On a Quadratic Eigenproblem Occuring in Regularized Total Least Squares

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Outline

- Regularized total least squares problems
- A quadratic eigenproblem
- Nonlinear maxmin characterization
- The maximum real solution of a quadratic eigenproblem

Total Least Squares Problem

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Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m \geq n$

Find $\tilde{A} \in \mathbb{R}^{m \times n}$, $\tilde{b} \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ such that

$$\|(A, b) - (\tilde{A}, \tilde{b})\|_F^2 = \min! \quad \text{subject to } \tilde{A}x = \tilde{b},$$

where $\| \cdot \|_F$ denotes the Frobenius norm.

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Using the orthogonal distance this problems can be rewritten as (cf. Golub, Van Loan 1980)

Find $x \in \mathbb{R}^n$ such that

$$\frac{\|Ax - b\|_2^2}{1 + \|x\|_2^2} = \min! \quad \text{subject to } \|Lx\|_2^2 \leq \delta^2.$$

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If $\delta > 0$ is chosen small enough (e.g. $\delta < \|Lx_{TLS}\|$ where x_{TLS} is the solution of the TLS problem), then the constraint $\|Lx\|_2^2 \le \delta^2$ is active, and the RTLS problem reads

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The first order optimality conditions are

$$B(x)x + \lambda L^T Lx = d(x), \quad \|Lx\|_2^2 = \delta^2$$

where

$$B(x) = \frac{1}{1+\|x\|_2^2}\left(A^TA - f(x)I_n\right), \ f(x) = \frac{\|Ax-b\|_2^2}{1+\|x\|_2^2}, \ d(x) = \frac{A^Tb}{1+\|x\|_2^2}.$$

Algorithm RTLSQEP: Sima, Van Huffel & Golub 2004

Initialization Let x^0 be a starting vector. Compute $B_0 := B(x^0)$ and $d_0 = d(x^0)$. Set $j = 0$

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stopping criterion if

$$\|B_{j+1} x^{j+1} + \lambda_{j+1} L^T L x^{j+1} - d_{j+1}\|_2 < \varepsilon$$

then STOP; else $j \leftarrow j + 1$ and go to step j.

A quadratic eigenproblem

Sima, van Huffel, Golub (2004)

The first order conditions can be solved via the maximal positive eigenvalue and corresponding eigenvector of a quadratic eigenproblem

$$\left((W + \lambda I)^2 - \delta^{-2} h h^T\right) u = 0 \qquad (QEP)$$

where $W \in \mathbb{R}^{k \times k}$ is symmetric, and $h \in \mathbb{R}^k$.

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$$\|Ax - b\|_2 = \min! \quad \text{subject to } \|x\|_2^2 = \delta^2$$

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For every fixed $x \in \mathbb{C}^n$, $x \neq 0$ assume that the real function

$$f(\cdot; x) \; : \; J \to \mathbb{R}, \; f(\lambda; x) := x^H T(\lambda) x$$

is continuously differentiable, and that the real equation

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Assume that

$$\frac{\partial}{\partial \lambda} f(\lambda; x)\Big|_{\lambda = p(x)} > 0 \quad \text{for every } x \in D.$$

maxmin characterization (V., Werner 1982)

Let $\sup_{v \in D} p(v) \in J$ and assume that there exists a subspace $W \subset \mathbb{C}^n$ of dimension ℓ such that

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- Then $T(\lambda)x = 0$ has at least ℓ eigenvalues in J, and for $j = 1, \dots, \ell$ the j-largest eigenvalue λ_j can be characterized by

$$\lambda_j = \max_{\substack{\dim V = j, \\ V \cap D \neq \emptyset}} \inf_{v \in V \cap D} p(v). \qquad (1)$$

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$$V \cap D \neq \emptyset \quad \text{and} \quad \lambda_j = \inf_{v \in V \cap D} p(v)$$

  is contained in $D \cup \{0\}$, and the maxmin characterization of λ_j can be replaced by

$$\lambda_j = \max_{\substack{\dim V = j, \\ V \setminus \{0\} \subset D}} \min_{v \in V \setminus \{0\}} p(v).$$

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is a parabola which attains its minimum at

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Let $J = (-\lambda_{\min}, \infty)$ where $\lambda_{\min}$ is the minimum eigenvalue of W. Then $f(\lambda, x) = 0$ has at most one solution $p(x) \in J$ for every $x \neq 0$. Hence, the Rayleigh functional p of (QEP) corresponding to J is defined, and the general conditions are satisfied.

Characterization of maximal real eigenvalue

Let $x_{\min}$ be an eigenvector of W corresponding to $\lambda_{\min}$. Then

$$f(-\lambda_{\min}, x_{min}) = x_{\min}^H (W - \lambda_{\min})^2 x_{\min} - |x_{\min}^H h|^2/\delta^2 = -|x_{\min}^H h|^2/\delta^2 \leq 0$$

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If $x_{min}^H h = 0$, and the minimum eigenvalue $\mu_{\min}$ of $T(-\lambda_{\min})$ is negative, then for the corresponding eigenvector y_{min} it holds

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$$f(-\lambda_{\min}, x) = x^H T(-\lambda_{\min}) x \geq 0 \quad \text{for every } x \neq 0,$$

and $D = \emptyset$.

Characterization of maximal real eigenvalue ct.

Assume that $D \neq \emptyset$. For $x^H h = 0$ it holds that

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Hence, D does not contain a two-dimensional subspace of $\mathbb{R}^n$, and therefore J contains at most one eigenvalue of (QEP).

If $\lambda \in \mathbb{C}$ is a non-real eigenvalue of (QEP) and x a corresponding eigenvector, then

$$x^H T(\lambda)x = \lambda^2 \|x\|_2^2 + 2\lambda x^H W x + \|Wx\|_2^2 - |x^H h|^2/\delta^2 = 0.$$

Hence, the real part of λ satisfies

$$\text{real}(\lambda) = -\frac{x^H W x}{x^H x} \leq -\lambda_{\min}.$$

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- Otherwise, the maximal real eigenvalue is the unique eigenvalue $\hat{\lambda}$ of (QEP) in $J = (-\lambda_{\text{min}}, \infty)$, and it holds

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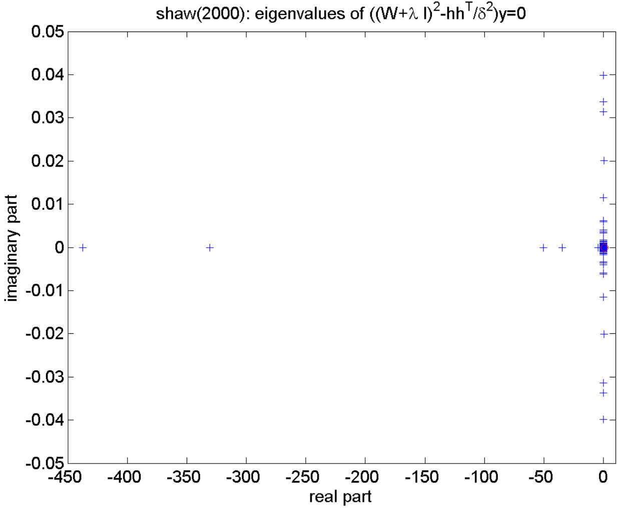
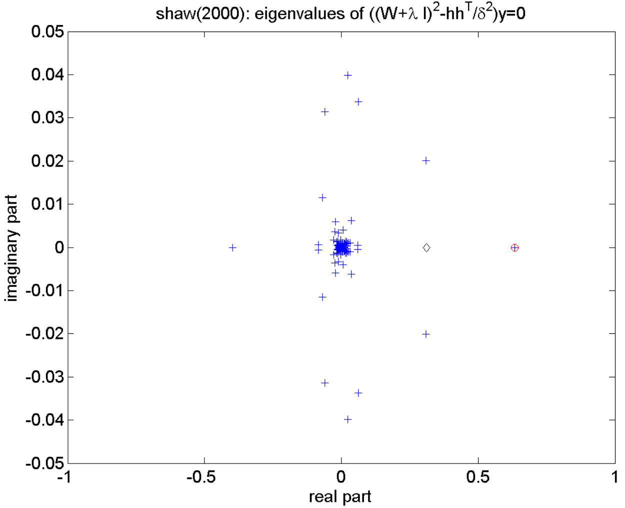
- $\hat{\lambda}$ is the right most eigenvalue of (QEP), i.e.

$$\text{real}(\lambda) \leq -\lambda_{\text{min}} \leq \hat{\lambda} \quad \text{for every eigenvalue } \lambda \neq \hat{\lambda} \text{ of (QEP).}$$

Example


shaw(2000): eigenvalues of ((W+λ I)^2-hh^T/δ^2)y=0
imaginary part
real part
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
-450
-400
-350
-300
-250
-200
-150
-100
-50
0

Example: close up



Positivity of $\hat{\lambda}$

Simplest counter–example: If W is positive definite with eigenvalue $\lambda_j > 0$, then $-\lambda_j$ are the only eigenvalues of the quadratic eigenproblem $(W + \lambda I)^2 x = 0$, and if the term $\delta^{-2} hh^T$ is small enough, then the quadratic problem will have no positive eigenvalue, but the right–most eigenvalue will be negative.

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However, in quadratic eigenproblems occurring in regularized total least squares problems δ and h are not arbitrary, but regularization only makes sense if $\delta \leq \|Lx_{\text{TLS}}\|$ where x_{TLS} denotes the solution of the total least squares problem without regularization.

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The following theorem characterizes the case that the right–most eigenvalue is negative.

Positivity of $\hat{\lambda}$ ct.

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The maximal real eigenvalue $\hat{\lambda}$ of the quadratic problem

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For the standard case $L = I$ the right-most eigenvalue $\hat\lambda$ is always nonnegative if $\delta < \|x_{TLS}\|_2$.

Convergence

Theorem (Sima et al.) Assume that

$$\min_{x\neq 0\,:\,Lx=0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \geq f(x^0). \qquad (*)$$

Then the algorithm provides a sequence of vectors $\{x^j\}$ for which the function f is monotonically decreasing:

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Under the condition (*) every limit point of $\{x^j\}$ is a global minimizer of

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Beck & Teboulle (2006) proved the convergence of Sima's algorithm if the equality constraint is replaced by the inequality constraint $\|Lx\|_2 \leq \delta$.