Flow control applied to transitional flows: input-output analysis, model reduction and control

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Outline

- Introduction with input-output configuration
- Matrix-free methods for flow stability using Navier-Stokes snapshots
  Edwards *et al.* (1994), ...
- Global modes and transient growth
  Cossu & Chomaz (1997), ...
- Input-output characteristics of Blasius BL and reduced order models based on balanced truncation
  Rowley (2005), ...
- LQG feedback control based on reduced order model
- Conclusions

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Message

- Need only snapshots from a Navier-Stokes solver (with adjoint) to perform stability analysis and control design for complex flows

- Main example Blasius BL, but many other more complex flows are now tractable ...

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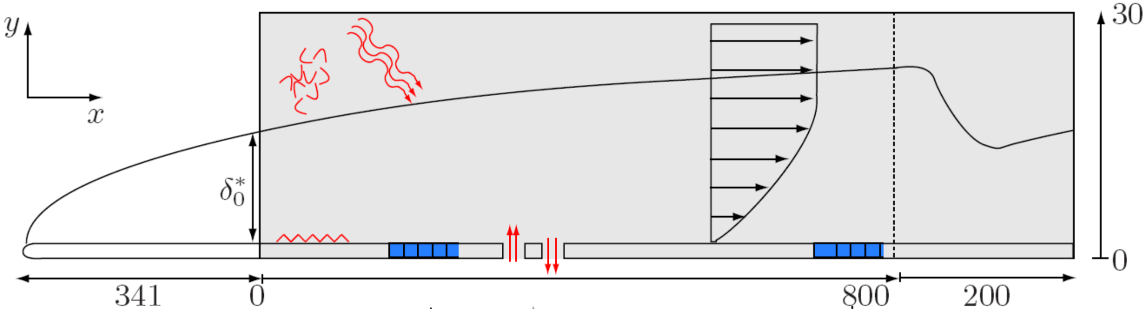
Linearized Navier-Stokes for Blasius flow



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Continuous formulation

$$\frac{\partial u}{\partial t} = \mathcal{A}u - \nabla p$$
$$0 = \nabla \cdot u$$
$$u = u_0 \quad \text{at} \quad t = 0$$

$$\mathcal{A} = -(U \cdot \nabla) - (\nabla U^T)^T + \frac{1}{Re}\nabla^2 + \lambda(x)$$

$$Re = \frac{U_\infty \delta_0^*}{\nu} = 1000$$

Discrete formulation

$$\frac{du}{dt} = Au$$
$$u = u_0 \quad \text{at} \quad t = 0$$

Input-output configuration for linearized N-S

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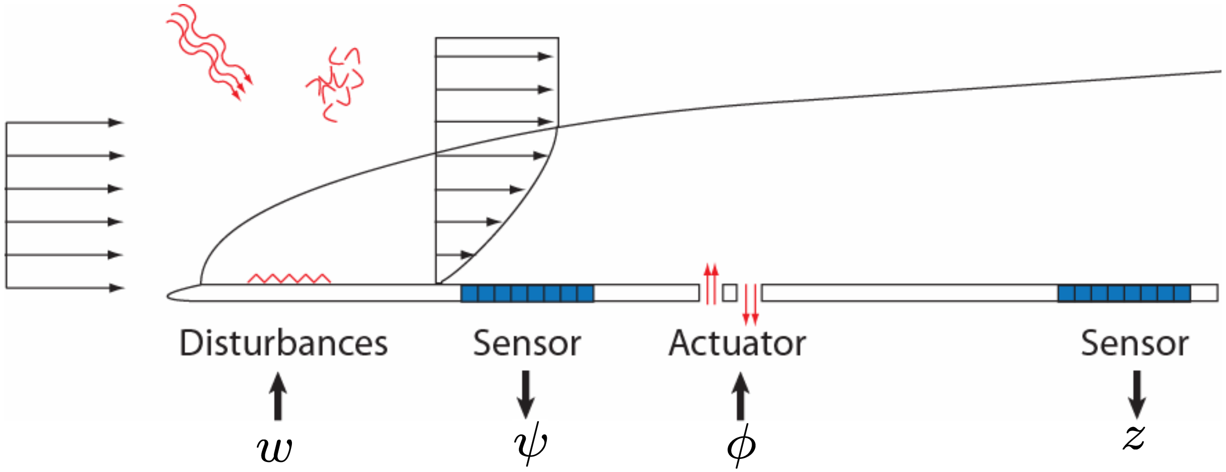

$$\begin{aligned} \frac{du}{dt} &= Au + Bf \\ y &= Cu \\ u &= u_0 \quad \text{at} \quad t = 0 \end{aligned}$$

$$\begin{aligned} B_1 f_1 = B_1 w &: \text{ incomming dist.} \\ B_2 f_2 = B_2 \phi &: \text{ actuator input} \\ y_1 = \psi = C_1 u &: \text{ sensor output} \\ y_2 = z = C_2 u &: \text{ objective function} \end{aligned}$$

$$\begin{aligned} B_1 &= h(x_i) = \text{"Gaussian"} \\ C_1 u &= (h, u) = h^T u \end{aligned}$$

Solution to the complete input-output problem

$$y(t) \;=\; C\, \underbrace{e^{At} u_0}_{\text{IVP}} + \underbrace{C \int_0^t e^{A(t-\tau)} B f(\tau)\, d\tau}_{\text{forced solution}}$$

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- Initial value problem: flow stability
- Forced problem: input-output analysis

The Initial Value Problem

- Asymptotic stability analysis:
  Global modes of the Blasius boundary layer

- Transient growth analysis:
  Optimal disturbances in Blasius flow

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Dimension of discretized system

	Base Flow	Inhomogeneous direction(s)	Dimension of $\boldsymbol{u}(t)$	Storage of A
Ginzburg-Landau	$U(x)$	1D	10^2	1 MB
Blasius	$U(x,y)$	2D	10^5	25 GB
Jet in crossflow	$U(x,y,z)$	3D	10^7	500 TB

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- Matrix A very large for complex spatially developing flows
- Consider eigenvalues of the matrix exponential, related to eigenvalues of A

$$\lambda_j = e^{\omega_j t}$$

- Use Navier-Stokes solver (DNS) to approximate the action of matrix exponential or evolution operator

$$u(t) = e^{At} u_0 = T(t) u_0$$

Krylov subspace with Arnoldi algorithm

- Krylov subspace created using NS-timestepper
- Orthogonal basis created with Gram-Schmidt
- Approximate eigenvalues from Hessenberg matrix H

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Krylov subspace: $\{v_1, e^{At}v_1, \cdots, (e^{At})^{m-1}v_1\}$

orthogonal basis: $V = \{v_1, v_2, \cdots, v_m\}$

$$\Rightarrow \quad e^{At} \approx VHV^T \qquad H : m \times m$$

eigenvalues: $H = E\tilde{\Lambda}E^{-1} \quad \Rightarrow \quad e^{At} \approx VE\tilde{\Lambda}E^{-1}V^T$

Global spectrum for Blasius flow

- Least stable eigenmodes equivalent using time-stepper and matrix solver
- Least stable branch is a global representation of Tollmien-Schlichting (TS) modes

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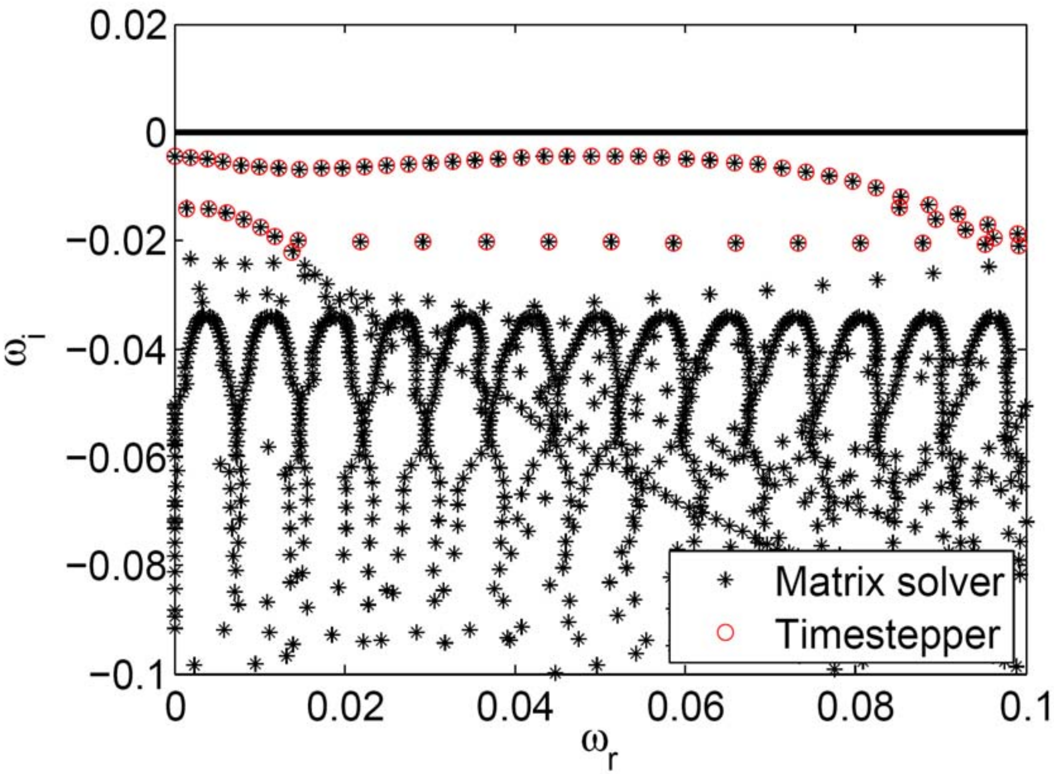
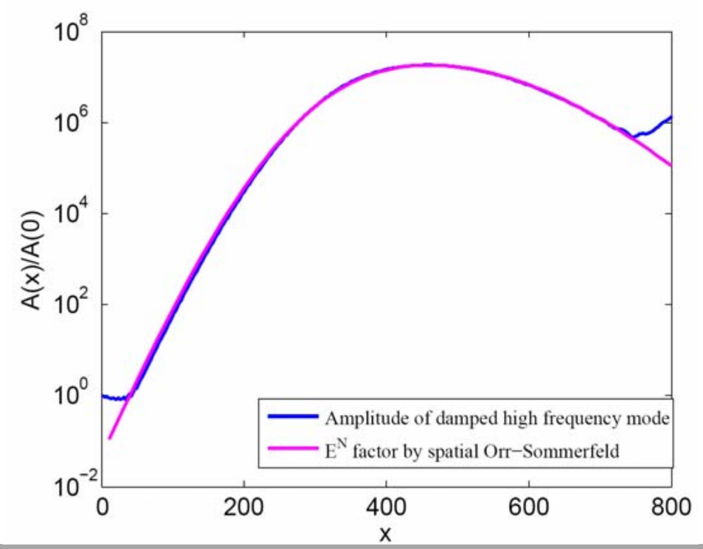
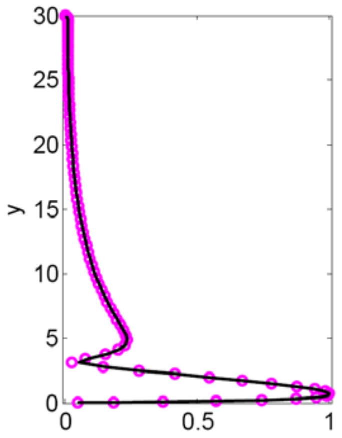


Global TS-waves

- Streamwise velocity of least damped TS-mode

- Envelope of global TS-mode identical to local spatial growth
- Shape functions of local and global modes identical





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Optimal disturbance growth

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- Optimal growth from eigenvalues of $T^*(t)T(t)$ $\qquad T(t) = e^{At}$

$$G(t) = \max_{\|u_0\|=1} (u(t), u(t)) = \max_{\|u_0\|=1} (u_0, T^*(t)T(t)u_0)$$

$$T^*(t)T(t)u_0 = \lambda_E u_0$$

- Krylov sequence built by forward-adjoint iterations

$$T^*(t)T(t) \approx VHV^T = VE\Lambda_E E^{-1}V^T$$

Evolution of optimal disturbance in Blasius flow

- Full adjoint iterations (black)
  sum of TS-branch modes only (magenta)

- Transient since disturbance propagates out of domain

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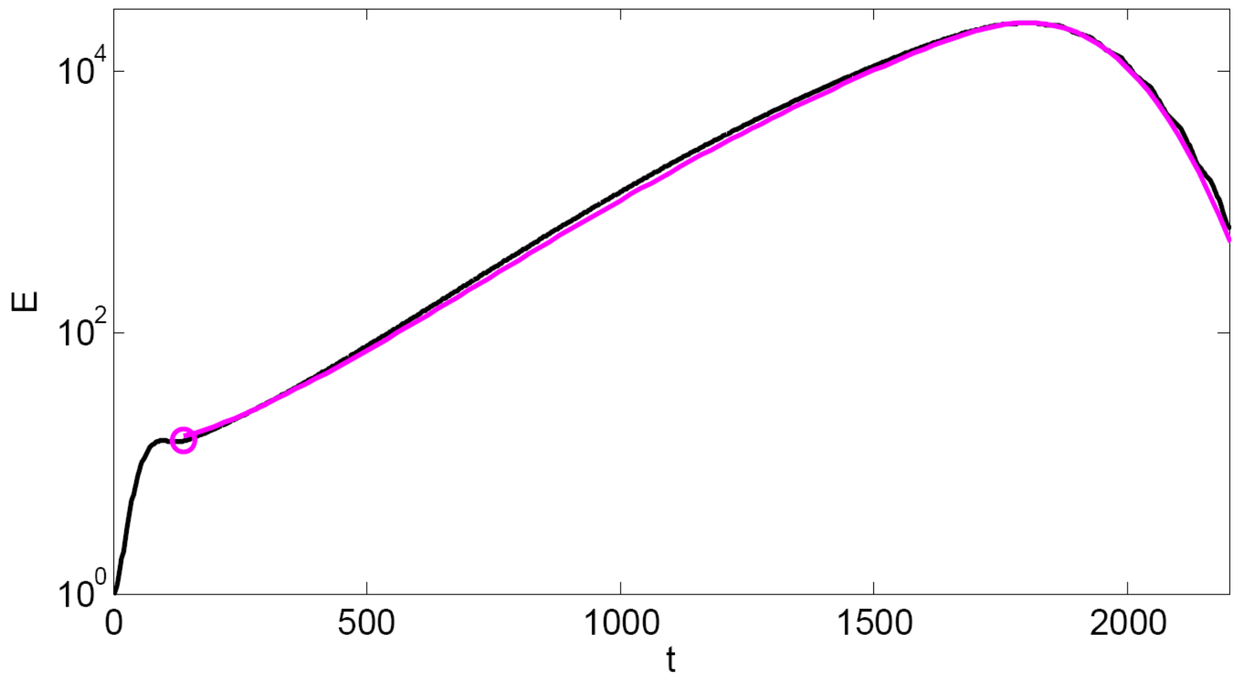

Snapshots of optimal disturbance evolution

- Initial disturbance leans against the shear raised up by Orr-mechanism into propagating TS-wavepacket

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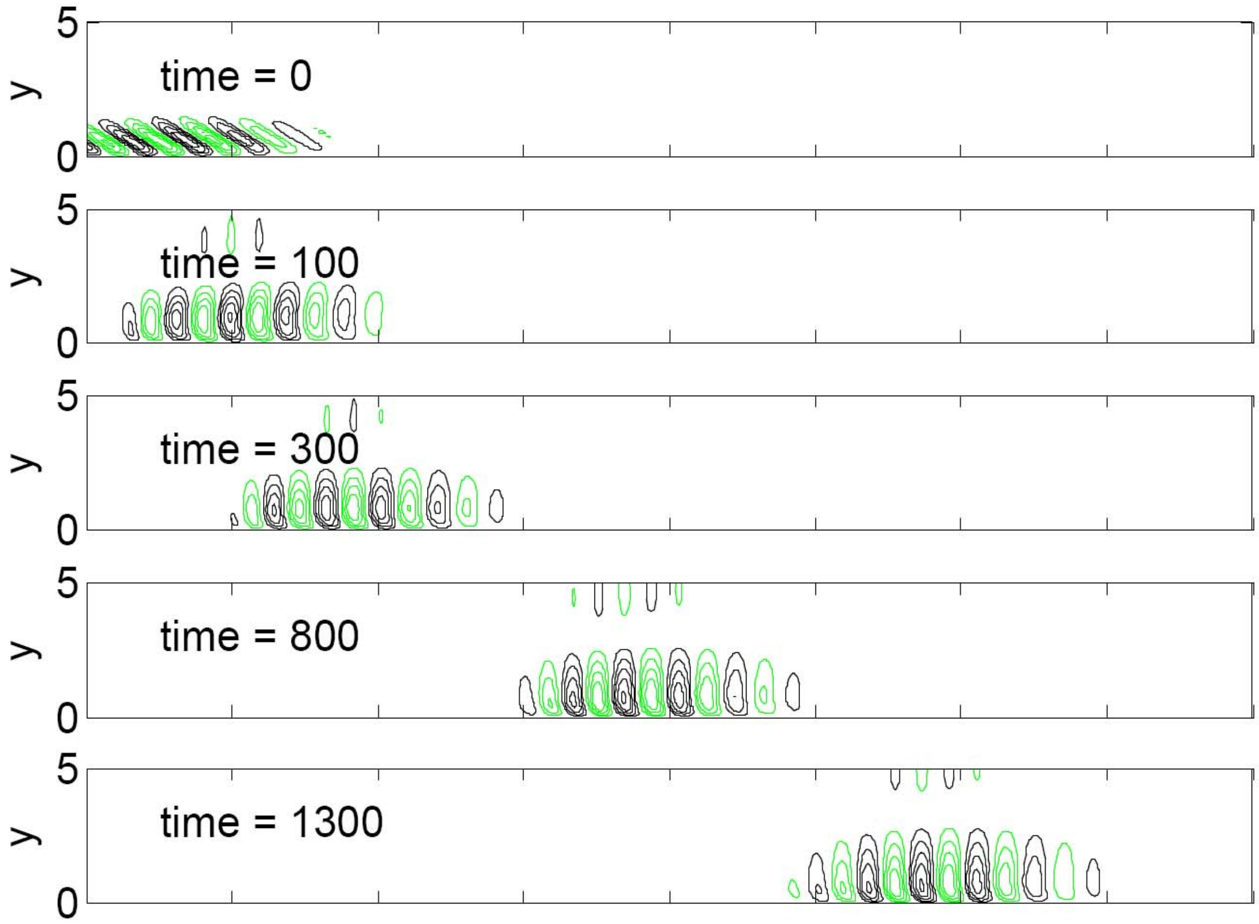

The forced problem: input-output

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- Ginzburg-Landau example
- Input-output for 2D Blasius configuration
- Model reduction

Ginzburg-Landau example

- Entire dynamics vs. input-output time signals

$$\frac{du}{dt} = Au + Bf$$
$$y = Cu$$

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bf(\tau)d\tau$$

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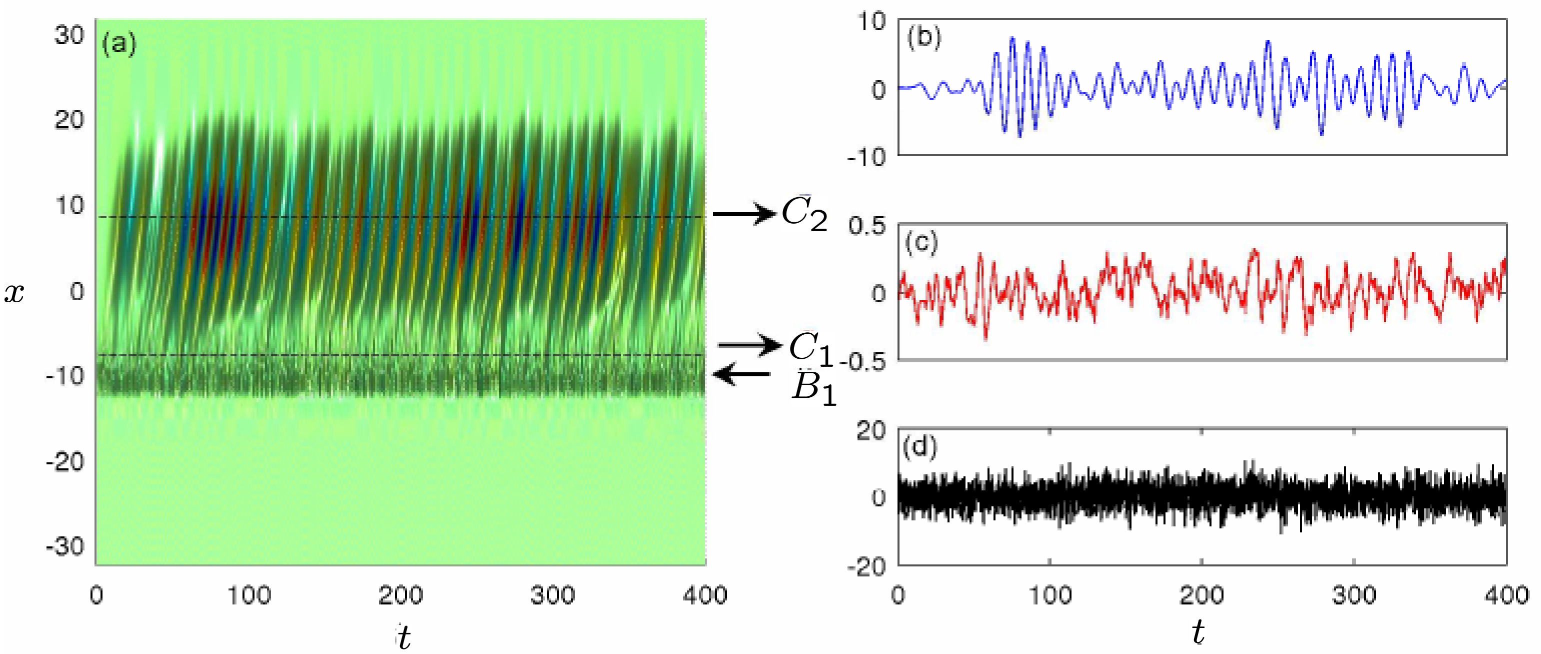
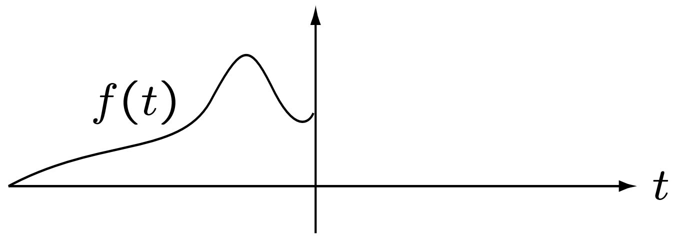


Input-output operators

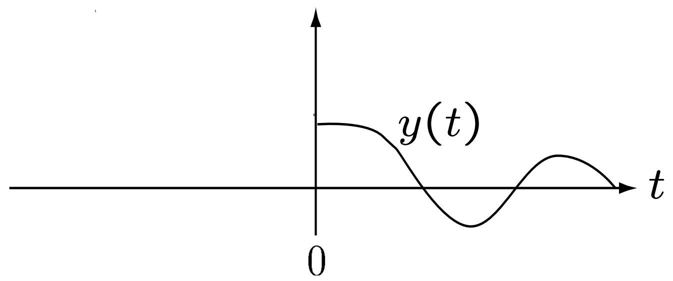
- Past inputs to initial state: class of initial conditions possible to generate through chosen forcing

$$u_0 = \mathcal{L}_c f = \int_{-\infty}^{0} e^{-A\tau} B f(\tau)\, \mathrm{d}\tau$$



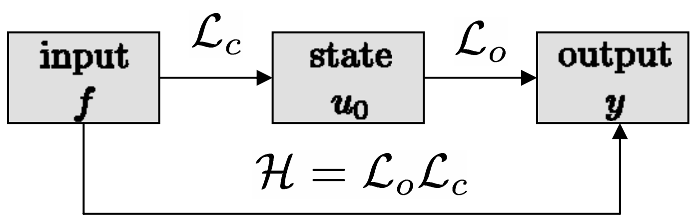
- Initial state to future outputs: possible outputs from initial condition

$$y(t) = \mathcal{L}_o u = C e^{At} u_0$$



- Past inputs to future outputs:

$$y = \mathcal{L}_o \mathcal{L}_c f(t) = \mathcal{H} f$$



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Most dangerous inputs and the largest outputs

- Eigenmodes of Hankel operator – balanced modes

$$\max_{\|f(t)\|=1} \|y(t)\|^2 = \max_{\|f(t)\|=1} (\mathcal{H}f, \mathcal{H}f) = \max_{\|f(t)\|=1} (f, \underbrace{\mathcal{L}_c^* \mathcal{L}_o^* \mathcal{L}_o \mathcal{L}_c}_{\mathcal{H}^*\mathcal{H}} f)$$

$$\underbrace{\mathcal{L}_c \mathcal{L}_c^*}_{P} \underbrace{\mathcal{L}_o^* \mathcal{L}_o}_{Q} \underbrace{\mathcal{L}_c f}_{u} = \sigma^2 \underbrace{\mathcal{L}_c f}_{u}$$

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- Controllability Gramian

$$P = \mathcal{L}_c \mathcal{L}_c^* = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau$$

- Observability Gramian

$$Q = \mathcal{L}_o^* \mathcal{L}_o = \int_\infty^0 e^{-A^T \tau} C^T C e^{-A\tau} d\tau$$

Controllability Gramian for GL-equation

$$P = \mathcal{L}_c \mathcal{L}_c^* = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau \approx X X^T, \quad X = [e^{At_1} B, \ldots, e^{At_m} B]$$

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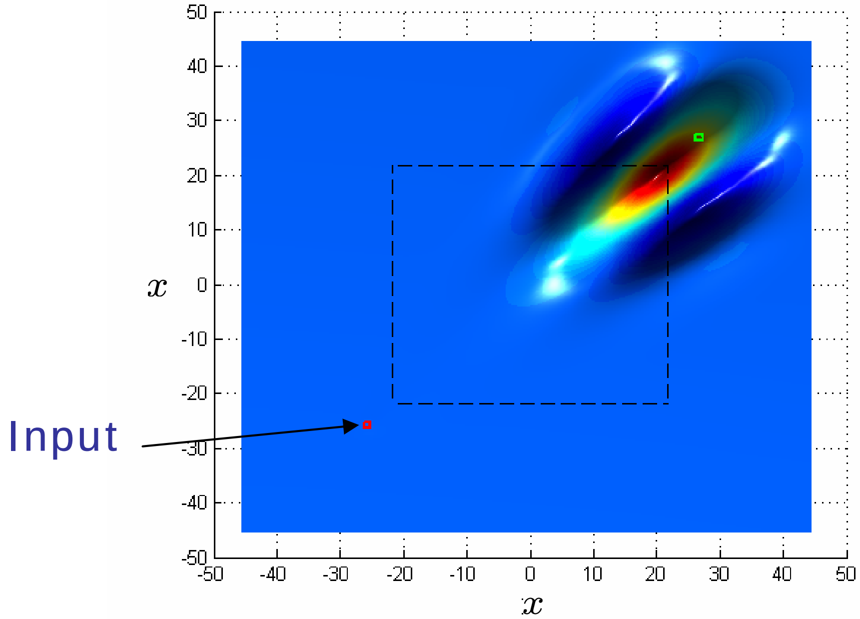
- Correlation of actuator impulse response in forward solution
- POD modes: $\mathrm{eig}\{XX^T\}$
- Ranks states most easily influenced by input
- Provides a means to measure controllability



Observability Gramian for GL-equation

$$Q = \mathcal{L}_o^* \mathcal{L}_o = \int_\infty^0 e^{-A^T \tau} C^T C e^{-A\tau} d\tau \approx YY^T, \quad Y = [e^{-A^T t_m} C^T, \ldots, e^{-A^T t_1} C^T]$$

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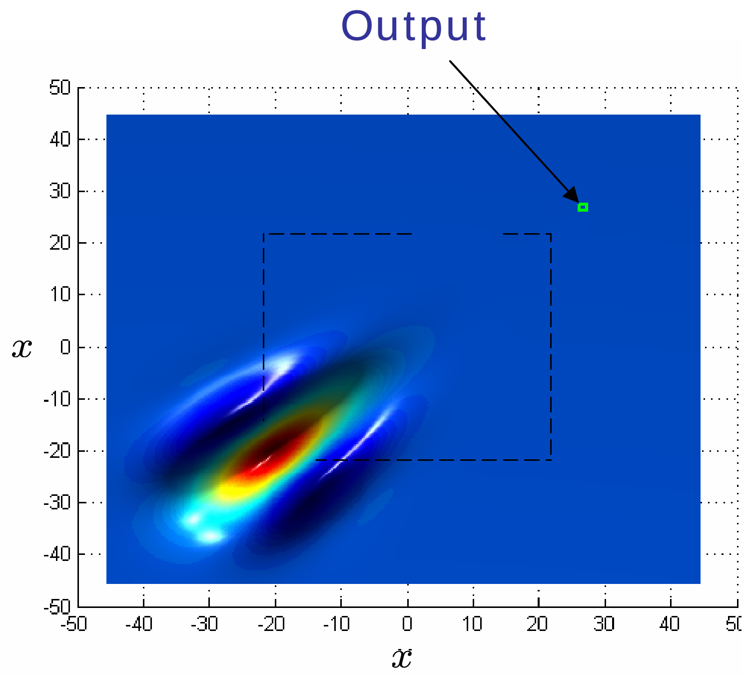
- Correlation of sensor impulse response in adjoint solution
- Adjoint POD modes: $\mathrm{eig}\{YY^T\}$
- Ranks states most easily sensed by output
- Provides a means to measure observability



Balanced modes: eigenvalues of the Hankel operator

- Combine snapshots of direct and adjoint simulation

$$\underbrace{\mathcal{L}_c\mathcal{L}_c^*}_{P}\underbrace{\mathcal{L}_o^*\mathcal{L}_o}_{Q} u = \sigma^2 u \qquad \Rightarrow \qquad \underbrace{XX^TYY^T}_{n\times n} u = \sigma^2 u$$

- Expand modes in snapshots to obtain smaller eigenvalue problem

$$u = Xa \quad \Rightarrow \quad X(\underbrace{XYY^TX}_{m\times m} a - \sigma^2 a) = 0$$

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Snapshots of direct and adjoint solution in Blasius flow

Direct simulation:

$$X = [u(t_1) \ldots u(t_m)]$$

Adjoint simulation:

$$Y = [p(t_m) \ldots p(t_1)]$$

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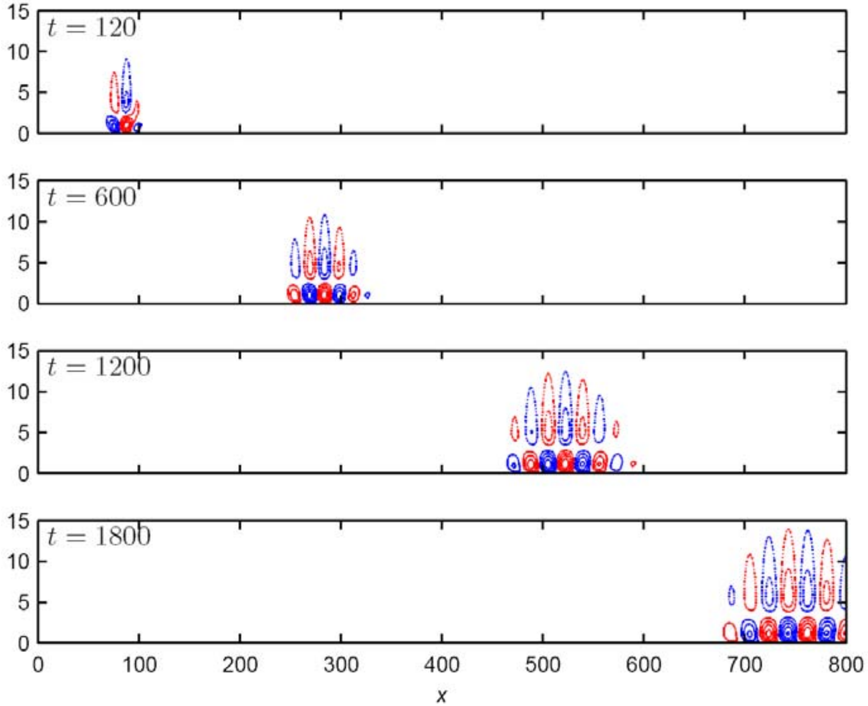

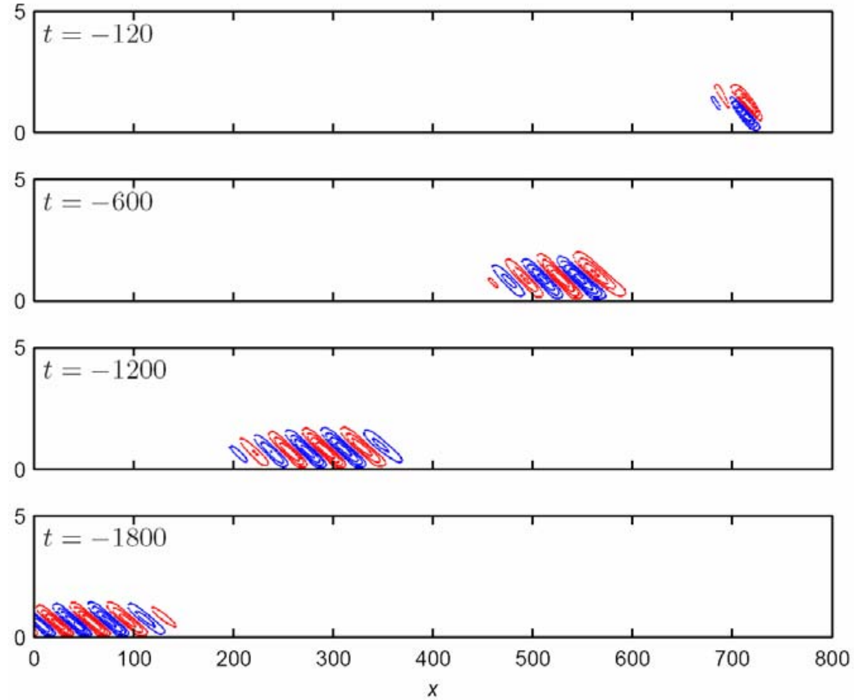

Balanced modes for Blasius flow

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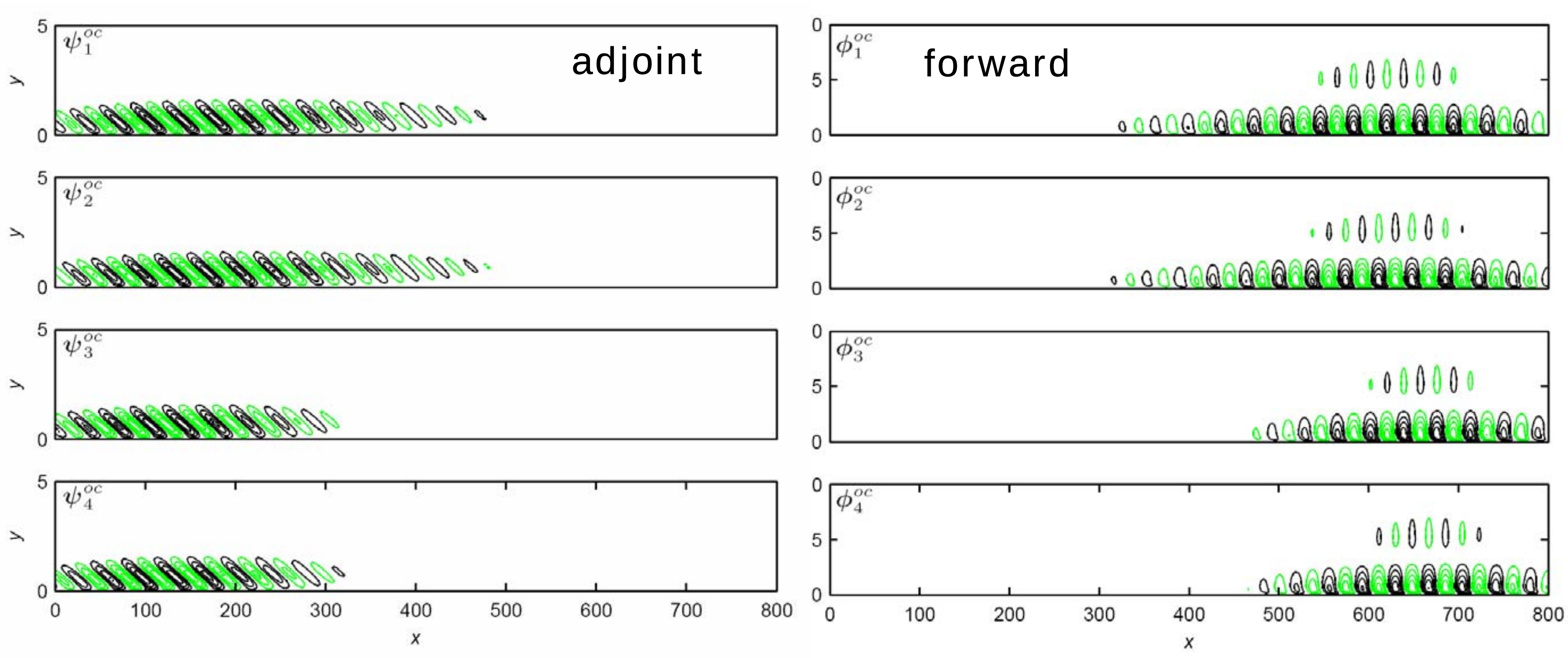

Properties of balanced modes

- Largest outputs possible to excite with chosen forcing
- Balanced modes diagonalize observability Gramian
- Adjoint balanced modes diagonalize controllability Gramian
- Ginzburg-Landau example revisited

$$\hat{Q} = \hat{P} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$$

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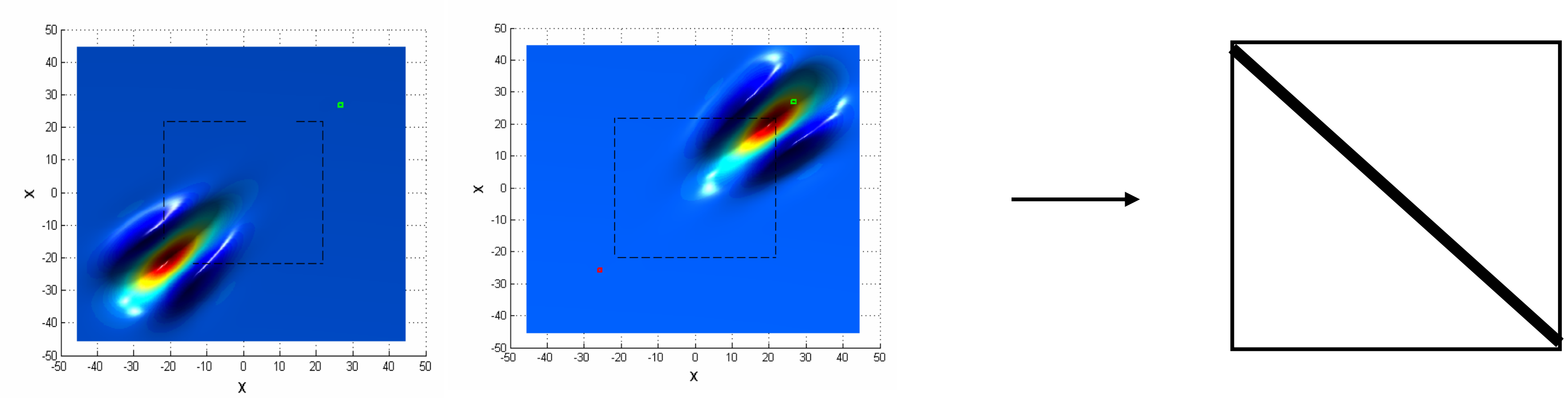

Model reduction

- Project dynamics on balanced modes using their biorthogonal adjoints

- Reduced representation of input-output relation, useful in control design

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$$\frac{d\kappa}{dt} = \mathsf{A}\kappa + \mathsf{B}f \qquad u \approx \sum_{j=1}^{r} \kappa_j(t)\, u_j$$

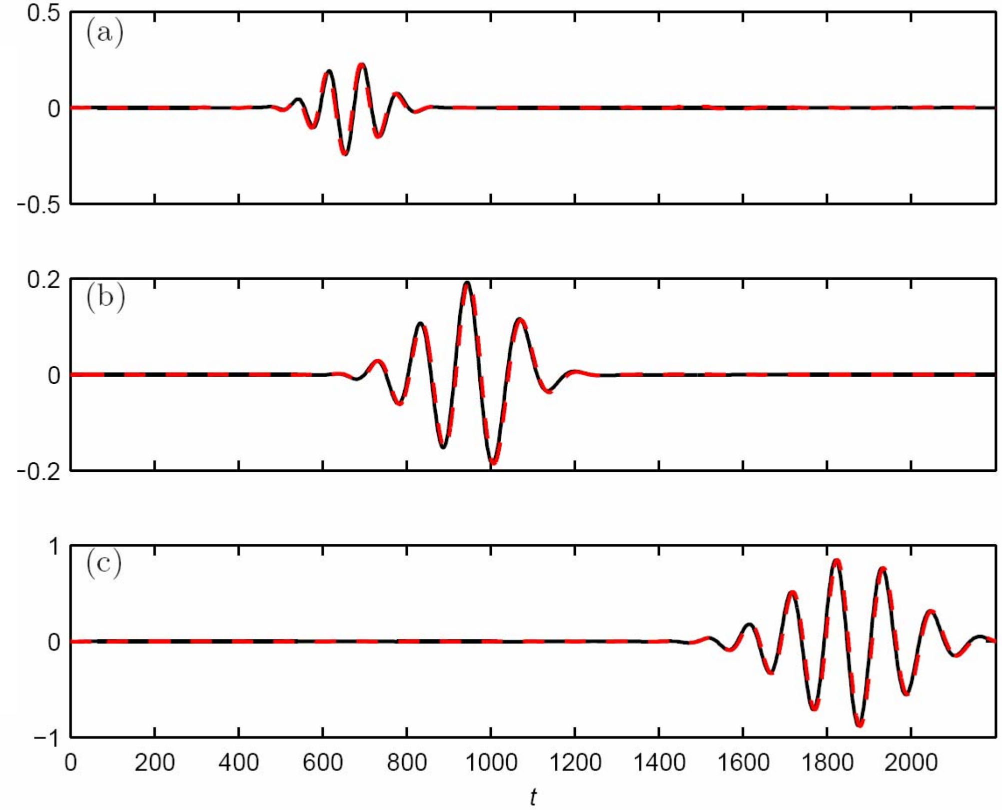
$$y = \mathsf{C}\kappa$$

$$\mathsf{A}_{ij} = (\psi_i, A\phi_i)$$

$$\mathsf{B}_{1j} = (\psi_j, B_1)$$

$$\mathsf{C}_{1j} = C_1\phi_j$$

Impulse response $f = \delta(0)$



Disturbance → Sensor

Actuator → Objective

Disturbance → Objective

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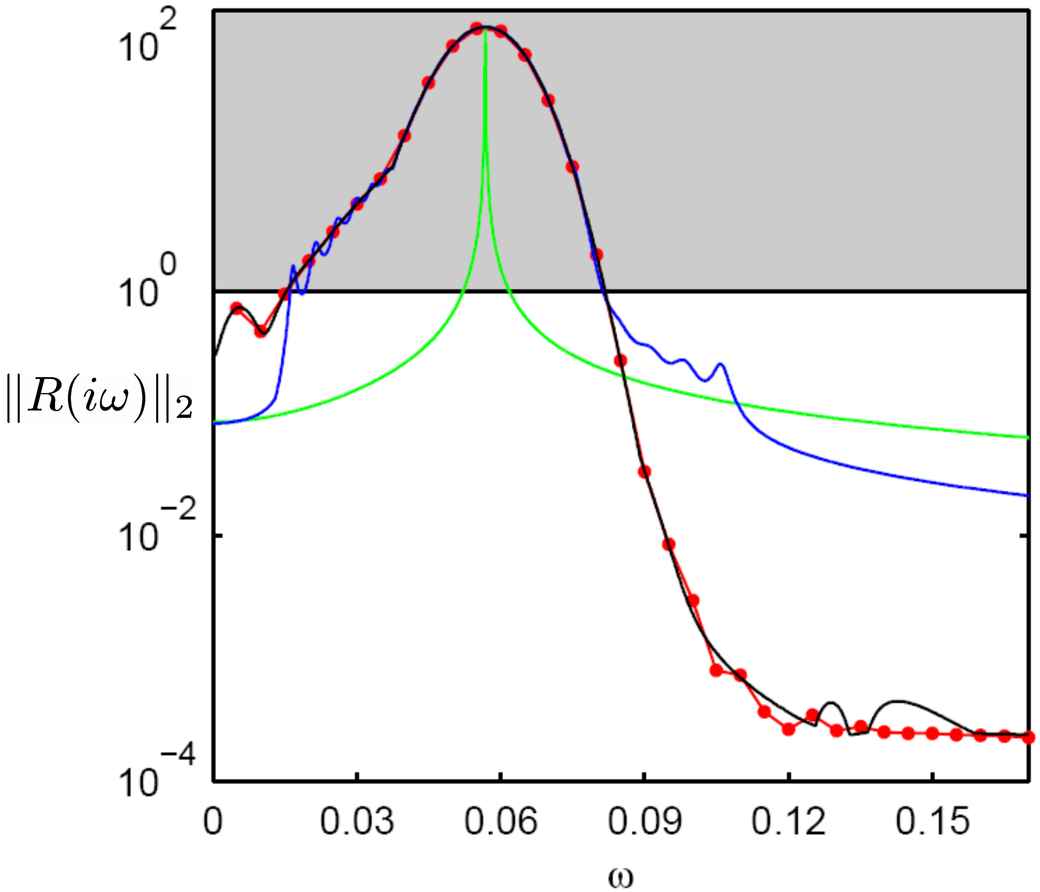
DNS: $n=10^5$ $y = Ce^{A\tau}Bf$

ROM: $m=50$ $y \approx Ce^{A\tau}Bf,$

Frequency response $f = e^{st}$



$$y = \underbrace{\int_0^\infty C e^{A(t-\tau)} B d\tau}_{R(s)} \, e^{st}$$

From all inputs to all outputs

DNS: $n = 10^5$
ROM: $m = 80$
$m = 50$
$m = 2$

Feedback control

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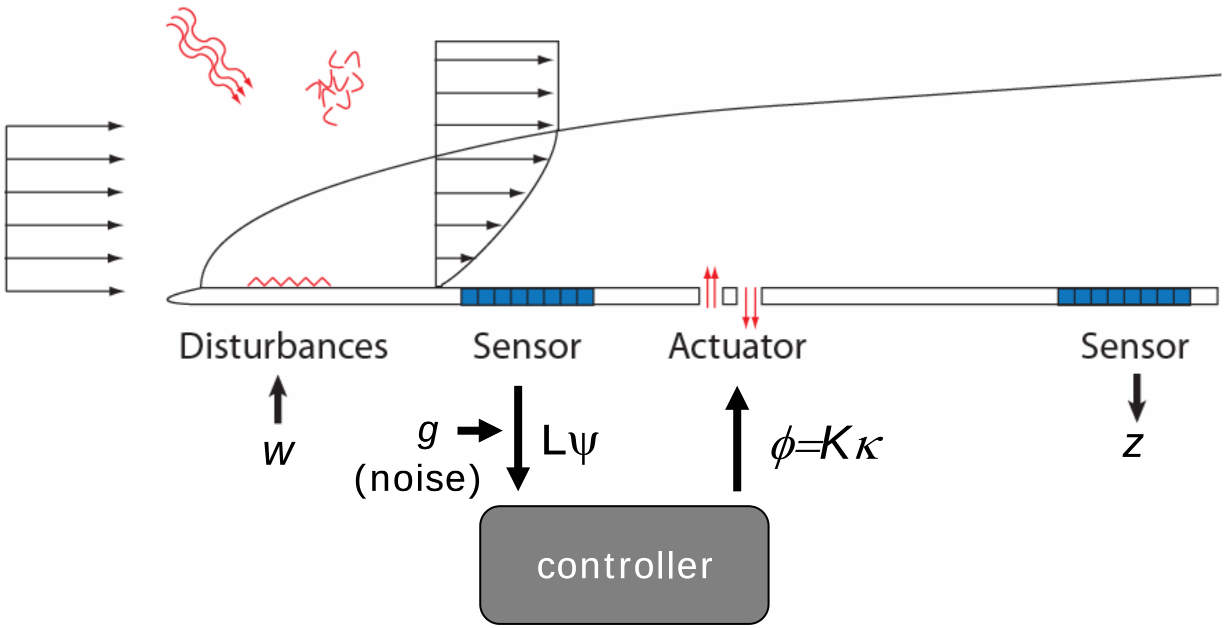
- LQG control design using reduced order model
- Blasius flow example

Optimal Feedback Control – LQG



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Find an optimal control signal $\phi(t)$ based on the measurements $\psi(t)$ such that in the presence of external disturbances $w(t)$ and measurement noise $g(t)$ the output $z(t)$ is minimized.

→ Solution: LQG/H2

LQG controller formulation with DNS

- Apply in Navier-Stokes simulation

$$\frac{du}{dt} = NS(u) + B_1 w + B_2 \phi$$
$$\psi = C_1 u$$

$$\frac{d\kappa_e}{dt} = (\mathsf{A} + \mathsf{B}_2\mathsf{K} + \mathsf{L}\mathsf{C}_2)\kappa_e - \mathsf{L}\psi$$
$$\phi = \mathsf{K}\kappa_e$$

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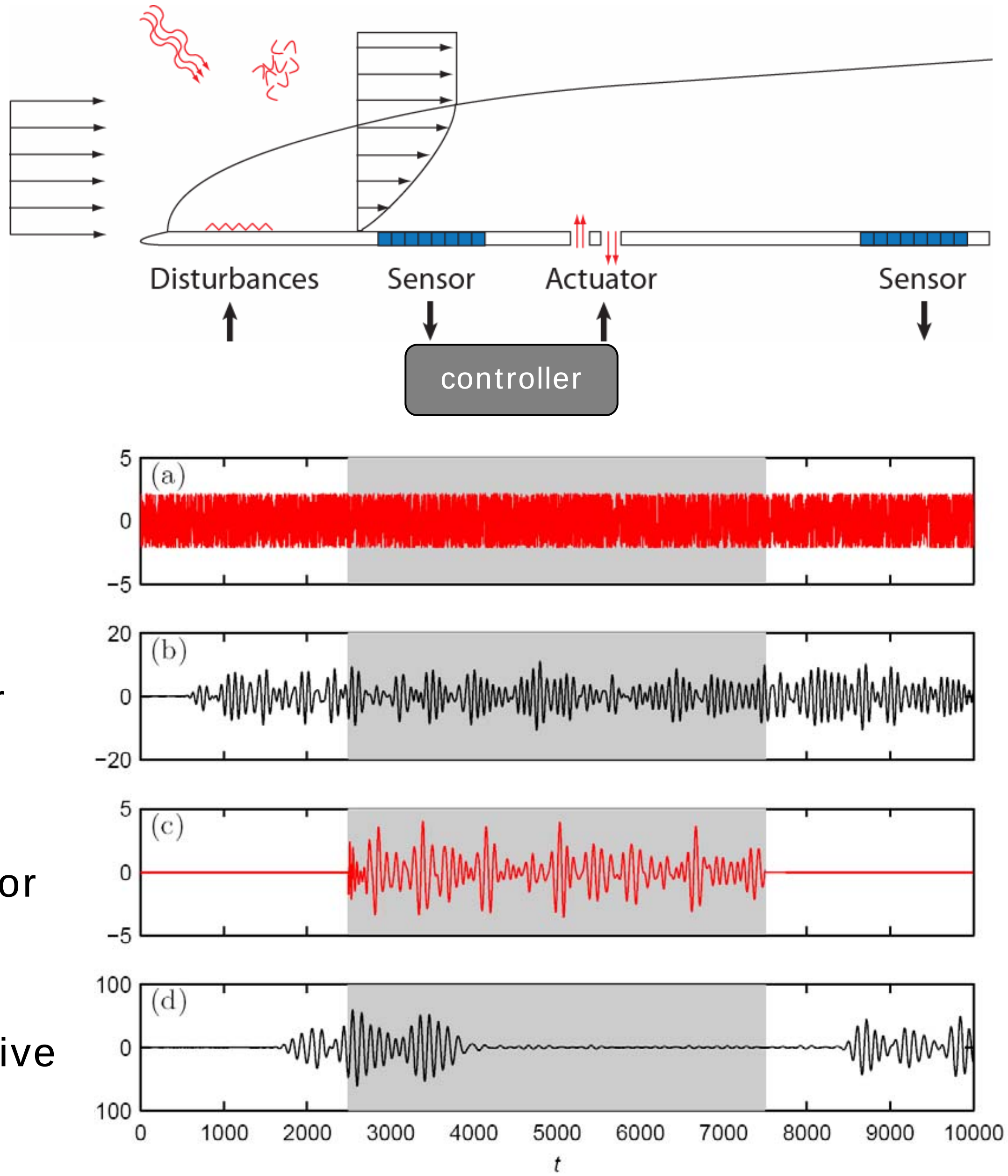
Performance of controlled system



Noise

Sensor

Actuator

Objective

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Performance of controlled system

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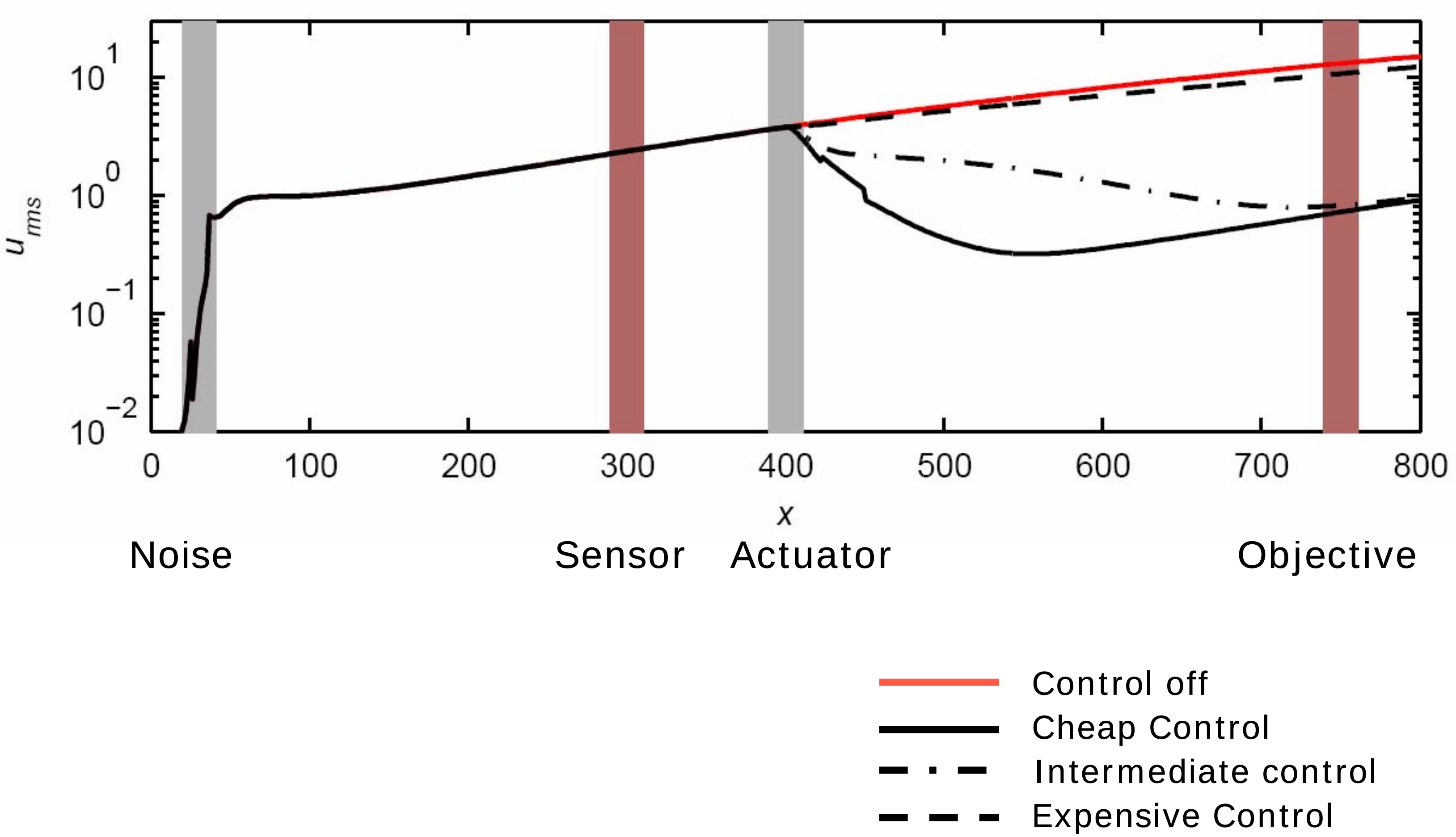

Conclusions

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- Complex stability/control problems solved using Krylov/Arnoldi methods based on snapshots of forward and adjoint Navier-Stokes solutions
- Optimal disturbance evolution brought out Orr-mechanism and propagating TS-wave packet automatically
- Balanced modes give low order models preserving input-output relationship between sensors and actuators
- Feedback control of Blasius flow
  - Reduced order models with balanced modes used in LQG control
  - Controller based on small number of modes works well in DNS
- Outlook: incorporate realistic sensors and actuators in 3D problem and test controllers experimentally

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