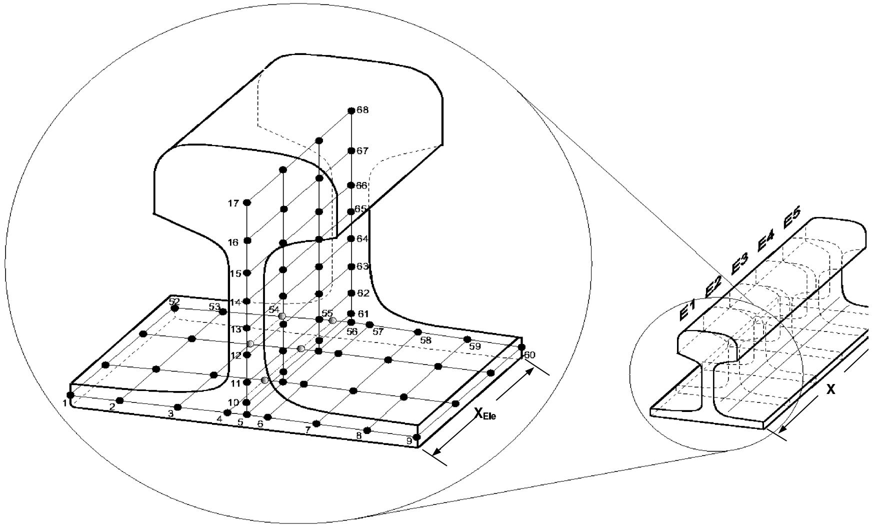
Palindromic polynomial eigenvalue problems

Application: vibration analysis of rail tracks excited by high speed trains



Finite element discretization leads to a **palindromic eigenvalue problem**

$$\left(\lambda^2 A_0^T + \lambda A_1 + A_0\right)x = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$, $A_1^T = A_1$, and A_0 is highly singular.

Palindromic polynomial eigenvalue problems

More applications:

- simulation of surface acoustic wave (SAW) filters (Zaglmayr 2002)
- computation of the Crawford number (Higham/Tisseur/Van Dooren 2002)

Important task: (sometimes) compute the deflating subspace associated with the eigenvalues inside (resp. outside) the unit circle

Question: What happens if there are eigenvalues close to the unit circle?

We need a perturbation theory for deflating subspaces!

Palindromics and symplectics

Helpful observation: Palindromic matrix polynomials are related to symplectic matrices (Schröder 2005)

Example: $P(\lambda) = \lambda^2 A + B + A^T$ can be linearized as

$$L_Z(\lambda) = \lambda Z + Z^T = \lambda \begin{bmatrix} A & B - A^T \\ A & A \end{bmatrix} + \begin{bmatrix} A^T & A^T \\ B - A & A^T \end{bmatrix}$$

if -1 is not an eigenvalue of $P(\lambda)$.

$Z^{-1}Z^T$ is similar to a symplectic matrix if 1 is not an eigenvalue of $P(\lambda)$.

Symplectic matrices

Symplectic matrices: $S \in \mathbb{F}^{2n\times 2n}$ is called **symplectic** if

$$S^T J S = J, \quad \text{where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Definition: (more general)

1) Let $J \in \mathbb{F}^{2n\times 2n}$ be a skew-symmetric invertible matrix.
   - A matrix $S \in \mathbb{F}^{2n\times 2n}$ is called J-**symplectic** if $S^T J S = J$.
2) Let $J \in \mathbb{C}^{m\times m}$ be a Hermitian invertible matrix.
   - A matrix $S \in \mathbb{C}^{m\times m}$ is called J-**unitary** if $S^* J S = J$.

Assumption: In 2), we assume J has n negative and n positive eigenvalues. Then J is congruent to

$$i \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Canonical forms for symplectic matrices

Three cases:

- Complex J-symplectics ($J^T = -J,\ S^T JS = J$);
- Real J-symplectics ($J^T = -J,\ S^T JS = J$);
- Complex J-unitaries ($J^* = J,\ S^* JS = J$);
  (and J has n negative and n positive eigenvalues);

Transformations that preserve structure:

- for J-symplectics: $(J, S) \mapsto (P^T JP,\ P^{-1}SP), \quad P$ invertible;
- for J-unitaries: $(J, S) \mapsto (P^* JP,\ P^{-1}SP), \quad P$ invertible;

Canonical forms for symplectic matrices

Case 1: Complex J-symplectics $\left(J^T = -J,\ S^T J S = J\right)$

- eigenvalues occur in reciprocal pairs: if λ is an eigenvalue, then so is λ^{-1} with the same algebraic multiplicity;
- the pairing also occurs with respect to the Jordan structure;
- the pairing degenerates for $\lambda = \pm 1$;
- the total number m_q of Jordan blocks of size $2q + 1$ associated with the eigenvalue $\lambda = +1$ is even;
- the total number m_r of Jordan blocks of size $2r + 1$ associated with the eigenvalue $\lambda = -1$ is even;

Canonical forms for symplectic matrices

Case 2: Complex J-unitaries $\left(J^* = J,\ S^*JS = J\right)$

- eigenvalues occur in conjugate reciprocal pairs: if λ is an eigenvalue, then so is $\overline{\lambda}^{-1}$ with the same algebraic multiplicity;
- the pairing also occurs with respect to the Jordan structure;
- the pairing degenerates for unimodular eigenvalues λ (i.e., $|\lambda| = 1$);
- unimodular eigenvalues have **signs** as additional invariants, e.g.,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix};$$

  S_1 and S_2 are similar, but the pairs (J, S_1) and (J, S_2) are not equivalent;
- the collection of signs is called the **sign characteristic** of (J, S);

Canonical forms for symplectic matrices

Case 3: Real J-symplectics $(J^T = -J,\ S^T JS = J)$

- eigenvalues occur in quadruplets: if λ is an eigenvalue, then so are $\overline{\lambda}$, λ^{-1}, and $\overline{\lambda}^{-1}$ with the same algebraic multiplicity;
- the pairing also occurs with respect to the Jordan structure;
- the pairing degenerates for unimodular eigenvalues λ (i.e., $|\lambda| = 1$);
- unimodular eigenvalues have **signs** as additional invariants;
- the sign characteristics associated with λ and $\overline{\lambda}$ are related;
- the total numbers $m_{q,\pm}$ of Jordan blocks of size $2q+1$ associated with the eigenvalues $\lambda = \pm 1$ are even;

Lagrangian subspaces

Definition:

1) Let $J \in \mathbb{F}^{2n \times 2n}$ be a skew-symmetric invertible matrix.

   - A subspace $\mathcal{M} \subseteq \mathbb{F}^{2n}$ is called **J-Lagrangian** if $\dim \mathcal{M} = n$ and

$$y^T J x = 0 \quad \text{for all } x, y \in \mathcal{M}.$$

2) Let $J \in \mathbb{C}^{2n \times 2n}$ be a Hermitian matrix with n positive and n negative eigenvalues.

   - A subspace $\mathcal{M} \subseteq \mathbb{C}^{2n}$ is called **J-Lagrangian** if $\dim \mathcal{M} = n$ and

$$y^* J x = 0 \quad \text{for all } x, y \in \mathcal{M}.$$

J-Lagrangian subspaces are maximal J-neutral subspaces.

Lagrangian subspaces

Aim: Study J-Lagrangian subspaces that are invariant for a J-symplectic S.

Examples:

1) $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathcal{M} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$;

   $\mathcal{M}$ is an S-invariant J-Lagrangian subspace;

2) $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, $S = \begin{bmatrix} S_1 & 0 \\ 0 & (S_1^{-1})^T \end{bmatrix}$, $\mathcal{M} = \text{span}\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix}\right)$;

   $\mathcal{M}$ is an S-invariant J-Lagrangian subspace;

Special case: If S has no unimodular eigenvalues, then the invariant subspace associated with the eigenvalues inside the unit circle is J-Lagrangian.

Stability of Lagrangian subspaces

$\mathcal{L}(J, S) := \{\mathcal{M} \subseteq \mathbb{F}^{2n} : \mathcal{M}$ is S-invariant and J-Lagrangian$\}$

Definition: Let S, S' be J-symplectic and let $\mathcal{M}$ be an S-invariant J-Lagrangian subspace.

1) $\mathcal{M}$ is called **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|S - S'\| < \delta \ \Rightarrow \ \exists \mathcal{M}' \in \mathcal{L}(J, S') \ : \ \mathrm{gap}\,(\mathcal{M}, \mathcal{M}) < \varepsilon.$$

2) $\mathcal{M}$ is called **conditionally stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|S - S'\| < \delta \text{ and } \mathcal{L}(J, S') \neq \emptyset \ \Rightarrow \ \exists \mathcal{M}' \in \mathcal{L}(J, S') \ : \ \mathrm{gap}\,(\mathcal{M}, \mathcal{M}) < \varepsilon.$$

Gap metric: $\mathrm{gap}\,(\mathcal{M}, \mathcal{N}) := \|P_{\mathcal{M}} - P_{\mathcal{N}}\|$;
($P_{\mathcal{M}}, P_{\mathcal{N}}$: orthogonal projections onto $\mathcal{M}$, $\mathcal{N}$);

Stability of Lagrangian subspaces

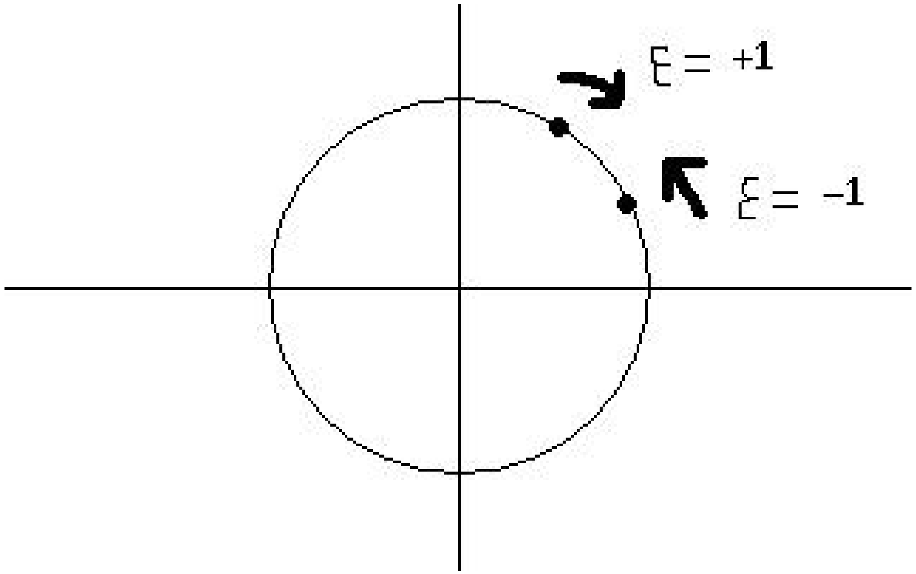
Case 1: Complex J-symplectics $(J^T = -J,\ S^T J S = J)$

Theorem: Every J-symplectic matrix S has an invariant J-Lagrangian subspace associated with eigenvalues in the closed unit disc.

Theorem: The following assertions are equivalent for an S-invariant J-Lagrangian subspace $\mathcal{M}$:

- $\mathcal{M}$ is stable;
- $\mathcal{M}$ is conditionally stable;
- dim Ker$(S+1) \leq 1$ and dim Ker$(S-1) \leq 1$

What happens under perturbations?



- let $S \in \mathbb{R}^{2n\times 2n}$ be J-symplectic, resp., let $S \in \mathbb{C}^{2n\times 2n}$ be J-unitary;
- let S have two close unimodular eigenvalues with opposite signs;
- if S is perturbed and the two eigenvalues meet, they generically form a Jordan block; then they may split off as a pair of nonunimodular reciprocal eigenvalues;

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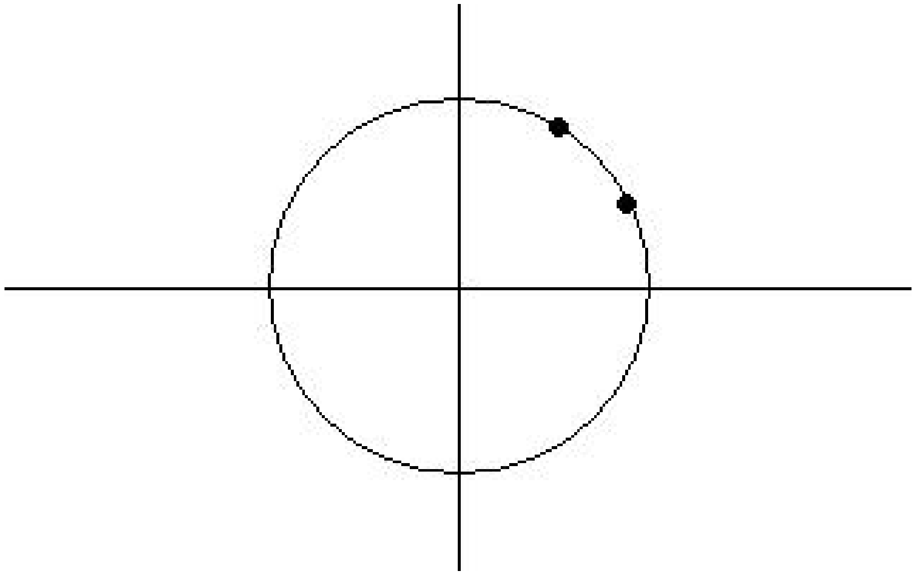

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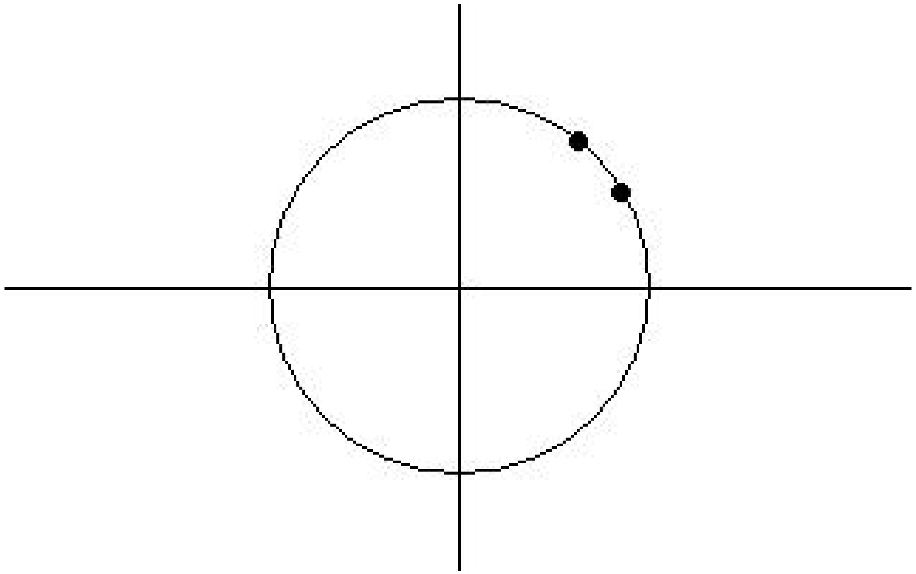

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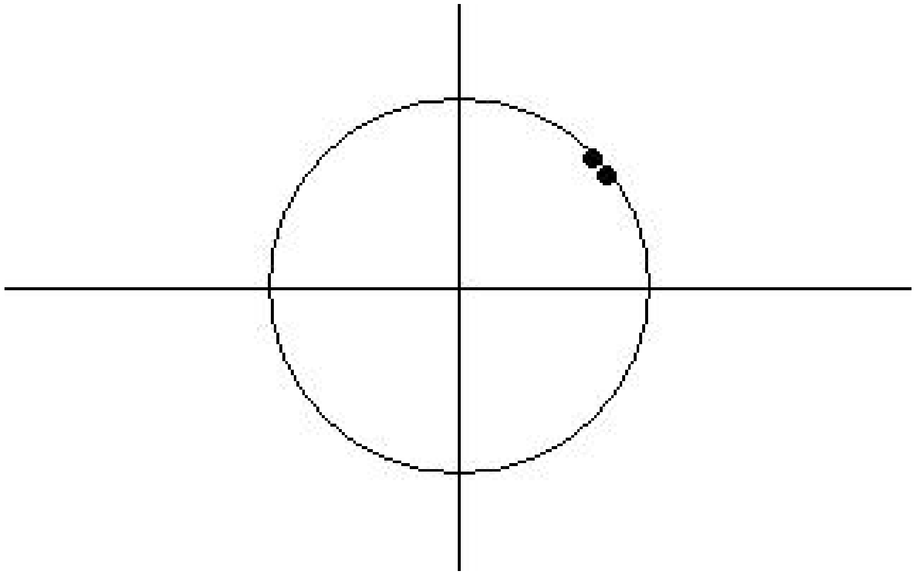

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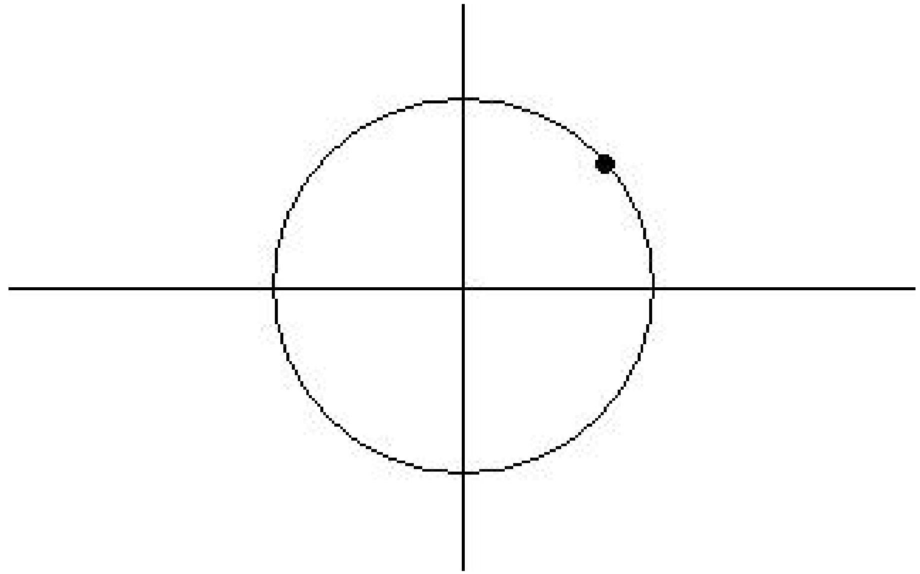

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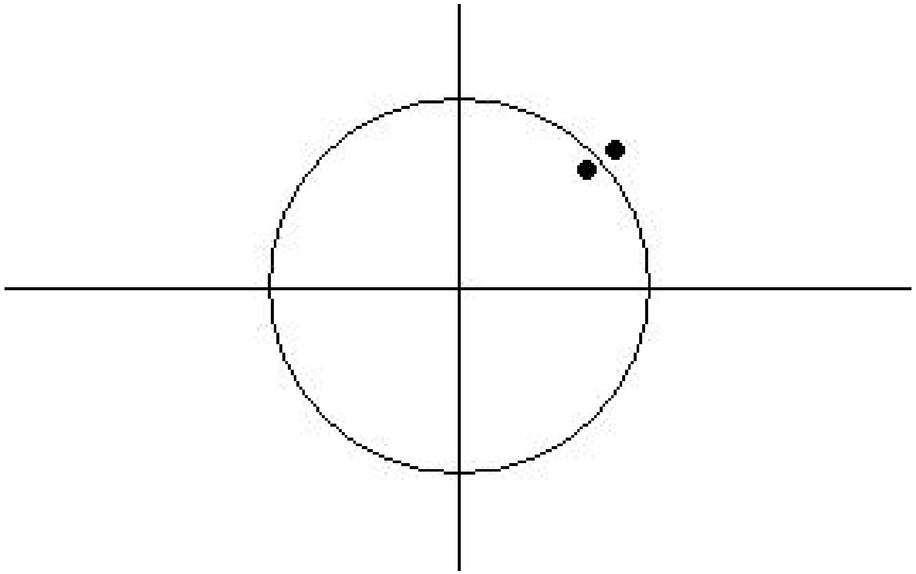

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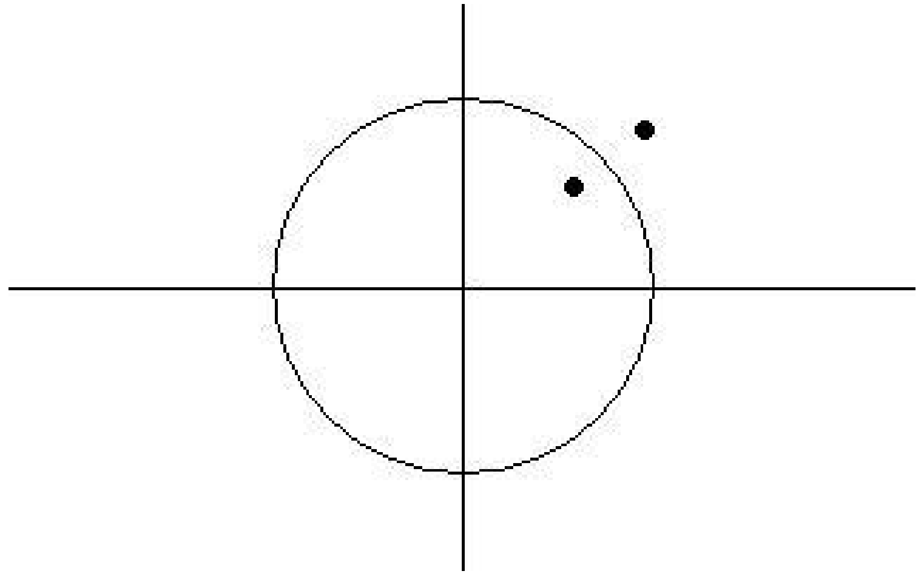

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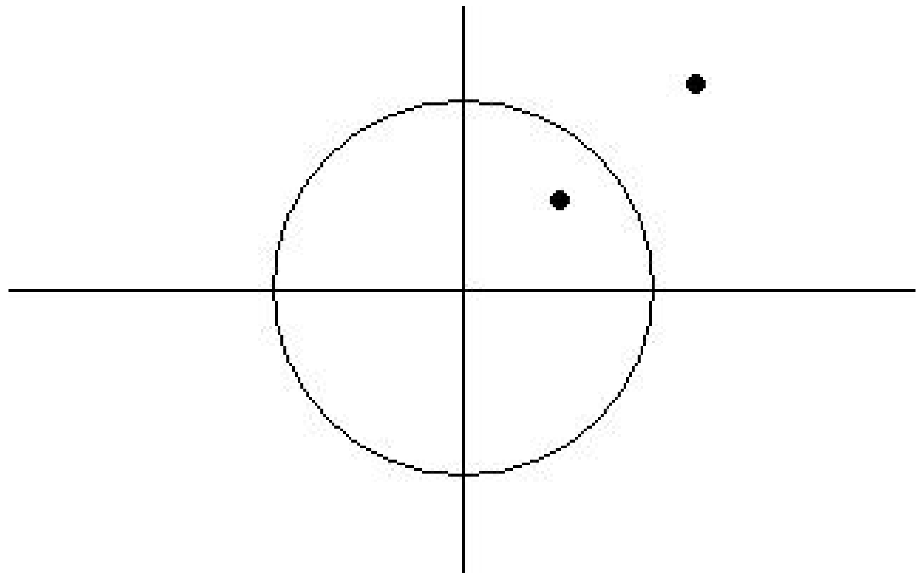

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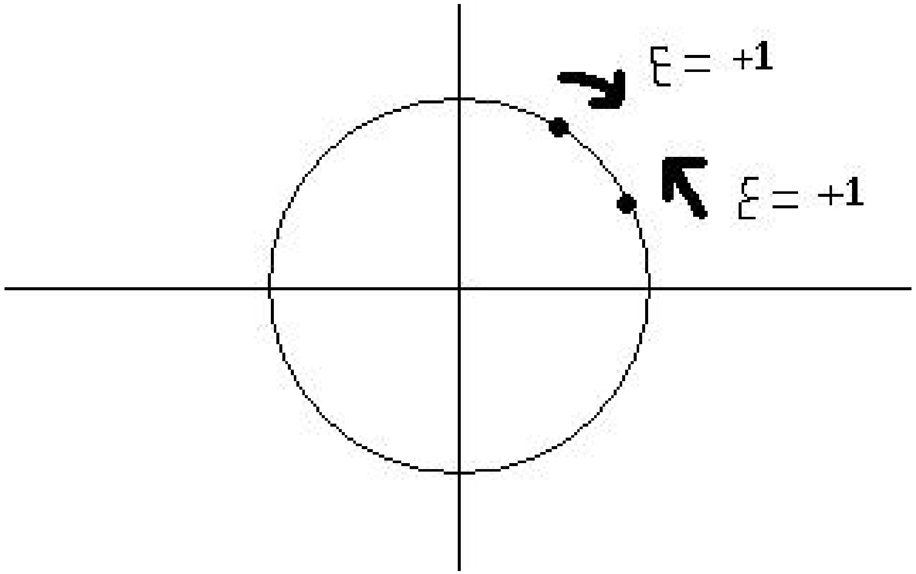

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- if S is perturbed and the two eigenvalues meet, they *cannot* form a Jordan block, and they *must* remain on the unit circle;

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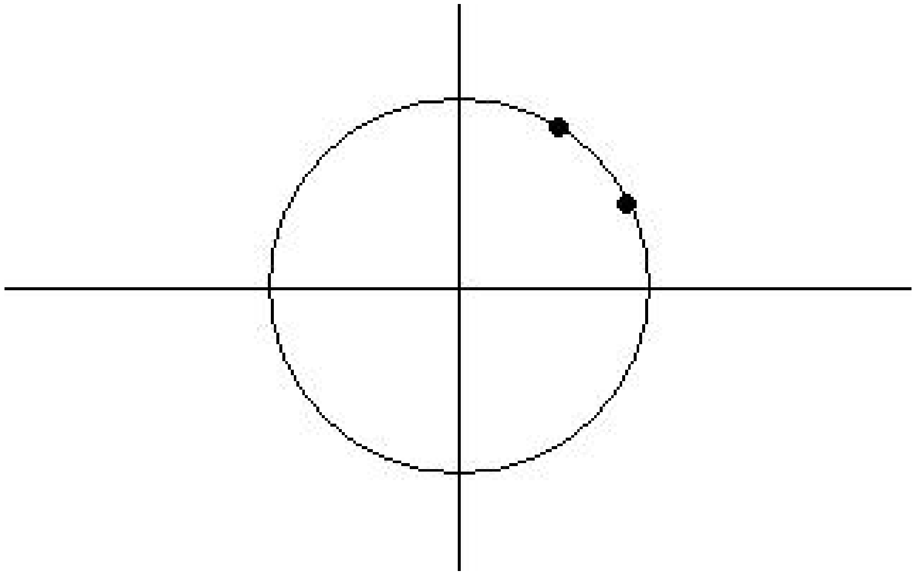

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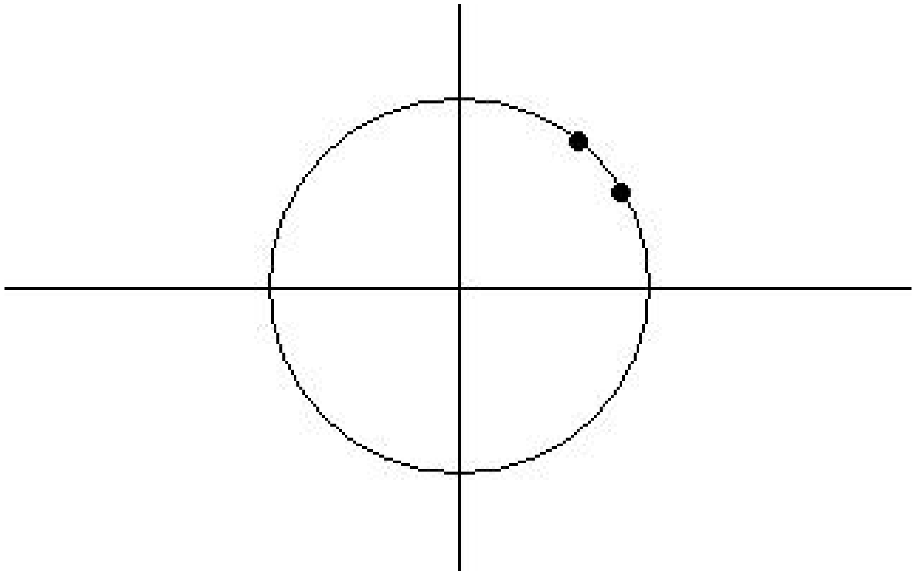

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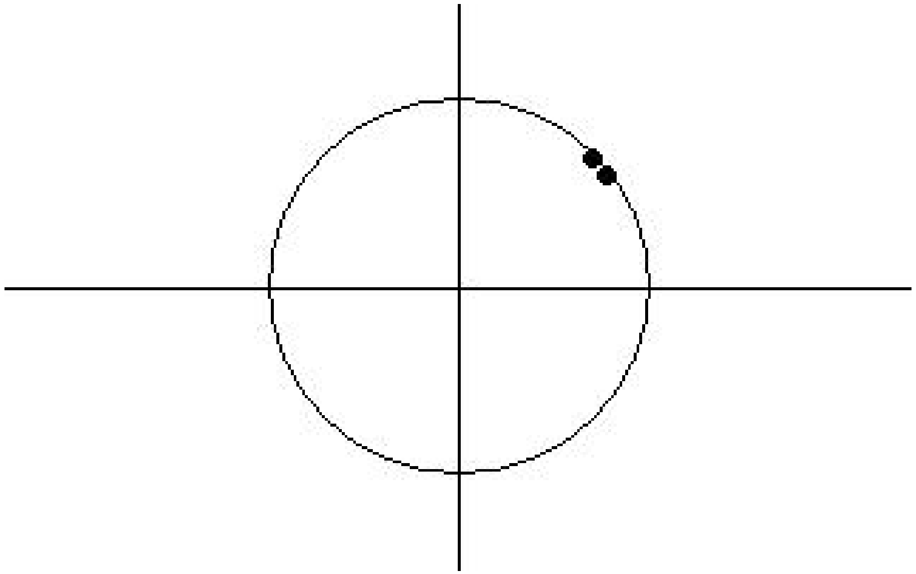

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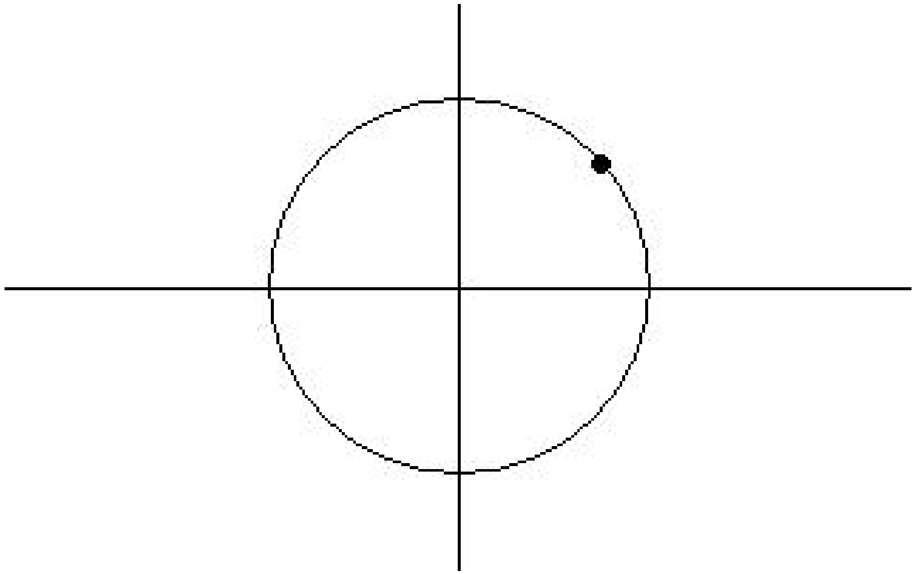

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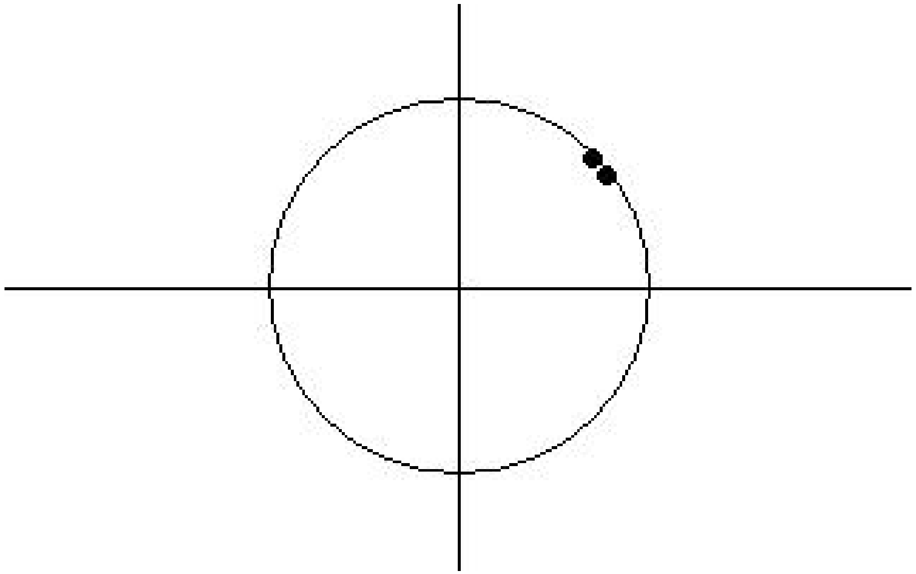

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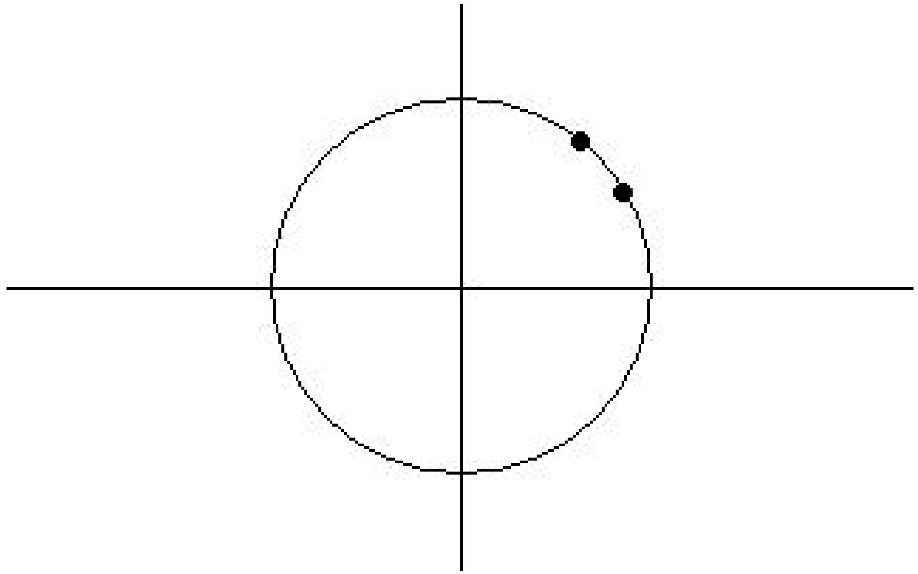

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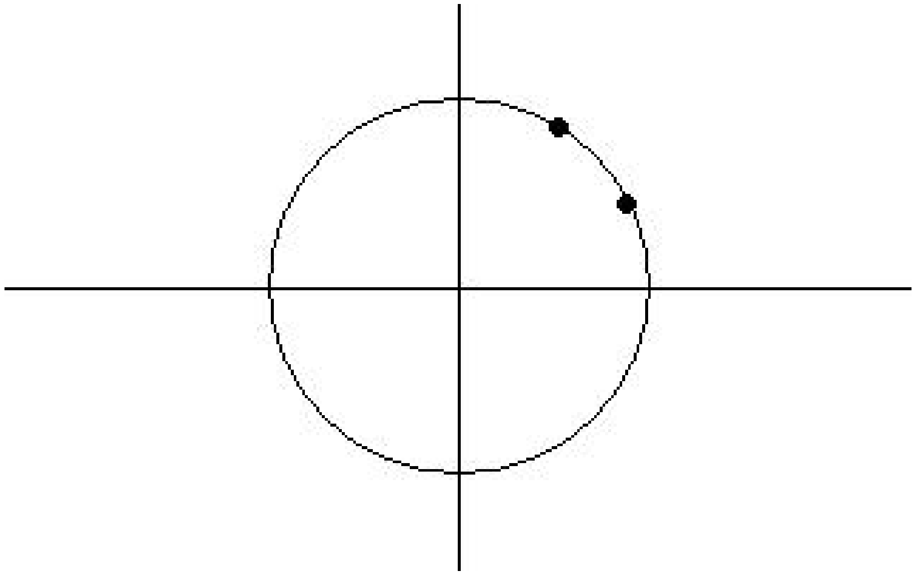

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Stability of Lagrangian subspaces

Case 2: Complex J-unitaries $(J^* = J,\ S^*JS = J)$

Theorem: Let S be J-unitary. There exists an S-invariant J-Lagrangian subspace if and only if for every unimodular eigenvalue $\omega \in \mathbb{C}$, $|\omega| = 1$, the number of odd partial multiplicities corresponding to ω is even, and the signs in the sign characteristic of S that correspond to these odd partial multiplicities sum up to zero.

Theorem: Let S be J-unitary. There exists a stable S-invariant J-Lagrangian subspace if and only if S has no unimodular eigenvalues.

Stability of Lagrangian subspaces

Case 2: Complex J-unitaries $(J^* = J,\ S^*JS = J)$

Theorem: Let S be J-unitary. There exists a conditionally stable S-invariant J-Lagrangian subspace if and only if every unimodular eigenvalue ω has only even partial multiplicities, and all the signs in the sign characteristic of S corresponding to ω are equal.

Stability of Lagrangian subspaces

Case 3: Real J-symplectics $(J^T = -J,\ S^T JS = J)$

Theorem: Let S be J-symplectic. There exists an S-invariant J-Lagrangian subspace if and only if for every unimodular eigenvalue $\omega \in \mathbb{C} \setminus \mathbb{R}$, $|\omega| = 1$, the number of odd partial multiplicities corresponding to ω is even, and the signs in the sign characteristic of S that correspond to these odd partial multiplicities sum up to zero.

Theorem: Let S be J-symplectic. There exists a stable S-invariant J-Lagrangian subspace if and only if S has no unimodular eigenvalues.

Stability of Lagrangian subspaces

Case 3: Real J-symplectics $(J^T = -J,\ S^TJS = J)$

Theorem: Let S be J-symplectic. There exists a conditionally stable S-invariant J-Lagrangian subspace if and only if:

1) every unimodular eigenvalue $\omega \neq \pm 1$ has only even partial multiplicities, and all the signs in the sign characteristic corresponding to ω are equal.

2) the eigenvalue 1 of S only has even partial multiplicities, say $2n_1, \ldots, 2n_p$, and if $\kappa_1, \ldots, \kappa_p$ are the corresponding signs, then

$$(-1)^{n_1}\kappa_1 = (-1)^{n_2}\kappa_2 = \cdots = (-1)^{n_p}\kappa_p.$$

3) the eigenvalue -1 of S only has even partial multiplicities, say $2n'_1, \ldots, 2n'_{p'}$, and if $\kappa'_1, \ldots, \kappa'_{p'}$ are the corresponding signs, then

$$(-1)^{n'_1}\kappa'_1 = (-1)^{n'_2}\kappa'_2 = \cdots = (-1)^{n'_{p'}}\kappa'_{p'}.$$

Index of stability of Lagrangian subspaces

Definition: Let S, S' be J-symplectic and let $\mathcal{M}$ be an S-invariant J-Lagrangian subspace.

1) $\mathcal{M}$ is called **α-stable** if there exists $\delta, K > 0$ such that

$$\|S - S'\| < \delta \Rightarrow \ \exists \mathcal{M}' \in \mathcal{L}(J, S') \ : \ \mathrm{gap}\,(\mathcal{M}, \mathcal{M}) \leq K\|S - S'\|^{1/\alpha}.$$

2) $\alpha_0 \geq 1$ is the **index of stability** of $\mathcal{M}$ if $\mathcal{M}$ is α_0-stable, but not α-stable for any $\alpha < \alpha_0$.

3) ... (analogously: **conditional α-stability** and **index of conditional α-stability**) ...

Open problem: For an α-stable S-invariant J-Lagrangian subspace determine the index of (conditional) α-stability if it exists. (So far: bounds and answers for special cases.)

Conclusions

- perturbation theory for J-Lagrangian invariant subspaces of J-symplectic matrices now complete;
- open problem: extend the results to palindromic matrix pencils and palindromic matrix polynomials;
- Reference: M., Mehrmann, Ran, Rodman. *Perturbation analysis of Lagrangian invariant subspaces of symplectic matrices*. MATHEON preprint, TU Berlin, 2006, available from `http://www.matheon.de`.