SOME ESTIMATES

FOR NON SELF-ADJOINT OPERATORS

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The self-adjoint case

Let us consider a linear operator $A \in \mathcal{L}(D(A), H)$ on a Hilbert space H. If this operator is self-adjoint, we can use all the power of spectral theory. In particular the spectrum $\sigma(A)$ is real, and we have the relation

$$\|r(A)\| = \sup_{x \in \sigma(A)} |r(x)|,$$

which holds for all rational functions bounded on $\sigma(A)$.

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which holds for all rational functions bounded on $\sigma(A)$.

This relation allows to define, via a density argument, $f(A) \in \mathcal{L}(H)$, for all continuous and bounded functions f on $\sigma(A)$, and the previous relation, with f instead of r, is still valid.

Example of application

If the linear operator $A \in \mathcal{L}(D(A), H)$ is self-adjoint and positive definite, we can in this way define $\cos(t\sqrt{A})$ and justify that

$$u(t) = \cos(t\sqrt{A})\, u_0, \qquad \text{with } u_0 \in D(A),$$

is the unique solution in $C^2(\mathbb{R}, H) \cap C^0(\mathbb{R}, D(A))$ of

$$\begin{cases} u''(t) + A\, u(t) = 0, \quad t \in \mathbb{R}, \\ u(0) = u_0, \quad u'(0) = 0. \end{cases}$$

Spectral sets, J. von Neumann, 1951

A set $X \subset \mathbb{C}$, with $\sigma(A) \subset X$, is called a spectral set for the operator A if the following inequality

$$\|r(A)\| \leq \sup_{z \in X} |r(z)|$$

holds for all rational functions bounded on $X \subset \mathbb{C}$.

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Remark. If the operator A is normal, then $\sigma(A)$ is a spectral set for A.

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von Neumann results

The unit disk $\mathbb{D} = \{z; |z| \leq 1\}$ is a spectral set for A iff $\|A\| \leq 1$.

The half-plane $\{z; \text{Re}\, z \geq 0\}$ is a spectral set for A if and only if

$$\text{Re}\langle Av, v\rangle \geq 0, \quad \text{for all } v \in D(A).$$

K-spectral sets

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holds for all rational functions bounded on $X \subset \mathbb{C}$.

It is called a K-spectral set for the operator A if the following inequality holds

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K-spectral sets

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holds for all rational functions bounded on $X \subset \mathbb{C}$.

Cauchy formula. Assume that the domain Ω contains $\sigma(A)$ (+ suitable hypotheses), we have

$$r(A) = \frac{1}{2\pi i} \int_{\partial\Omega} r(\sigma)(\sigma - A)^{-1} d\sigma,$$

This shows that Ω is a K-spectral set for the operator A with

$$K = \frac{1}{2\pi} \int_{\partial\Omega} \|(\sigma - A)^{-1}\| \, |d\sigma|.$$

TWO USEFUL LEMMATA

Lemma 1. We assume that

$dm(t)$ is a bounded, complex-valued measure on E,

$M(t) \in \mathcal{L}(H), \quad M(t) = M^*(t) \geq 0,$ in E,

r is a rational function bounded by 1 on E.

Then we have

$$\Big\| \int_E r(t) M(t)\, dm(t) \Big\| \leq \Big\| \int_E M(t)\, |dm(t)| \Big\|.$$

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Proof. We have

$$|\langle M(t)u, v\rangle| \leq \langle M(t)u, u\rangle^{1/2} \langle M(t)v, v\rangle^{1/2}$$

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Thus, with

. $\mathcal{A} = \int_E r(t) M(t))\, dm(t)$, and $\mathcal{B} = \int_E M(t)\, |dm(t)|$,

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$$|\langle \mathcal{A}u, v\rangle| \leq \langle \mathcal{B}u, u\rangle^{1/2} \langle \mathcal{B}v, v\rangle^{1/2},$$

which yields $\|\mathcal{A}\| \leq \|\mathcal{B}\|$.

Lemma 2. We assume that

$dm(t)$ is a bounded, complex-valued measure on E,

$M(t), N(t) \in \mathcal{L}(H), \quad N(t) = N^*(t), \quad$ in $E, \quad \alpha > 0$

$\mathrm{Re}\, M(t) = \frac{1}{2}(M(t) + M(t)^*) \geq N(t) \geq \alpha, \quad$ in E,

r is a rational function bounded by 1 on E.

Then we have

$$\left\| \int_E r(t)(M(t))^{-1}\, dm(t) \right\| \leq \left\| \int_E (N(t))^{-1}\, |dm(t)| \right\|.$$

Lemma 2. We assume that r is bounded by 1,

$$\operatorname{Re} M(t) = \tfrac{1}{2}(M(t) + M(t)^*) \geq N(t) \geq \alpha, \quad \text{in } E.$$

Then we have

$$\left\| \int_E r(t)(M(t))^{-1}\, dm(t) \right\| \leq \left\| \int_E (N(t))^{-1}\, |dm(t)| \right\|.$$

Proof.

The proof is similar by noticing that

$$|\langle M(t)^{-1}u, v\rangle| \leq \langle N(t)^{-1}u, u\rangle^{1/2} \langle N(t)^{-1}v, v\rangle^{1/2}.$$

Application : a problem in an annulus $C = \{z\,;\rho^{-1} \leq |z| \leq \rho\}$.

In 1974, Shields have shown that, if an operator A satisfies $\|A\| \leq \rho$ and $\|A^{-1}\| \leq \rho$, then for all rational functions r which is bounded by 1 on C, we have,

$$\|r(A)\| \leq 2 + \sqrt{\frac{\rho^2+1}{\rho^2-1}}.$$

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But, if $\rho = 1$, then A is a unitary operator, and then $\|r(A)\| \leq 1$.

Application : a problem in an annulus $C = \{z\,;\rho^{-1} \le |z| \le \rho\}$.

We now want to explain how the two previous lemmata allow to show that, if an operator A satisfies $\|A\| < \rho$ and $\|A^{-1}\| < \rho$, then for all rational functions r which is bounded by 1 on C, we have,

$$\|r(A)\| \le 2 + \sqrt{\frac{\rho^2 + 2\rho + 1}{\rho^2 + \rho + 1}} \le 2 + 2/\sqrt{3}.$$

Application : a problem in an annulus $C = \{z\,; \rho^{-1} \leq |z| \leq \rho\}$.

Let us consider an operator A with $\|A\| < \rho$ and $\|A^{-1}\| < \rho$, and a rational function r which is bounded on C. We have, from the Cauchy formula,

$$r(A) = \frac{1}{2\pi i}\int_{\partial C} r(\sigma)(\sigma - A)^{-1}\, d\sigma.$$

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Let us consider an operator A with $\|A\| < \rho$ and $\|A^{-1}\| < \rho$, and a rational function r which is bounded on C. We have, from the Cauchy formula,

$$r(A) = \frac{1}{2\pi i}\int_{\partial C} r(\sigma)(\sigma - A)^{-1}d\sigma = R_1 + R_2 + R_3 + R_4,$$

with

$$R_1 = \int_{|\sigma|=\rho} r(\sigma)\tfrac{1}{2\pi i}\Big((\sigma - A)^{-1}d\sigma - (\bar\sigma - A^*)^{-1}d\bar\sigma - \sigma^{-1}d\sigma\Big),$$

$$R_2 = \int_{|\sigma|=\rho^{-1}} r(\sigma)\tfrac{1}{2\pi i}\Big((\sigma - A)^{-1}d\sigma - (\bar\sigma - A^*)^{-1}d\bar\sigma - \sigma^{-1}d\sigma\Big),$$

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We write $\sigma = \rho e^{i\theta}$, thus $d\sigma = i\sigma d\theta$, and note that

$$\frac{1}{2\pi i}\Big((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} - \sigma^{-1} d\sigma \Big) =$$
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$$= \frac{1}{2\pi}(\sigma-A)^{-1}(\rho^2-AA^*)(\bar{\sigma}-A^*)^{-1}d\theta \geq 0.$$

Indeed $\|A\| \leq \rho$ implies $\rho^2-AA^* \geq 0$.

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We can apply Lemma 1 and we get, if $|r| \leq 1$ on C,

$$\|R_1\| \leq \Big\| \int_{|\sigma|=\rho} \frac{1}{2\pi i}\Big((\sigma - A)^{-1} d\sigma - (\bar{\sigma} - A^*)^{-1} d\bar{\sigma} - \sigma^{-1} d\sigma\Big)\Big\|$$
$$\leq \|1 + 1 - 1\| = 1.$$

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$$\leq \|1 + 1 - 1\| = 1.$$

We similarly get the estimate $\|R_2\| \leq 1$.

Estimate of $R_3 + R_4$.

We have

$$R_3 = \int_{|\sigma|=\rho} r(\sigma) \frac{1}{2\pi i} \Big((\bar{\sigma} - A^*)^{-1} d\bar{\sigma} + \sigma^{-1} d\sigma \Big).$$

We use $\bar{\sigma} = \rho^2/\sigma$, thus $d\bar{\sigma} = -\rho^2/\sigma^2 d\sigma$, and get

$$R_3 = -\frac{1}{2\pi i} \int_{|\sigma|=\rho} r(\sigma) A^* (\rho^2 - \sigma A^*)^{-1} d\sigma$$

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We use $\bar\sigma = \rho^2/\sigma$, thus $d\bar\sigma = -\rho^2/\sigma^2 d\sigma$, and get

$$\begin{aligned} R_3 &= -\tfrac{1}{2\pi i} \int_{|\sigma|=\rho} r(\sigma) A^*(\rho^2 - \sigma A^*)^{-1} d\sigma \\ &= -\tfrac{1}{2\pi i} \int_{|\sigma|=1} r(\sigma) A^*(\rho^2 - \sigma A^*)^{-1} d\sigma. \end{aligned}$$

Indeed $r(\sigma) A^*(\rho^2 - \sigma A^*)^{-1}$ is holomorphic in σ, for $\rho^{-1} \le |\sigma| \le \rho$, which allows to replace the path $|\sigma| = \rho$ by the path $|\sigma| = 1$.

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$$= -\frac{1}{2\pi i} \int_{|\sigma|=1} r(\sigma) A^* (\rho^2 - \sigma A^*)^{-1} d\sigma.$$

Similarly we obtain

$$R_4 = \frac{1}{2\pi i} \int_{|\sigma|=1} r(\sigma) A^* (\rho^{-2} - \sigma A^*)^{-1} d\sigma.$$

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We have obtained

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$$= -\frac{\rho^2 - \rho^{-2}}{2\pi}\int_0^{2\pi} r(e^{i\theta})M(\theta, A^*)^{-1}\, d\theta,$$

with

$$M(\theta, A^*) = \rho^2 + \rho^{-2} - e^{i\theta}A^* - e^{-i\theta}A^{-*}.$$

Estimate of $R_3 + R_4$.

We set $A^* = UG$, with unitary U and self-adjoint $G \geq 0$, and note that $\|A\| \leq \rho$ and $\|A^{-1}\| \leq \rho$ imply $\rho^{-1} \leq G \leq \rho$. Thus

$$2 \leq G + G^{-1} \leq \rho + \rho^{-1} = 2\tau,$$

by setting $\tau = (\rho + \rho^{-1})/2$.

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by setting $\tau = (\rho + \rho^{-1})/2$, and then

$$\|G + G^{-1} - 1 - \tau\| \leq \tau - 1,$$

Estimate of $R_3 + R_4$.

We note that

$$\operatorname{Re} M(\theta, A^*) = \rho^2 + \rho^{-2} - (1+\tau)\operatorname{Re}(e^{i\theta}U) - \operatorname{Re}(e^{i\theta}U(G + G^{-1} - 1 - \tau)),$$

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This yields to $\operatorname{Re} M(\theta, A^*) \geq N(\theta, U)$ with

$$N(\theta, U) := \rho^2 + \rho^{-2} - (1+\tau)\operatorname{Re}(e^{i\theta}U) + 1 - \tau$$

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$$\begin{aligned} N(\theta, U) &:= \rho^2 + \rho^{-2} - (1+\tau)\,\text{Re}(e^{i\theta}U) + 1 - \tau \\ &\ge \rho^2 + \rho^{-2} - 2\tau > 0. \end{aligned}$$

Estimate of $R_3 + R_4$.

Thus we may apply Lemma 2 and obtain

$$\|R_3 + R_4\| \le \frac{\rho^2 - \rho^{-2}}{2\pi} \left\| \int_0^{2\pi} N(\theta, U)^{-1} d\theta \right\|$$

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But the integral may be computed, and we get

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Finally, we have obtained

$$\|r(A)\| \leq \|R_1\| + \|R_2\| + \|R_3 + R_4\| \leq 2 + \frac{2}{\sqrt{3}}.$$

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Let us consider $D_1 = \{z\,;\,|z| \leq \rho\}$ and $D_2 = \{z\,;\,|z| \geq \rho^{-1}\}$.

We have shown that, if D_1 and D_2 are spectral sets for an operator A, then $C = D_1 \cap D_2$ is a $2 + 2/\sqrt{3}$-spectral set for A.

The previous result is a particular case of a more general one, obtained with C. Badea and B. Beckermann

If D_1, D_2,. . ., D_n are n disks of the Riemann sphere, and
if D_1, D_2,. . ., D_n are spectral sets for an operator A,
then $X = D_1 \cap D_2 \cap \cdots \cap D_n$ is a (complete) K-spectral set for A,
with $K \le n + n(n-1)/\sqrt{3}$.

A similar proof allows to show that

If the numerical range of an operator A

$$W(A) := \{\langle Av, v\rangle\,;\, v \in D(A),\ \ \|v\| = 1\}$$

is contained in a conic domain X, then X is a (complete) K-spectral set for A, with $K \leq 2 + 2/\sqrt{3}$.

A more intricate proof allows to show that

The numerical range of an operator A

$$W(A) := \{\langle Av, v\rangle\, ; v \in D(A),\ \|v\| = 1\}$$

is a (complete) K-spectral set for A, with $K \leq 11.08$.

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