SPECTRAL THEORY OF LARGE FINITE DIRECTED GRAPHS

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following work with M Hager

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Perturbations of Jordan Matrices

The simple matrix

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

has spectrum $\{0\}$. The same applies if one has several Jordan blocks.

Perturbations of Jordan Matrices

The situation changes entirely if one adds a few extra entries as in

$$A = \begin{pmatrix} 0 & 1 & & & 1 & 0 \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}$$

The Associated Graph

The graph of any matrix is defined by putting

$$X = \{1, 2, \ldots, n\}$$

and if $x, y \in X$ then

$$x \to y \text{ iff } A_{x,y} \neq 0.$$

The Associated Graph

The graph of any $n \times n$ matrix is defined by putting

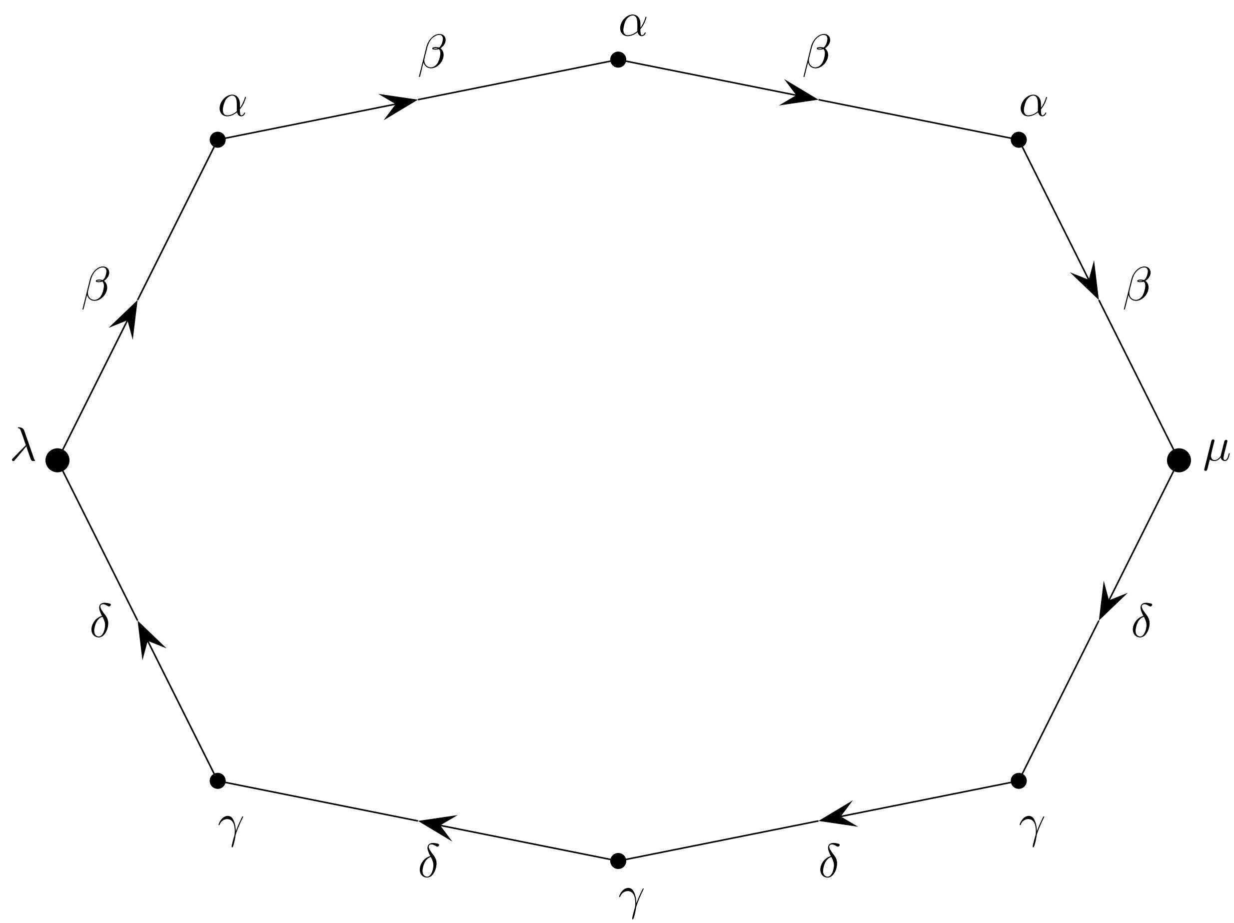
$$S = \{1, 2, \dots, n\}$$

and if $x, y \in S$ then

$$x \to y \text{ iff } A_{x,y} \neq 0.$$

We will assume that this graph is irreducible in the sense that any point is accessible from any other.

One can regard the graph as weighted by attaching the number $A_{x,y}$ to the edge $x \to y$ and the number $A(x, x)$ to the vertex x.


α
β
α
β
α
β
β
λ
μ
δ
δ
γ
δ
γ
δ
γ

Channels

We define C to be the set of all $s \in S$ that have indegree 1 and outdegree 1. We assume that C and $J = S \backslash C$ are both non-empty.

The set C can be written as the union of disjoint 'channels' C_i, which we define as subsets T of S that can be identified with $\{b = 1, 2, ..., e-1, e\}$ in such a way that

1. every $x \in T$ has outdegree 1; if $x < e$ then $x \to x+1$; moreover $e \to \tilde{e} \in J$;
2. every $x \in T$ has indegree 1; if $x > 1$ then $x - 1 \to x$; moreover $J \ni \tilde{b} \to b$.

Asymptotics

We will assume that the length of each channel is a multiple of n and investigate the spectral asymptotics as $n \to \infty$. The geometry of the graph and the matrix entries are otherwise unchanged.

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Different channels may have a different pair of constants.

The Eigenvalue Problem

$$\det(A_n - zI) = F_n(z) = \sum_{r=1}^{R} a_r(z) f_r(z)^n$$

where each $a_r(z)$ and $f_r(z)$ is a polynomial.

The task is to find where the zeros of such an expression lie in the limit $n \to \infty$.

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One can generalize by allowing the functions to be analytic rather than polynomial.

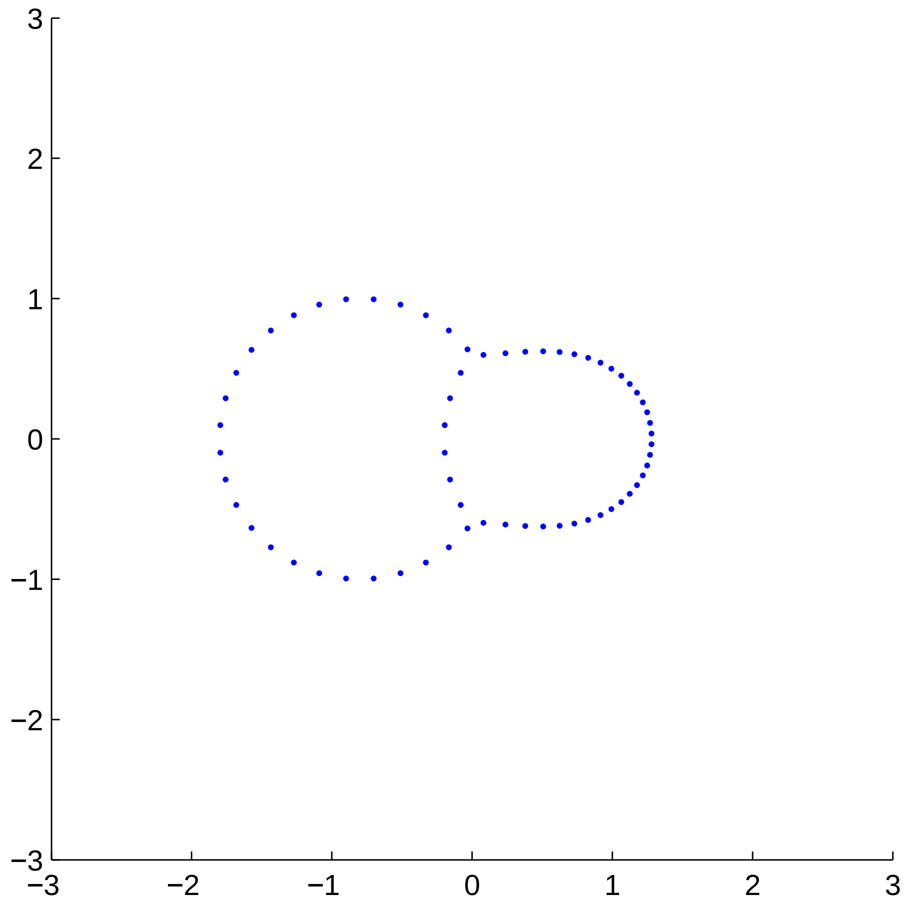
An Example

Consider the function

$$F(z) = (z-a)^n(z+a)^n + \alpha(z-a)^n + \gamma.$$

for which

$$\begin{aligned} f_1(z) &= (z-a)(z+a), \\ f_2(z) &= (z-a), \\ f_3(z) &= 1. \end{aligned}$$


3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

Excluded Regions

We can exclude the regions

$$U_r = \{z : |f_r(z)| > |f_s(z)| \text{ for all } s \neq r\}.$$

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except for certain isolated points.

These are the points where $a_r(z) = 0$.

The Remaining Eigenvalues

Most of the eigenvalues must lie close to the curves

$$C_{r,s} = \{z : |f_r(z)| = |f_s(z)| > |f_t(z)| \text{ for all } t \neq r, s\}.$$

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Each curve starts and ends at a point of E.

Another Example

Consider the $(3n+3) \times (3n+3)$ matrix

$$A_{i,j} = \begin{cases} i/2 & \text{if } 1 \leq i = j \leq n, \\ -2 & \text{if } i = j = n+1, \\ -i/2 & \text{if } n+2 \leq i = j \leq 2n+1, \\ 2 & \text{if } i = j = 2n+2, \\ 3/2 & \text{if } 2n+3 \leq i = j \leq 3n+2, \\ 0 & \text{if } i = j = 3n+3, \\ 1 & \text{if } i+1 = j, \\ 1 & \text{if } i = 3n+3,\ j = 1, \\ 1 & \text{if } i = n+1,\ j = 3n+3, \\ 0 & \text{otherwise.} \end{cases}$$

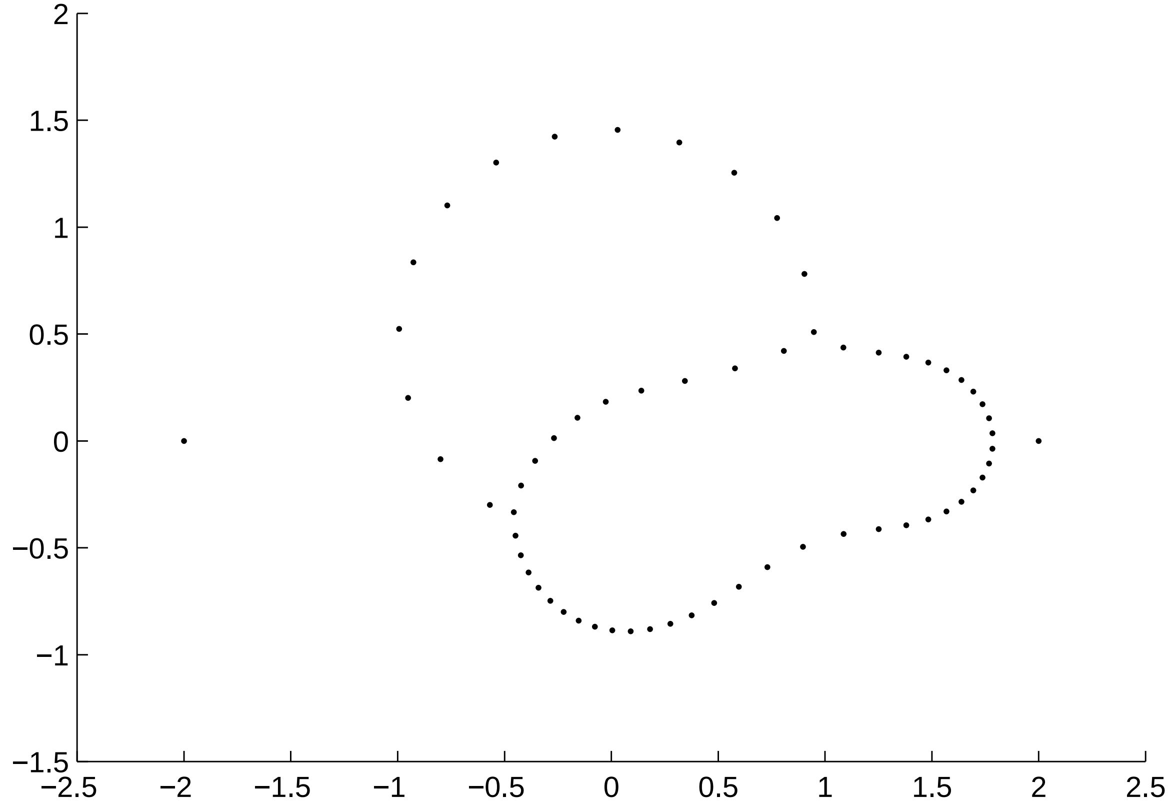
It corresponds to a graph with three channels of length n and two junctions. It has characteristic polynomial

$$\begin{aligned} \det(zI - A) &= z(z^2-4)(z-i/2)^n(z+i/2)^n(z-3/2)^n \\ &\quad -(z-2)(z+i/2)^n(z-3/2)^n - 1. \end{aligned}$$

Also

$$\begin{aligned} f_1(z) &= (z-i/2)(z+i/2)(z-3/2) \\ f_2(z) &= (z+i/2)(z-3/2) \\ f_3(z) &= -1. \end{aligned}$$

The Case $n = 20$



The Eigenvalue Density

One finally has to prove that the eigenvalues close to the curves

$$C_{r,s} = \{z : |f_r(z)| = |f_s(z)| > |f_t(z)| \text{ for all } t \neq r, s\}.$$

have an asymptotic density.

For every interval J on the curve the number of eigenvalues close to that interval is of the form $c(J)n + o(n)$ where $c(J)$ is the integral of a certain real-analytic function over the interval.

Pseudospectra