The quadratic numerical range: A new concept for localizing spectra and more ...

Christiane Tretter

University of Bern, Switzerland

Manchester, May 15, 2008

(jointly with H. Langer, A. Markus, V. Matsaev, M. Wagenhofer)

Classical numerical range (NR)

$\mathcal{H}$ Hilbert space, $\mathcal{A} \in L(\mathcal{H})$ (bounded linear operator)

Numerical range:

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Classical numerical range (NR)

$\mathcal{H}$ Hilbert space, $\mathcal{A} \in L(\mathcal{H})$ (bounded linear operator)

Numerical range:

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Well-known:

- $W(\mathcal{A})$ convex,

Classical numerical range (NR)

$\mathcal{H}$ Hilbert space, $\mathcal{A} \in L(\mathcal{H})$ (bounded linear operator)

Numerical range:

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Well-known:

- $W(\mathcal{A})$ convex,
- $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}),\ \ \sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$,

Classical numerical range (NR)

$\mathcal{H}$ Hilbert space, $\mathcal{A} \in L(\mathcal{H})$ (bounded linear operator)

Numerical range:

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Well-known:

- $W(\mathcal{A})$ convex,
- $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \;\; \sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$,
- $\lambda_0 \in W(\mathcal{A})$ corner $\implies \lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$,

Classical numerical range (NR)

$\mathcal{H}$ Hilbert space, $\mathcal{A} \in L(\mathcal{H})$ (bounded linear operator)

Numerical range:

$$W(\mathcal{A}) := \big\{(\mathcal{A}x, x) : x \in \mathcal{H}, \|x\| = 1\big\}.$$

Well-known:

- $W(\mathcal{A})$ convex,
- $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W(\mathcal{A}), \;\; \sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$,
- $\lambda_0 \in W(\mathcal{A})$ corner $\implies \lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$,
- $\big\|(\mathcal{A}-\lambda)^{-1}\big\| \le \dfrac{1}{\text{dist}\,\big(\lambda, W(\mathcal{A})\big)}, \;\; \lambda \notin \overline{W(\mathcal{A})}$,
- $\mathcal{A} = \mathcal{A}^* \ge \alpha \implies \lambda_n = \min\limits_{\dim \mathcal{L}=n} \max\limits_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \big(\mathcal{A}x, x\big)$
- ......

1. The quadratic numerical range (QNR)

Now: $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$,

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \times \mathcal{H}_2$$

with $A \in L(\mathcal{H}_1)$, $B \in L(\mathcal{H}_2, \mathcal{H}_1)$, $C \in L(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in L(\mathcal{H}_2)$.

1. The quadratic numerical range (QNR)

Now: $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$,

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \times \mathcal{H}_2$$

with $A \in L(\mathcal{H}_1)$, $B \in L(\mathcal{H}_2, \mathcal{H}_1)$, $C \in L(\mathcal{H}_1, \mathcal{H}_2)$, and $D \in L(\mathcal{H}_2)$.

Definition. (LT'98)

$$W^2(\mathcal{A}) := \Bigg\{ \lambda \in \mathbb{C} \, : \, \det \overbrace{\begin{pmatrix} (Ax_1, x_1) - \lambda & (Bx_2, x_1) \\ (Cx_1, x_2) & (Dx_2, x_2) - \lambda \end{pmatrix}}^{=: \mathcal{A}_{x_1, x_2} - \lambda} = 0 \text{ for some}$$

$$x_1 \in \mathcal{H}_1, \, x_2 \in \mathcal{H}_2, \, \|x_1\| = \|x_2\| = 1 \Bigg\}$$

$$= \bigcup_{\|x_1\| = \|x_2\| = 1} \sigma_{\mathrm{p}}(\mathcal{A}_{x_1, x_2})$$

is called quadratic numerical range (QNR) of $\mathcal{A}$.

2. Spectral inclusion **[LMMT'01]**

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

2. Spectral inclusion [LMMT'01]

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Theorem 2. $\sigma_{\rm p}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$.

2. Spectral inclusion [LMMT'01]

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Theorem 2. $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$.

Proof. (for $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$)

$$\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0.$$

2. Spectral inclusion [LMMT'01]

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Theorem 2. $\sigma_{\mathrm{p}}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$.

Proof. (for $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A})$)

$$\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0.$$

Then

$$(Ax_1, \widehat{x}_1) + (Bx_2, \widehat{x}_1) = \lambda(x_1, \widehat{x}_1),$$
$$Cx_1, \widehat{x}_2) + (Dx_2, \widehat{x}_2) = \lambda(x_2 \qquad ,$$

2. Spectral inclusion [LMMT'01]

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Theorem 2. $\sigma_{\rm p}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$.

Proof. (for $\boldsymbol{\lambda} \in \boldsymbol{\sigma}_{\mathbf{p}}(\boldsymbol{\mathcal{A}})$)

$$\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0.$$

Then

$$(Ax_1, \widehat{x}_1) + (Bx_2, \widehat{x}_1) = \lambda(x_1, \widehat{x}_1),$$
$$(Cx_1, \widehat{x}_2) + (Dx_2, \widehat{x}_2) = \lambda(x_2, \widehat{x}_2),$$

with $x_i =: \|x_i\| \widehat{x}_i$, $\|\widehat{x}_i\| = 1$, for $i = 1, 2$. Hence

2. Spectral inclusion [LMMT'01]

Theorem 1. $W^2(\mathcal{A}) \subset W(\mathcal{A})$.

Theorem 2. $\sigma_{\rm p}(\mathcal{A}) \subset W^2(\mathcal{A}), \quad \sigma(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$.

Proof. (for $\lambda \in \sigma_{\rm p}(\mathcal{A})$)

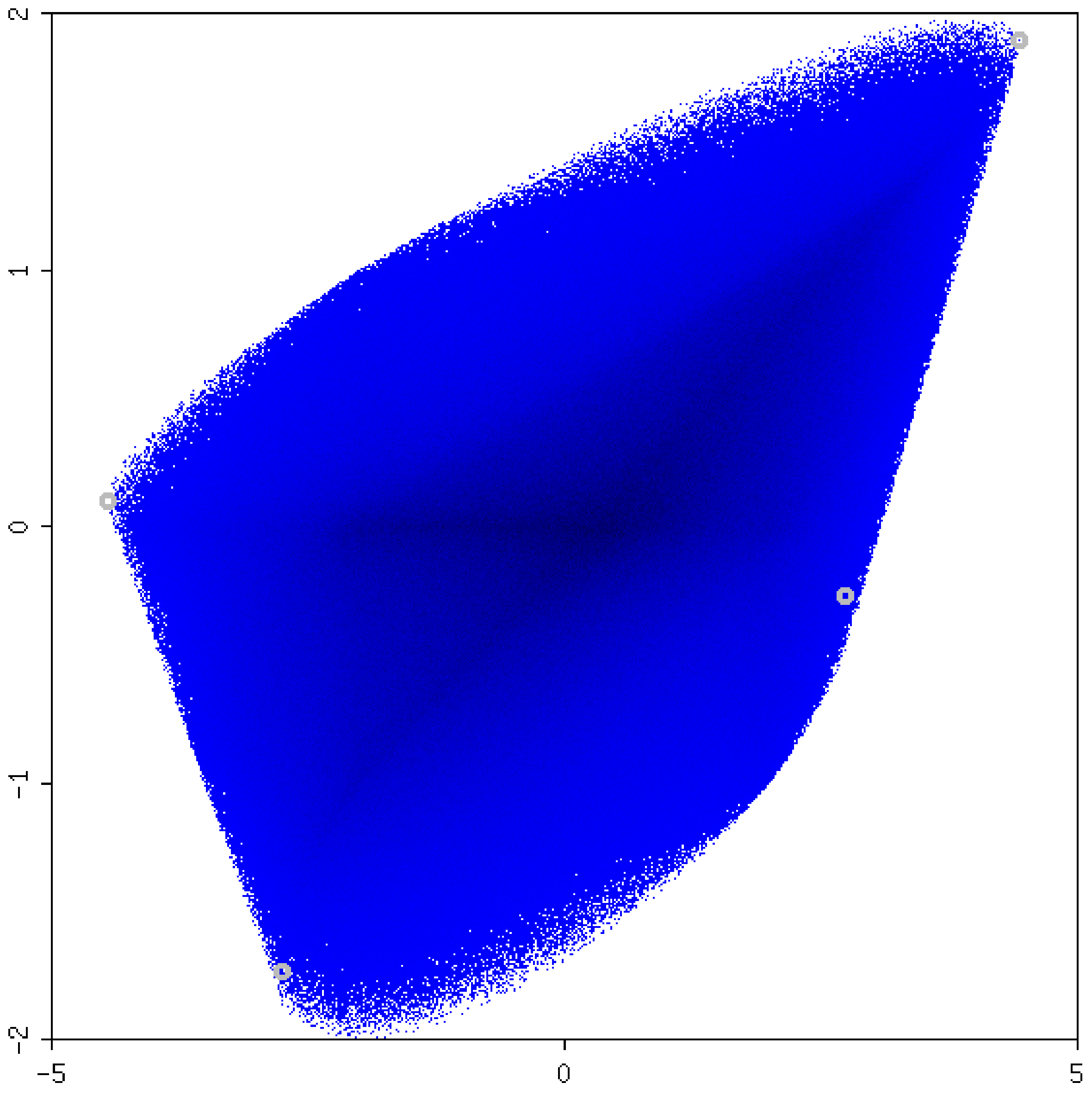
$$\mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq 0.$$

Then

$$(Ax_1, \widehat{x}_1) + (Bx_2, \widehat{x}_1) = \lambda(x_1, \widehat{x}_1),$$
$$(Cx_1, \widehat{x}_2) + (Dx_2, \widehat{x}_2) = \lambda(x_2, \widehat{x}_2),$$

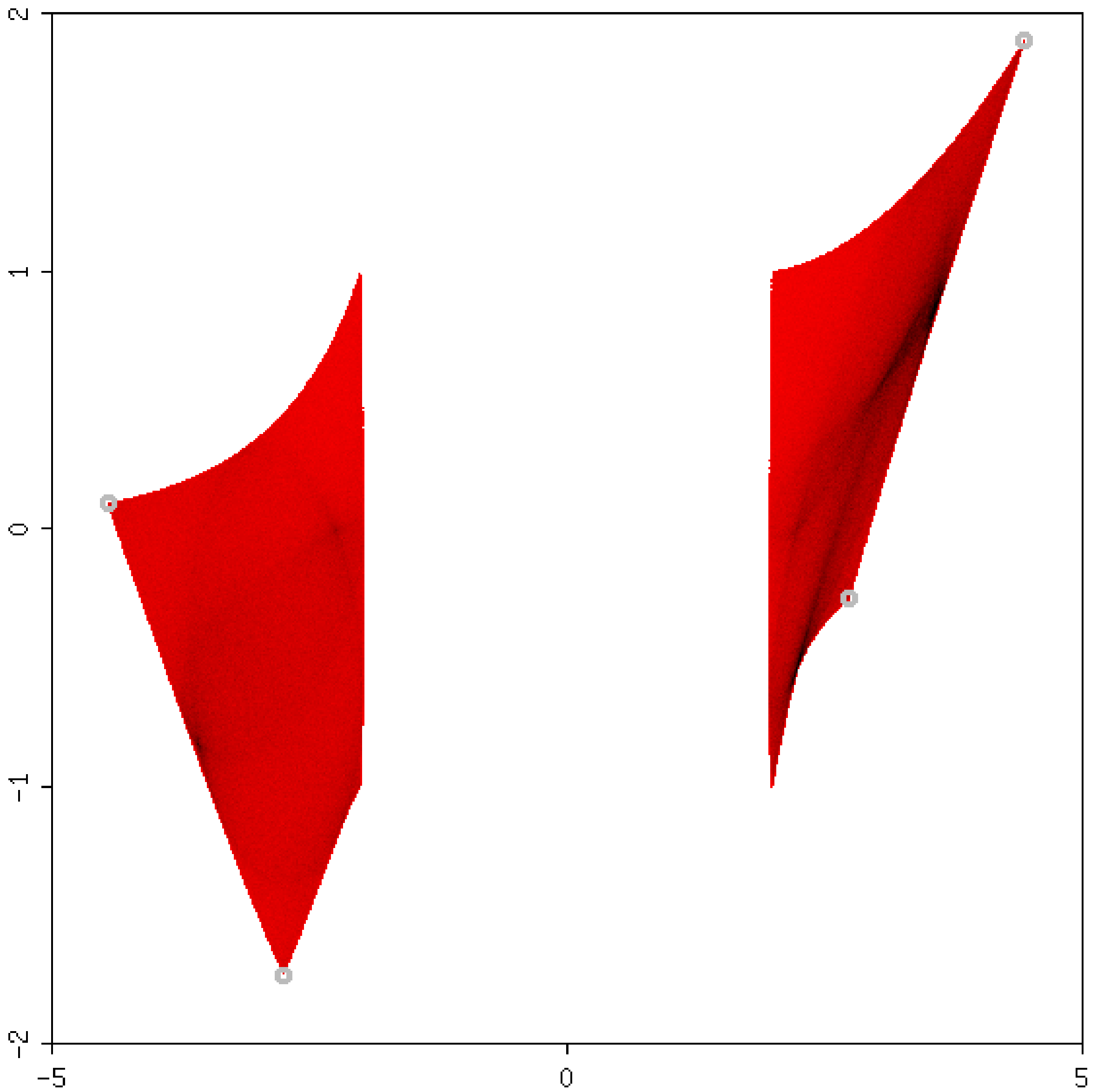
with $x_i =: \|x_i\| \widehat{x}_i$, $\|\widehat{x}_i\| = 1$, for $i = 1, 2$. Hence

$$\Longrightarrow \quad \mathcal{A}_{\widehat{x}_1, \widehat{x}_2} \begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix} = \lambda \begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix} \quad \Longrightarrow \quad \lambda \in \sigma_{\rm p}(\mathcal{A}_{\widehat{x}_1, \widehat{x}_2}) \subset W^2(\mathcal{A}). \quad \Box$$



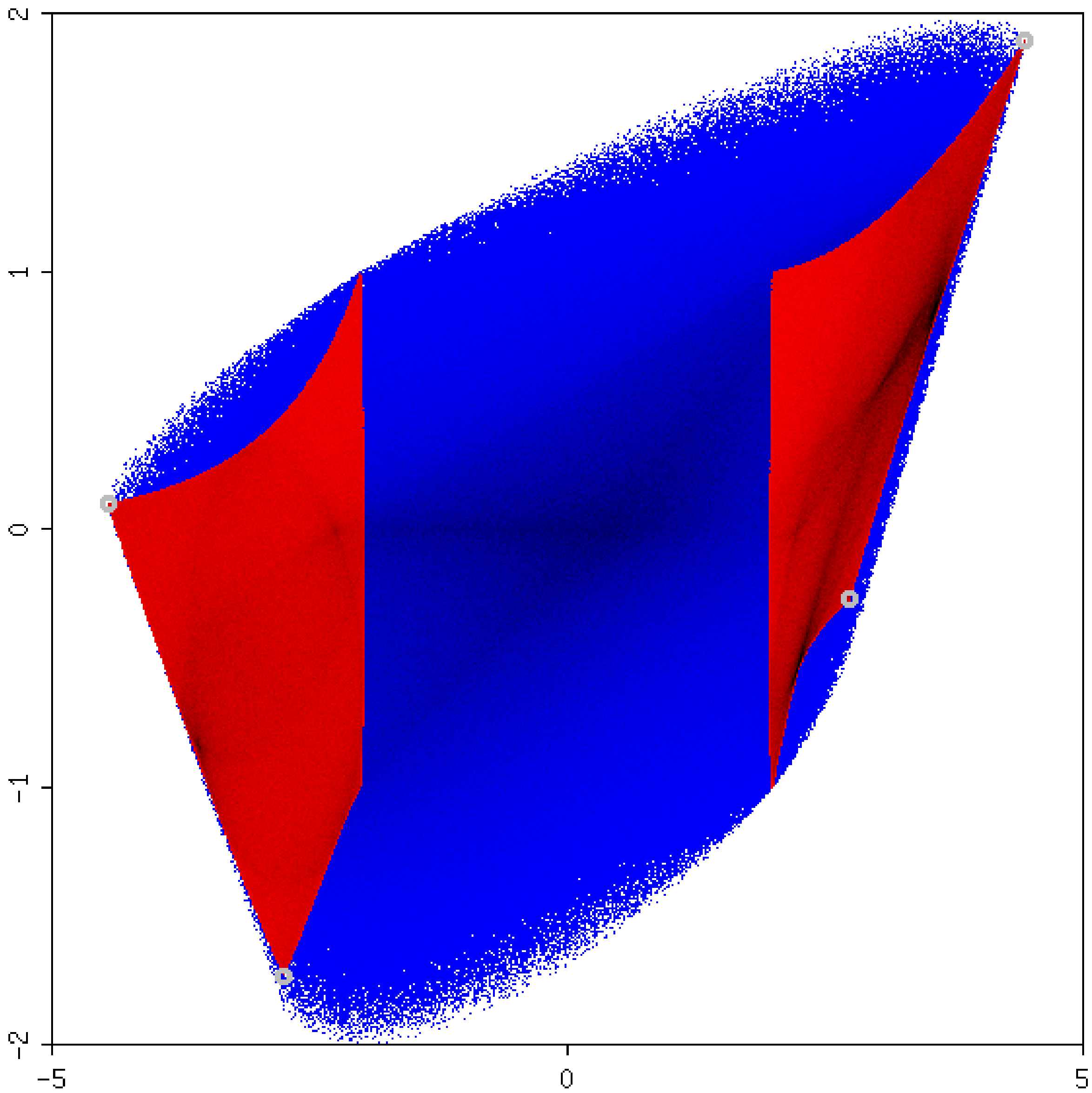
Numerical range of

$$\begin{pmatrix} 2 & i & 1 & 3+i \\ i & 2 & 3+i & 1 \\ 1 & 3+i & -2 & i \\ 3+i & 1 & i & -2 \end{pmatrix}$$



QNR of

$$\left(\begin{array}{cc|cc} 2 & \mathrm{i} & 1 & 3+\mathrm{i} \\ \mathrm{i} & 2 & 3+\mathrm{i} & 1 \\ \hline 1 & 3+\mathrm{i} & -2 & \mathrm{i} \\ 3+\mathrm{i} & 1 & \mathrm{i} & -2 \end{array}\right)$$

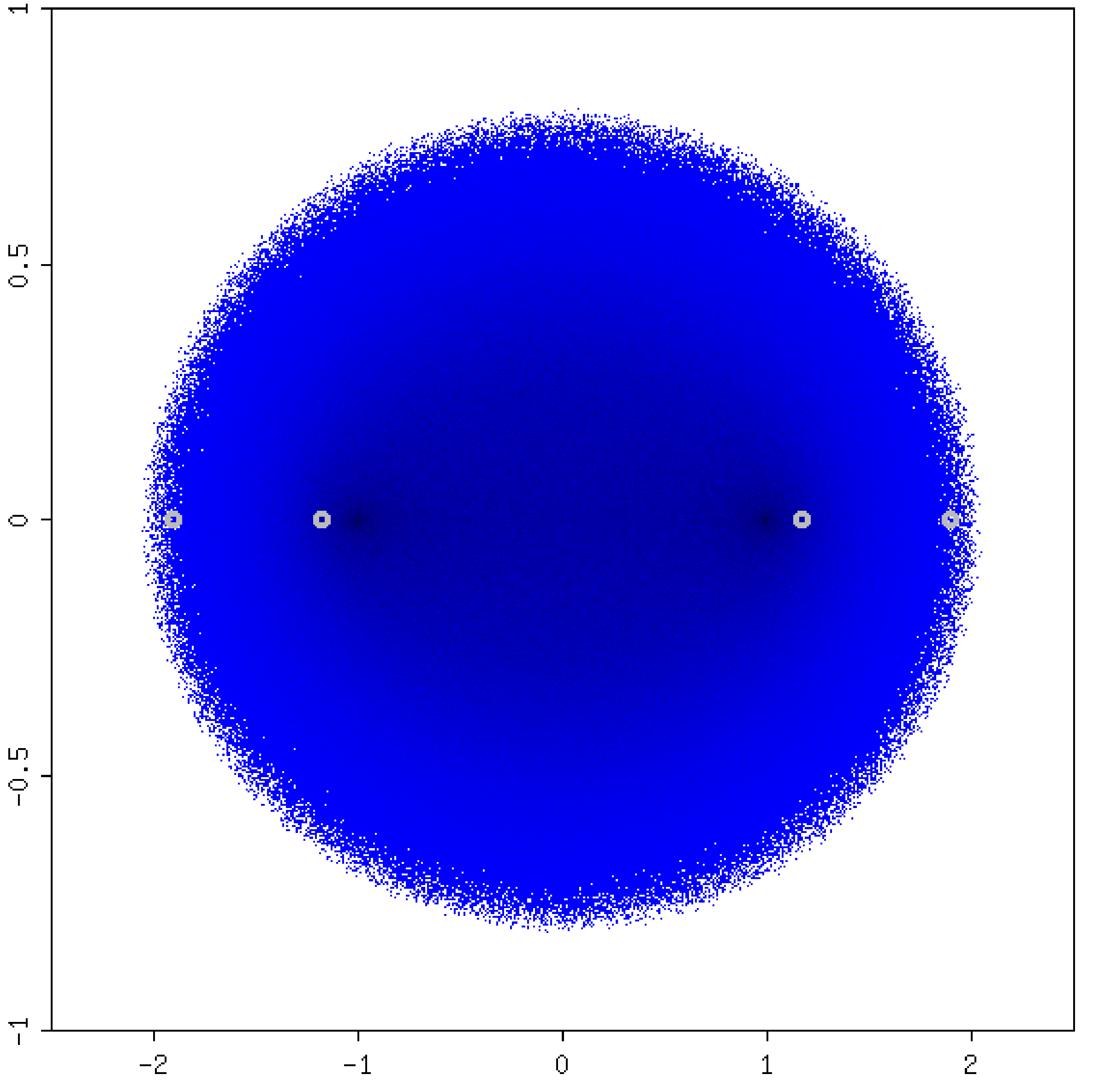


Numerical range of

$$\begin{pmatrix} 2 & i & 1 & 3+i \\ i & 2 & 3+i & 1 \\ 1 & 3+i & -2 & i \\ 3+i & 1 & i & -2 \end{pmatrix}$$

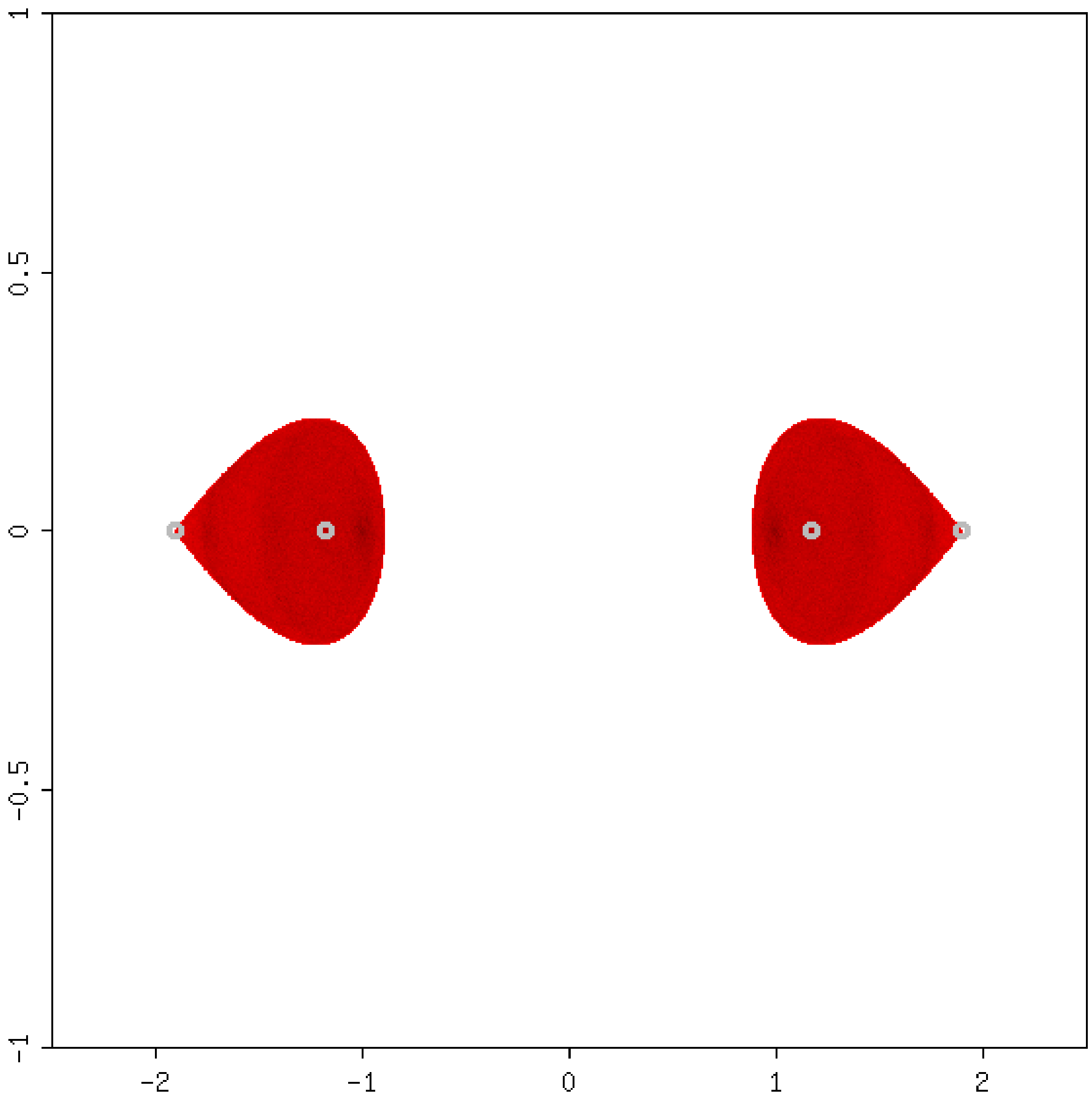
QNR of

$$\left(\begin{array}{cc|cc} 2 & i & 1 & 3+i \\ i & 2 & 3+i & 1 \\ \hline 1 & 3+i & -2 & i \\ 3+i & 1 & i & -2 \end{array}\right)$$



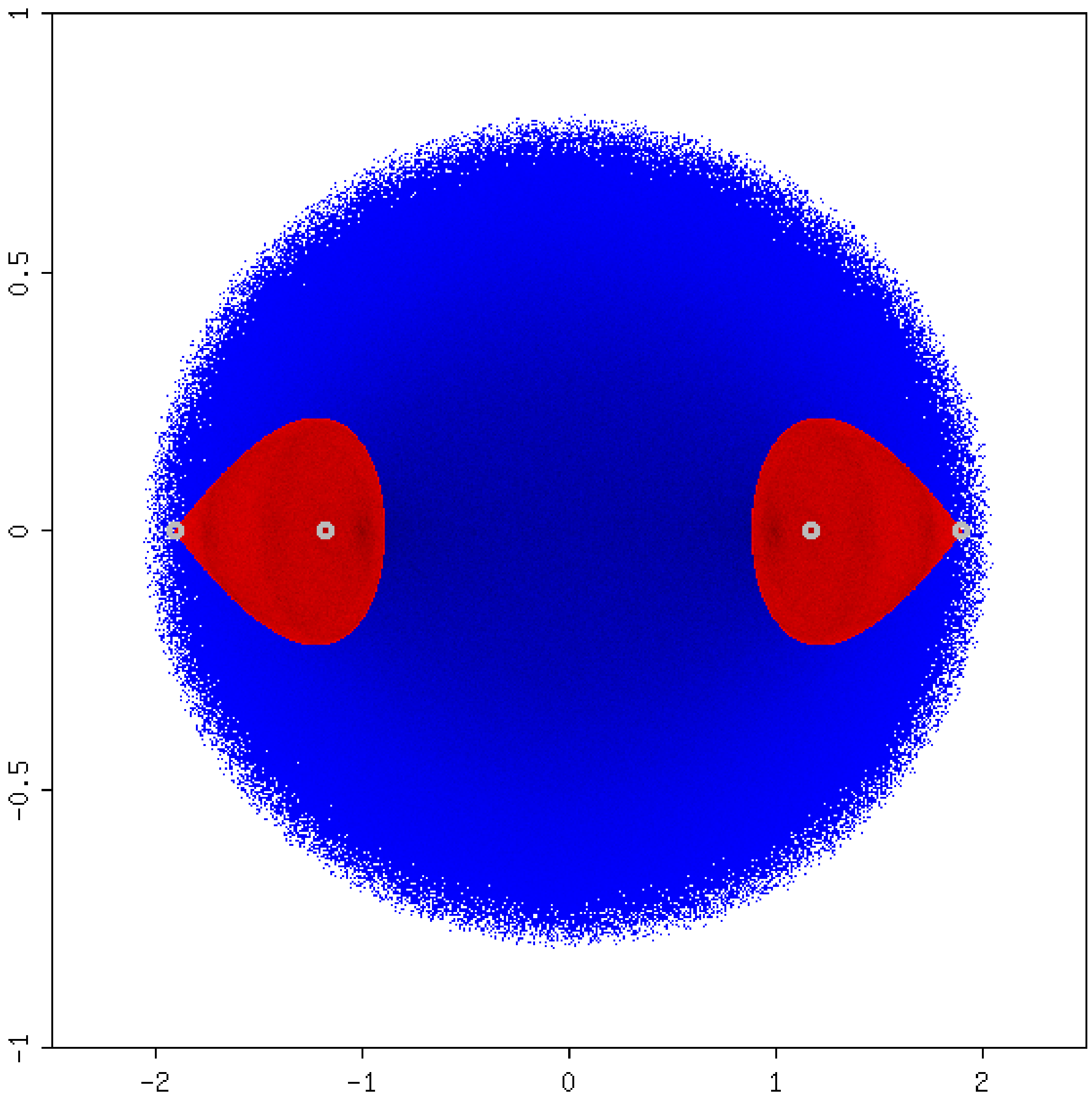
Numerical range of

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$



QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

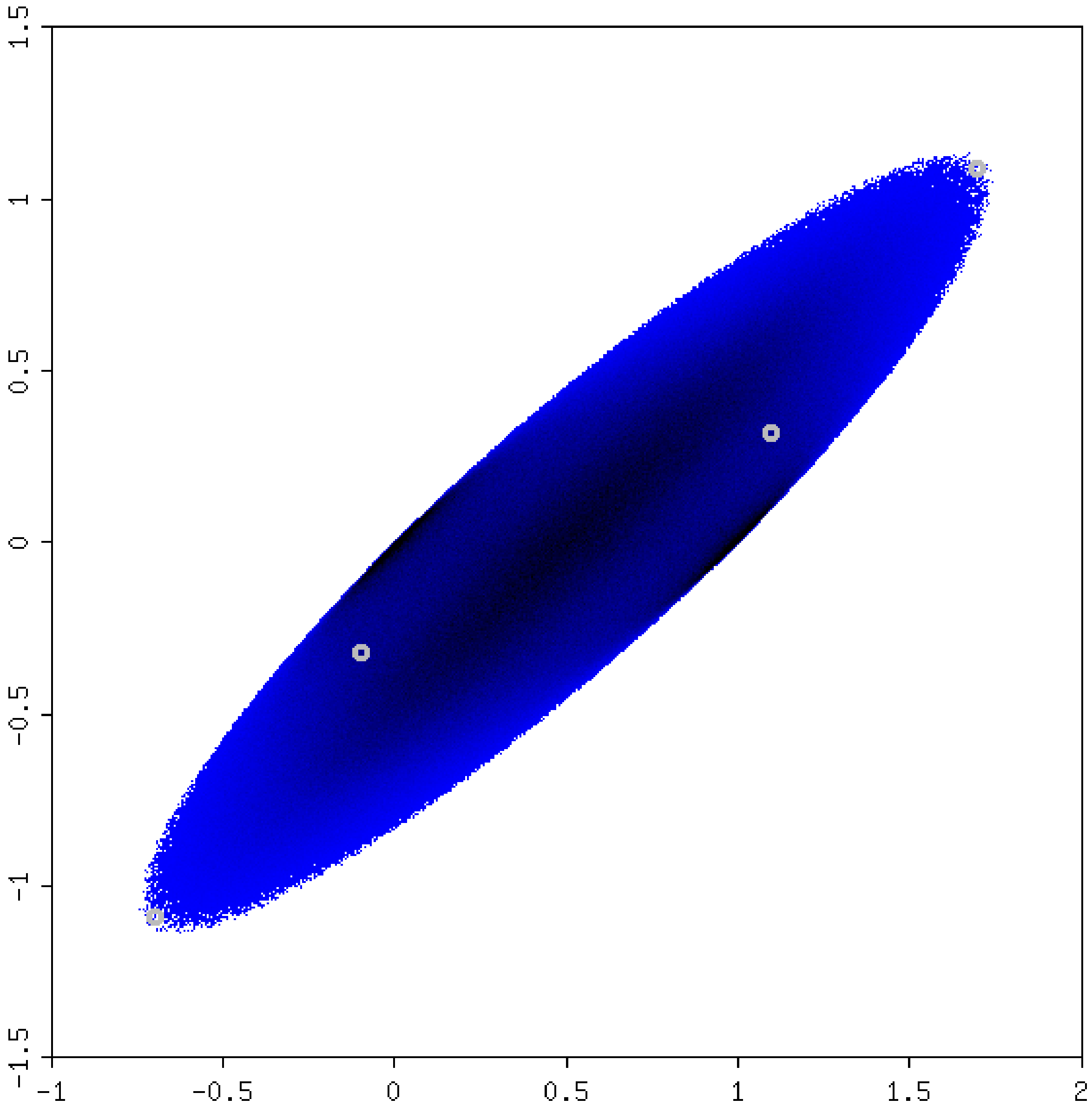


Numerical range of

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

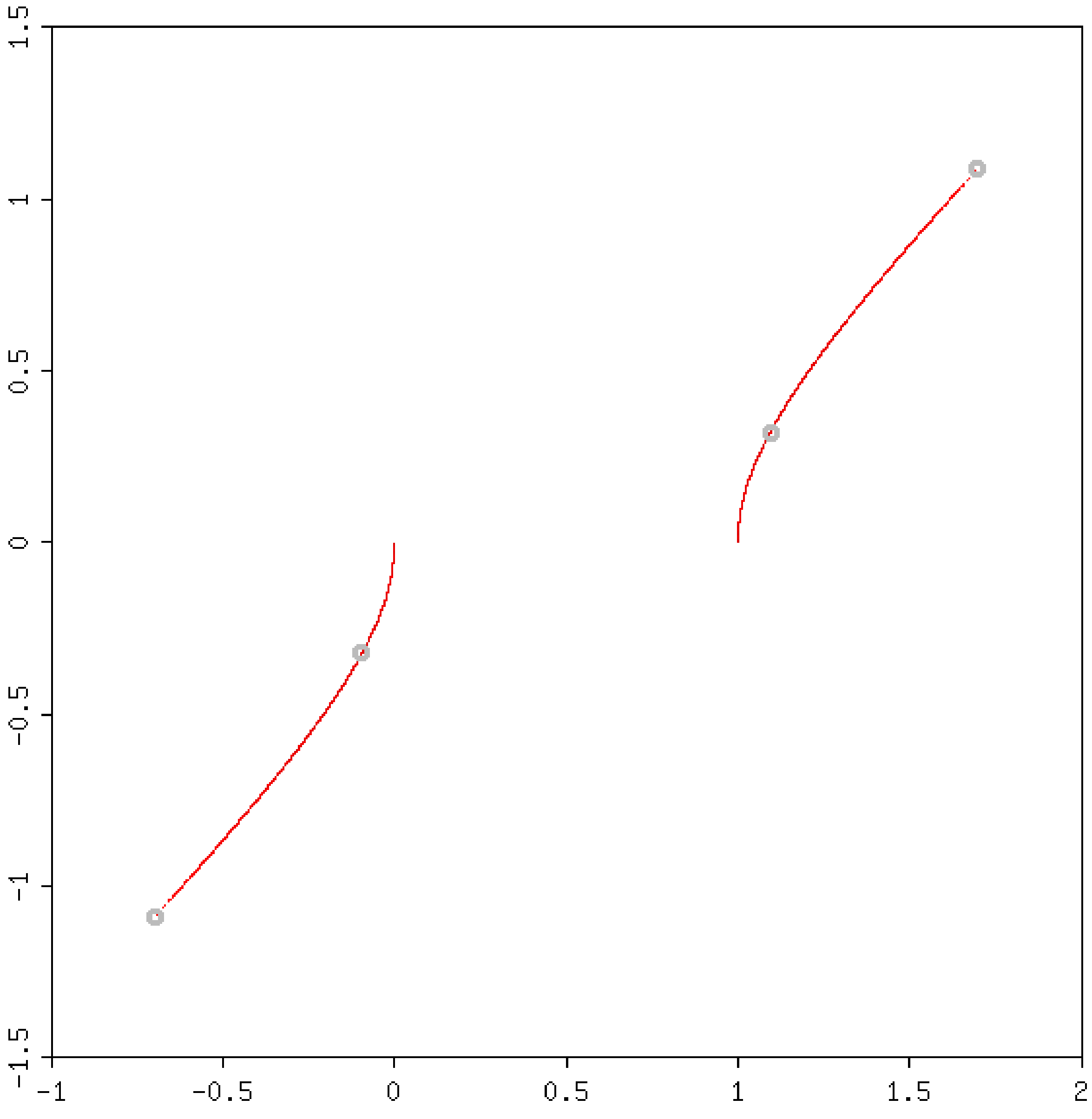
QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right)$$



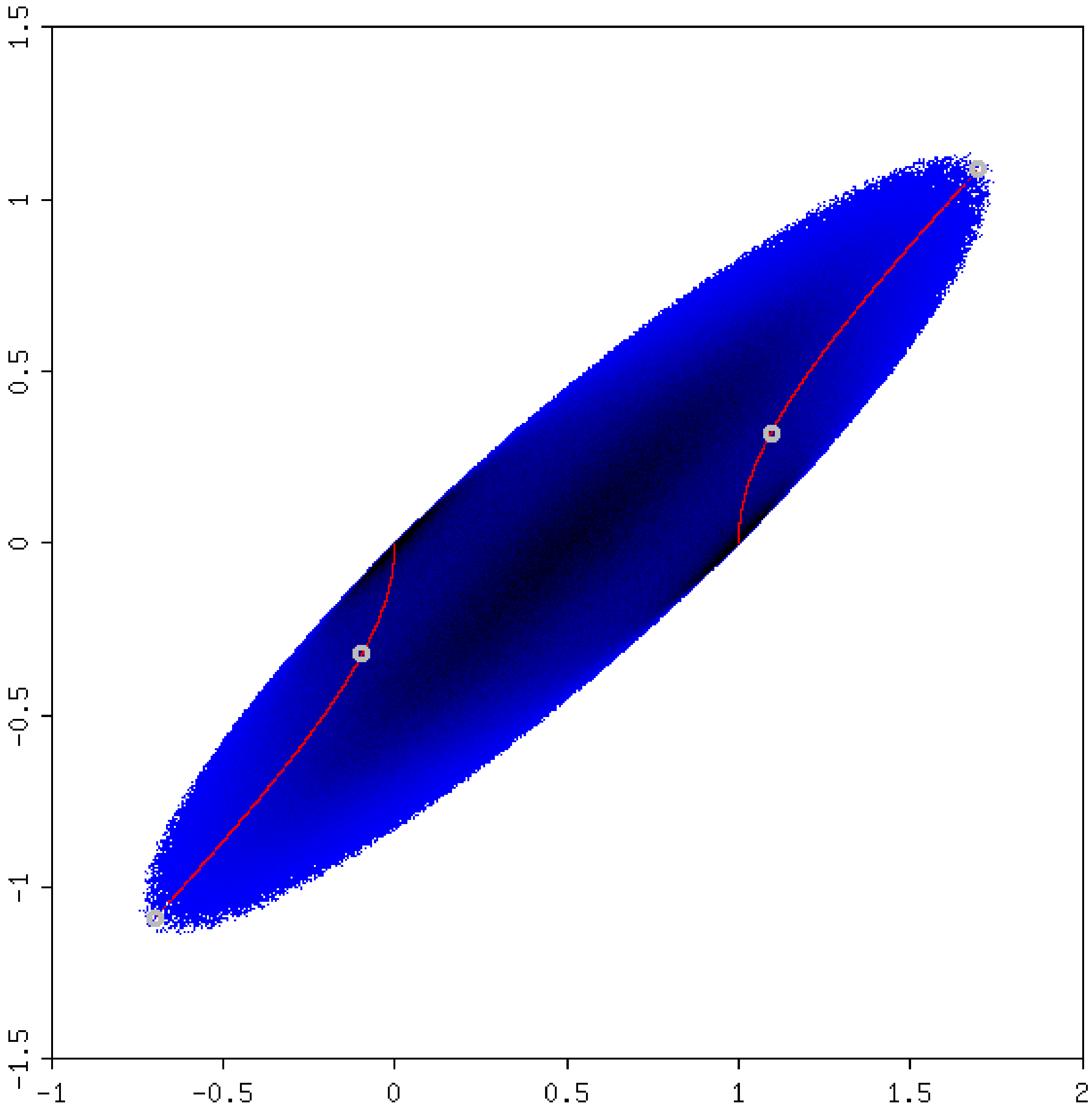
Numerical range of

$$\begin{pmatrix} 1 & 0 & 1 & i \\ 0 & 1 & 0 & 1 \\ i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \end{pmatrix}$$



QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & i \\ 0 & 1 & 0 & 1 \\ \hline i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \end{array}\right)$$

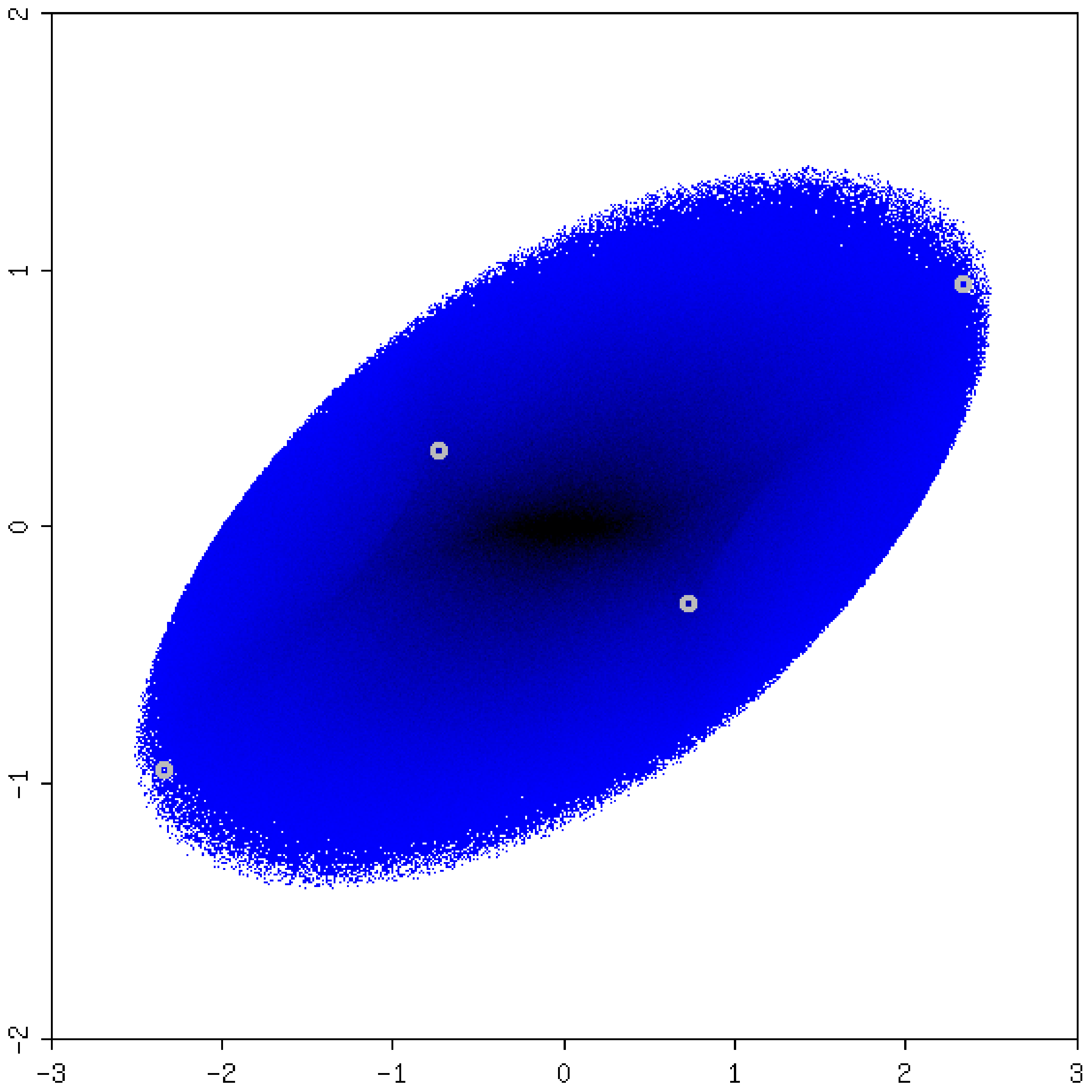


Numerical range of

$$\begin{pmatrix} 1 & 0 & 1 & i \\ 0 & 1 & 0 & 1 \\ i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \end{pmatrix}$$

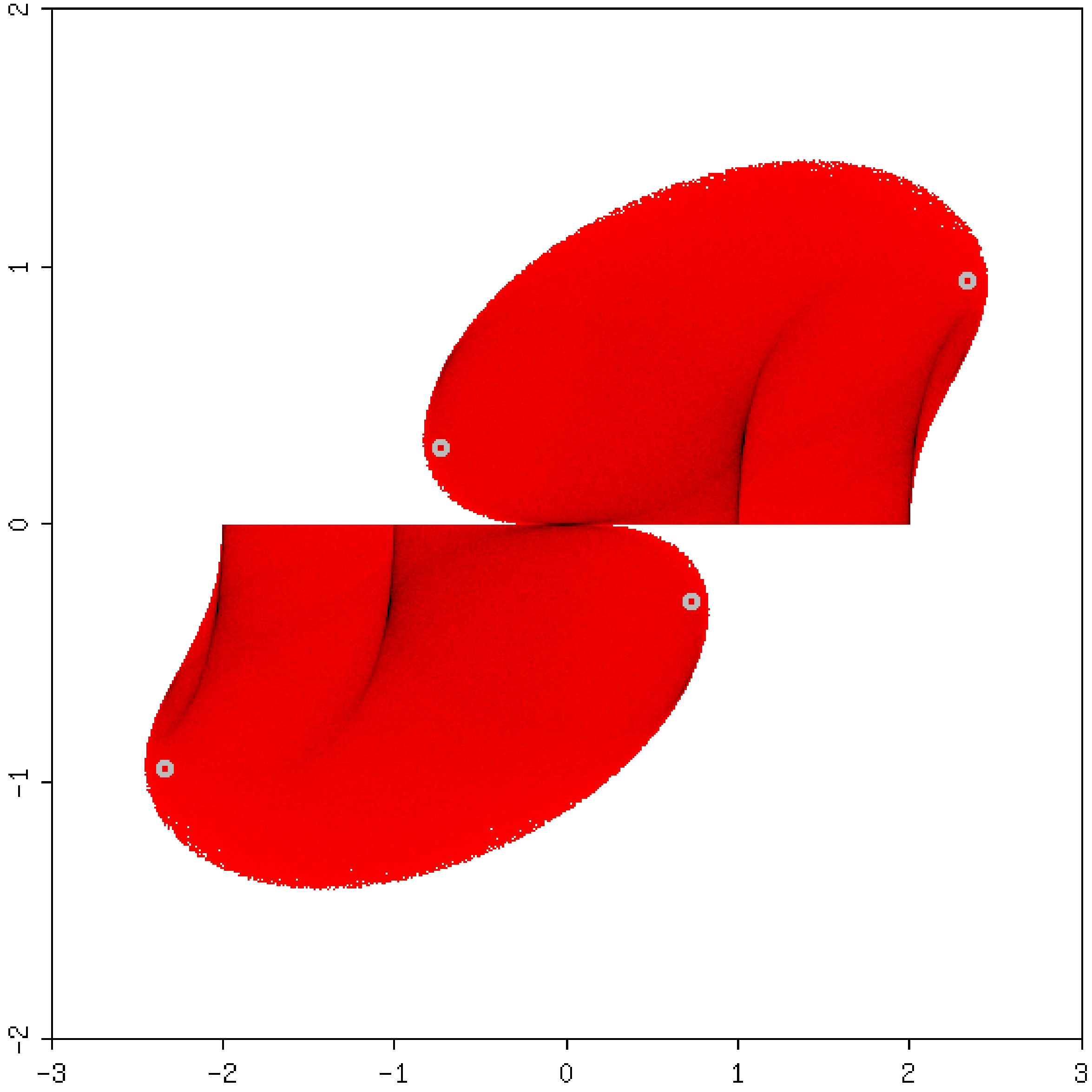
QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & i \\ 0 & 1 & 0 & 1 \\ \hline i & 0 & 0 & 0 \\ 1 & i & 0 & 0 \end{array}\right)$$



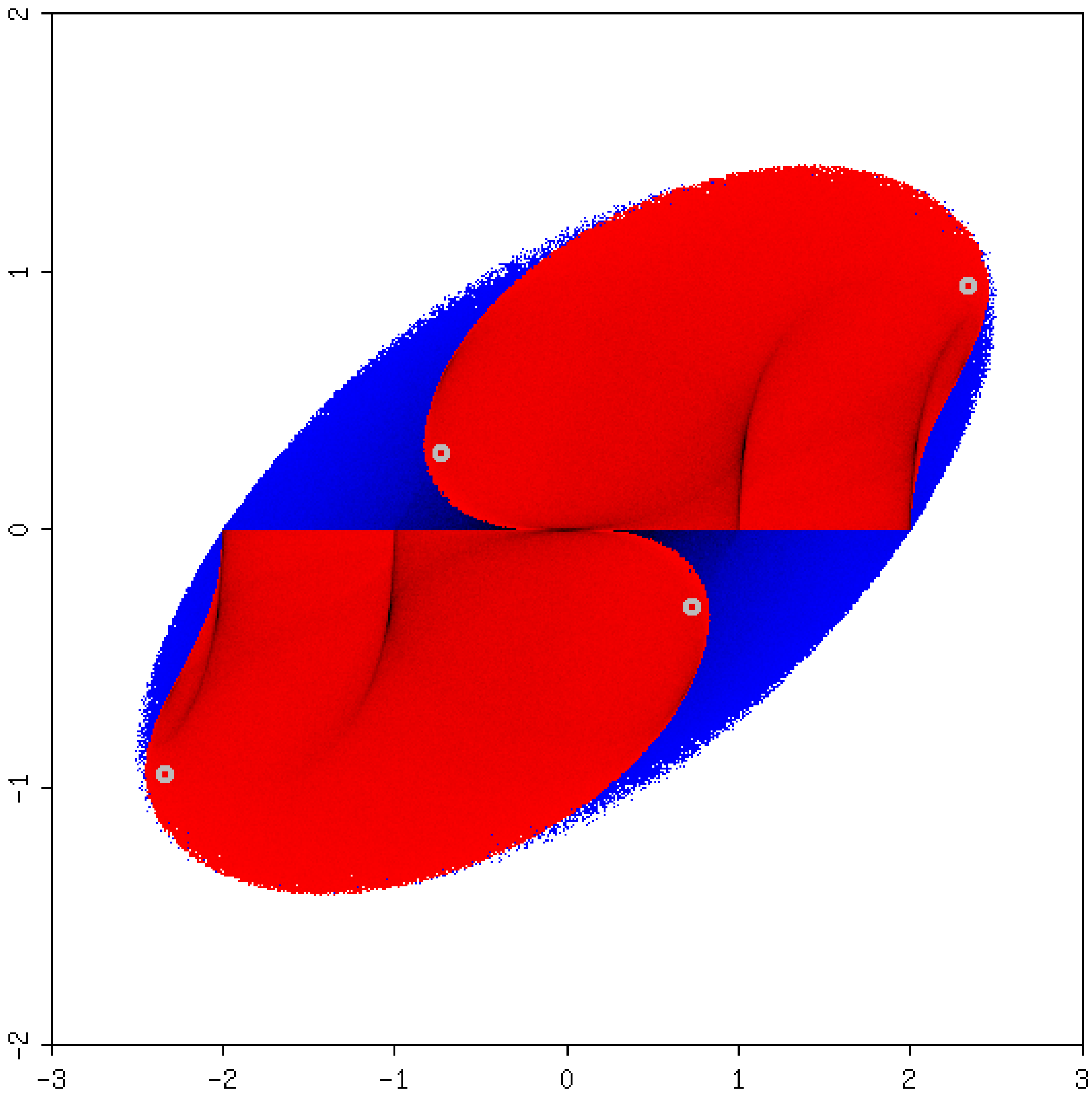
Numerical range of

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \mathrm{i} & \mathrm{i} & 1 & 0 \\ \mathrm{i} & \mathrm{i} & 0 & -1 \end{pmatrix}$$



QNR of

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \hline \mathrm{i} & \mathrm{i} & 1 & 0 \\ \mathrm{i} & \mathrm{i} & 0 & -1 \end{pmatrix}$$

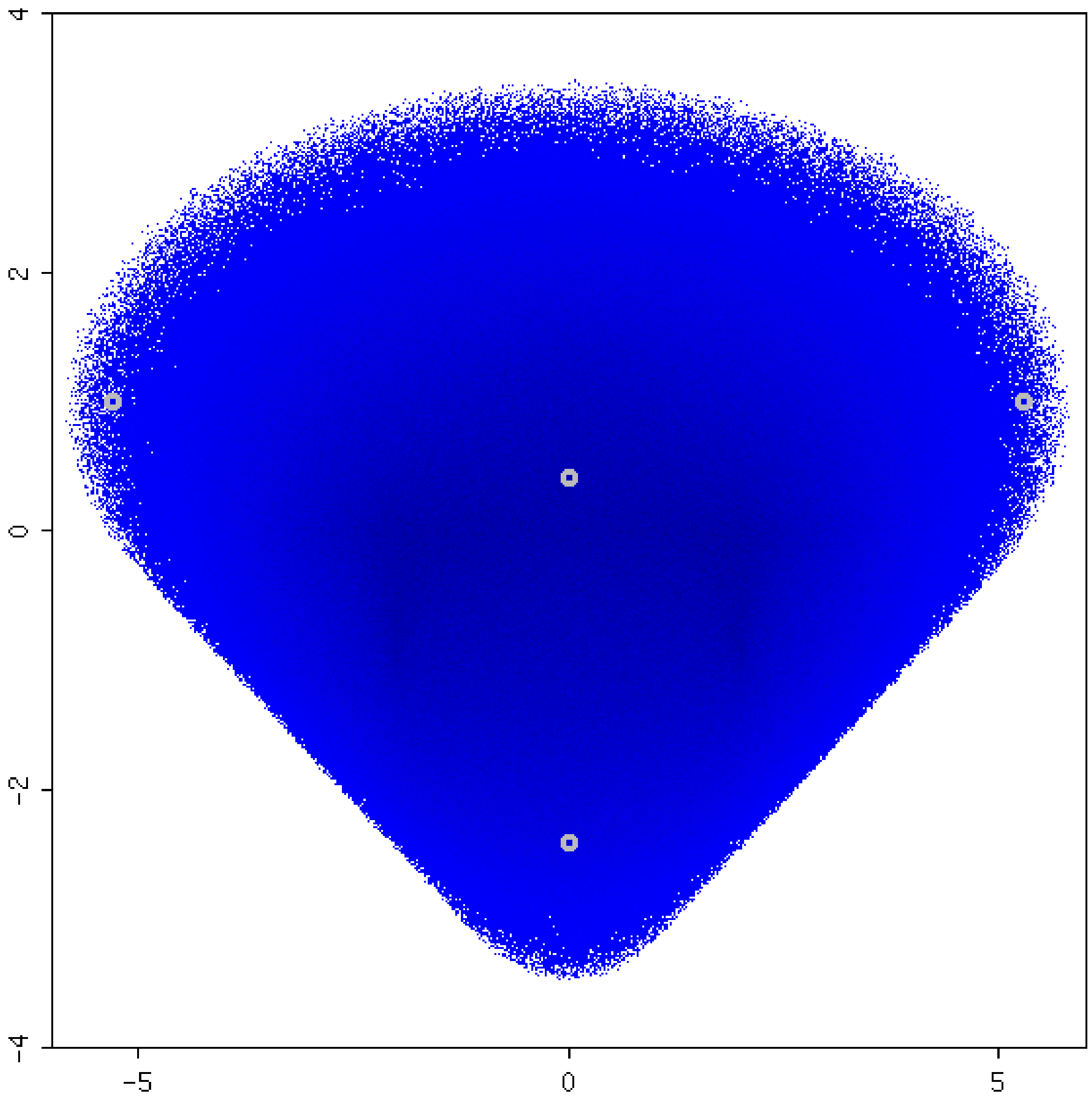


Numerical range of

$$\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ i & i & 1 & 0 \\ i & i & 0 & -1 \end{pmatrix}$$

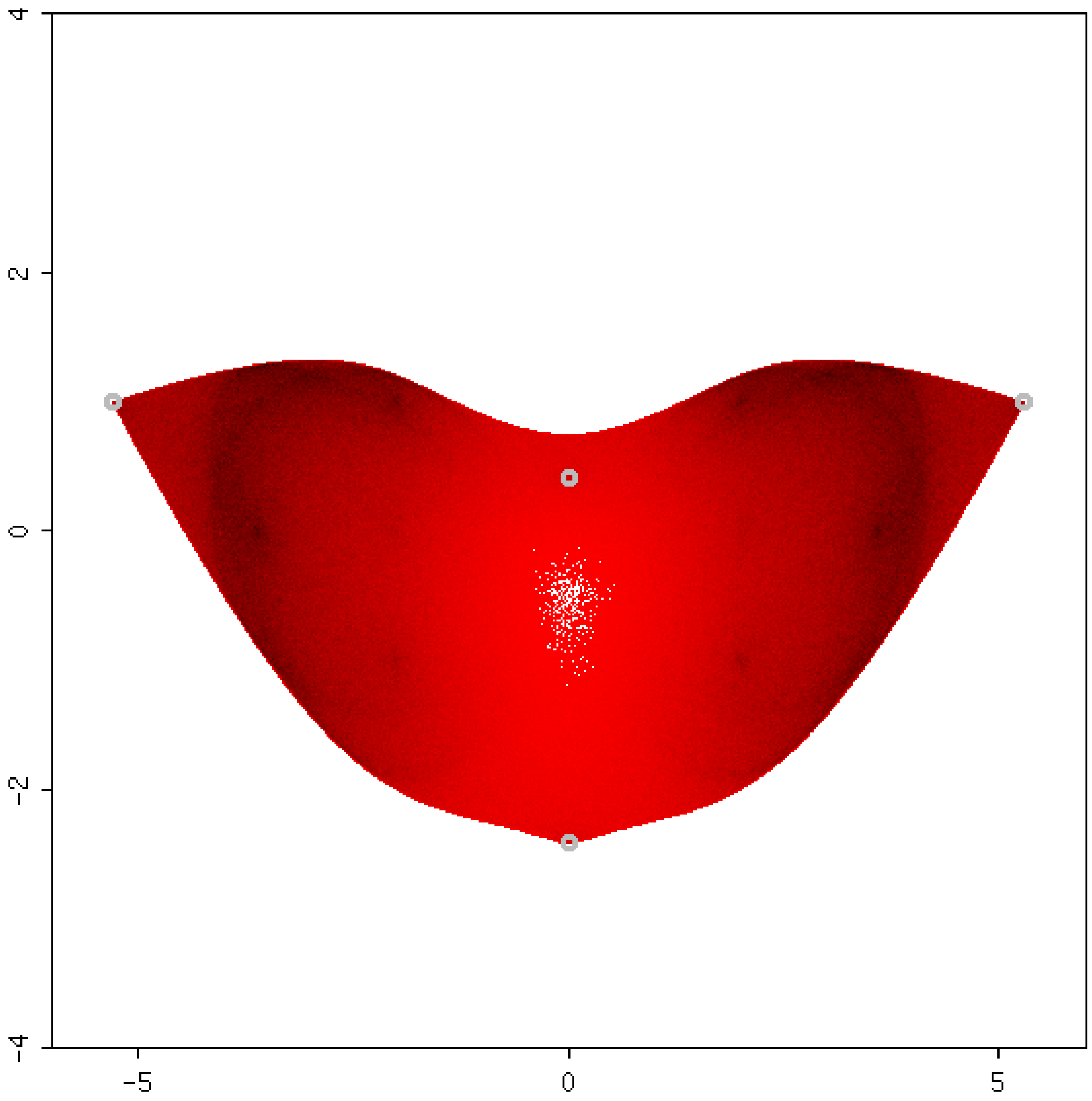
QNR of

$$\left(\begin{array}{cc|cc} 2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \hline i & i & 1 & 0 \\ i & i & 0 & -1 \end{array}\right)$$



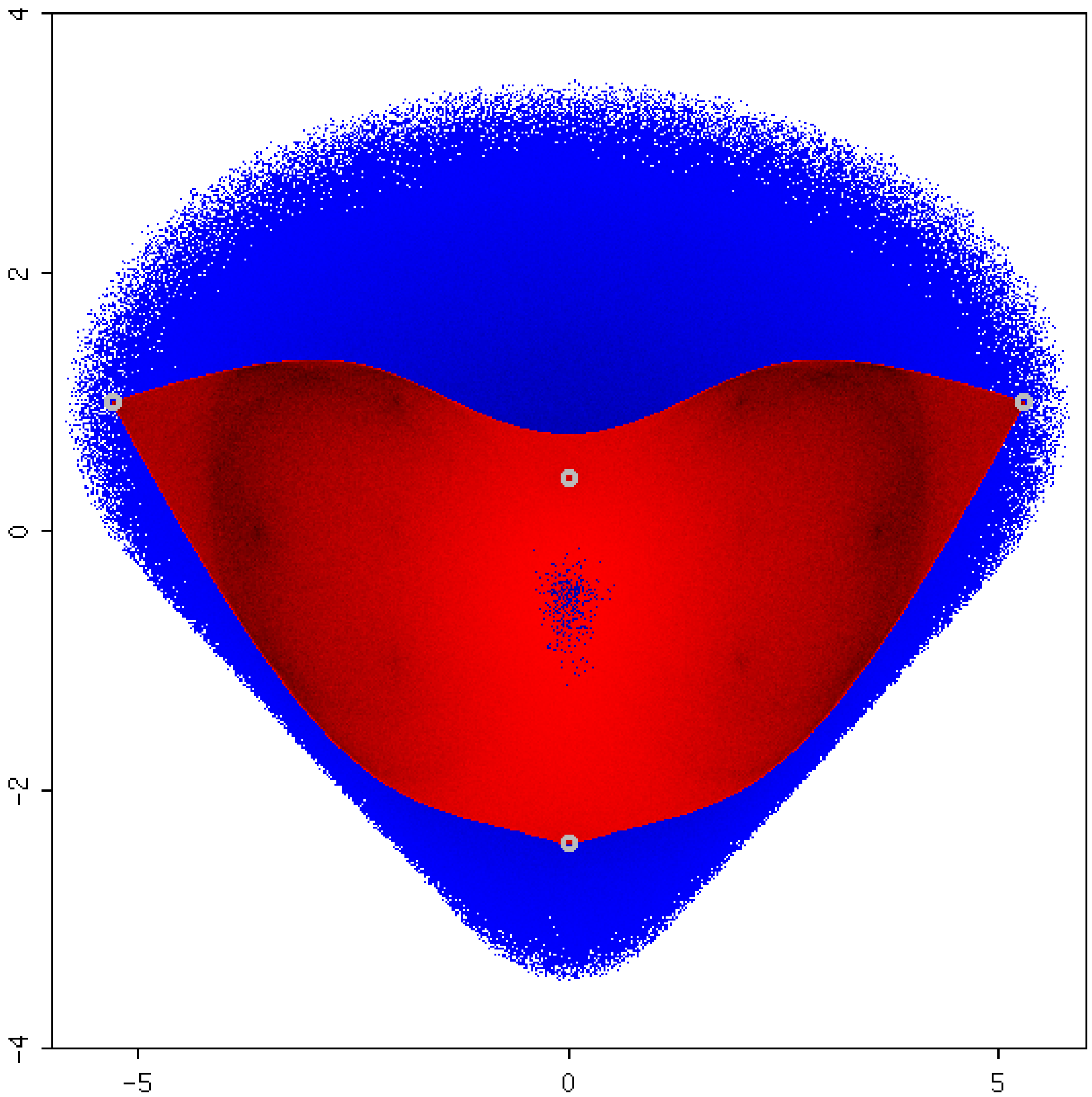
Numerical range of

$$\begin{pmatrix} 2 & i & 0 & 3 \\ i & 2 & 3 & 0 \\ 5 & 3 & -2 & i \\ 3 & 5 & i & -2 \end{pmatrix}$$



QNR of

$$\left(\begin{array}{cc|cc} 2 & i & 0 & 3 \\ i & 2 & 3 & 0 \\ \hline 5 & 3 & -2 & i \\ 3 & 5 & i & -2 \end{array}\right)$$

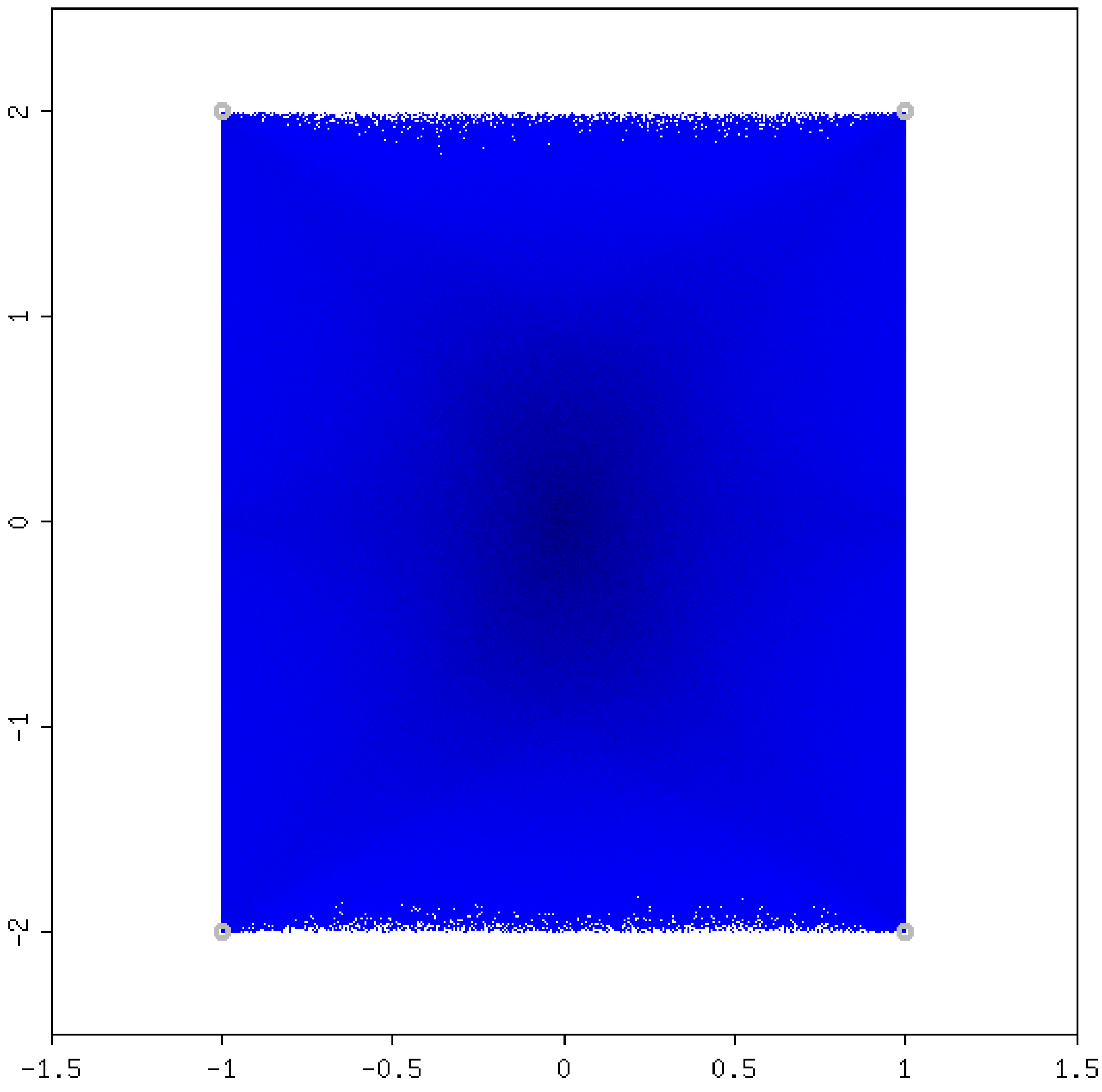


Numerical range of

$$\begin{pmatrix} 2 & i & 0 & 3 \\ i & 2 & 3 & 0 \\ 5 & 3 & -2 & i \\ 3 & 5 & i & -2 \end{pmatrix}$$

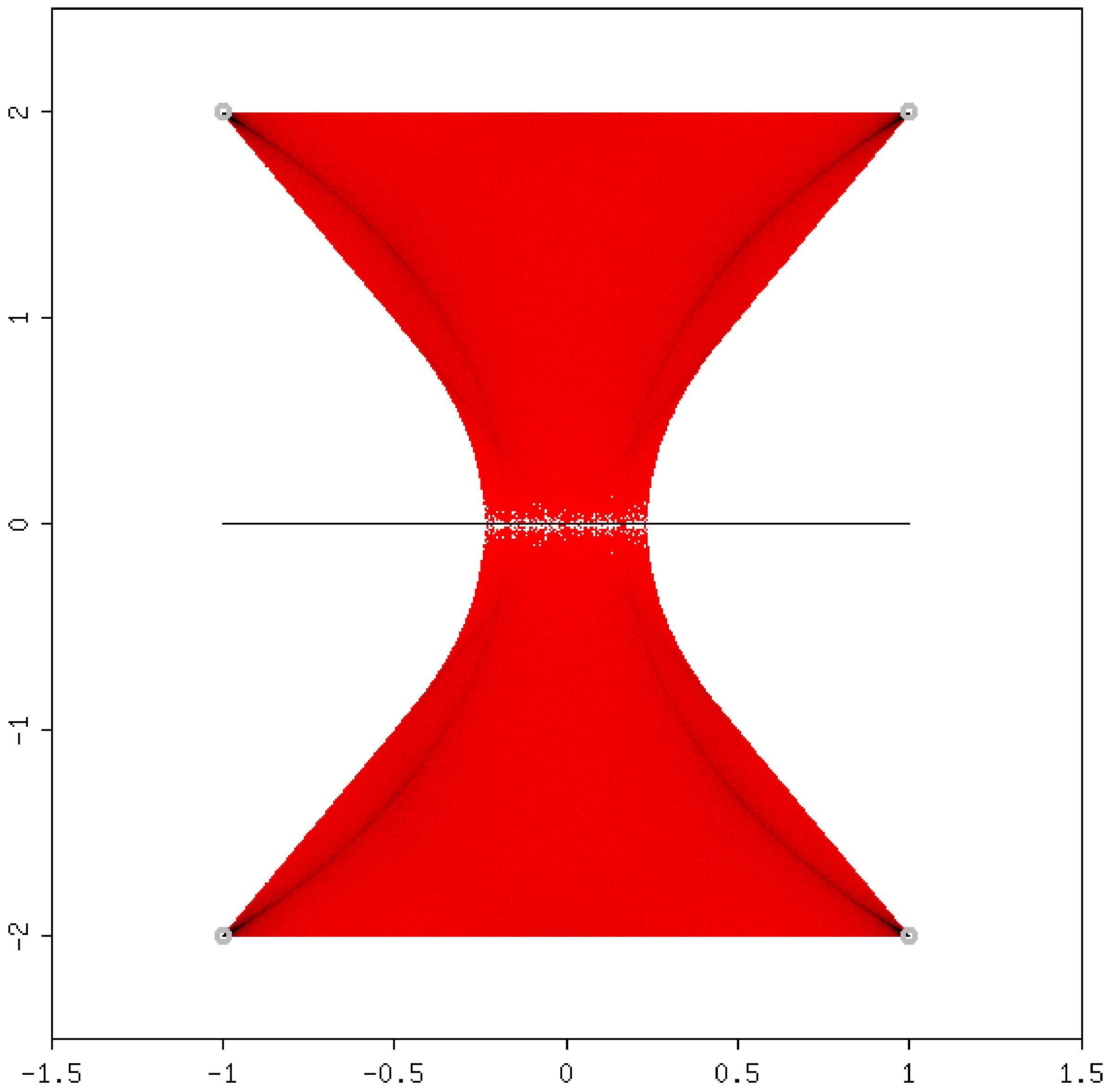
QNR of

$$\left(\begin{array}{cc|cc} 2 & i & 0 & 3 \\ i & 2 & 3 & 0 \\ \hline 5 & 3 & -2 & i \\ 3 & 5 & i & -2 \end{array}\right)$$



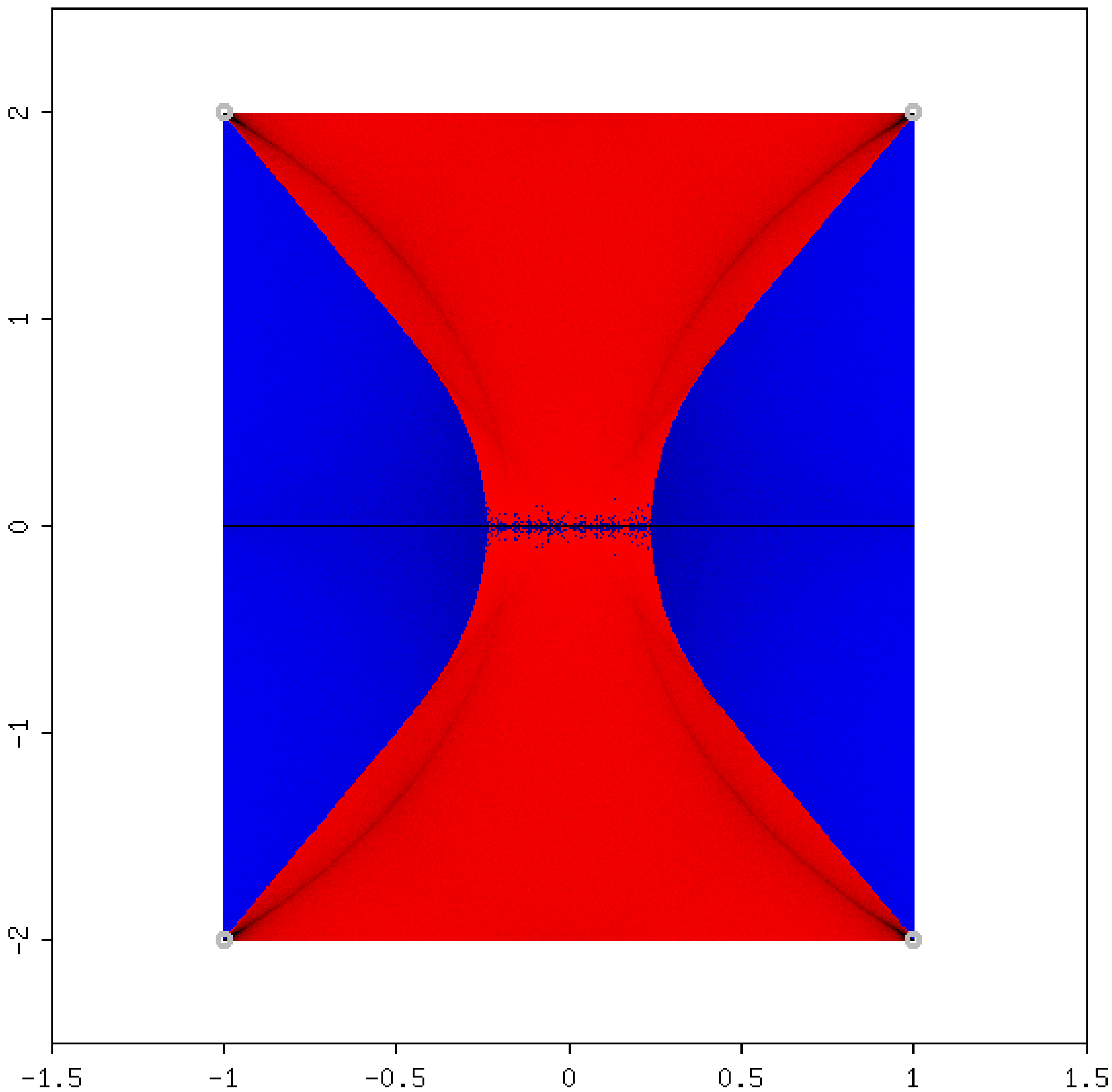
Numerical range of

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix}$$



QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right)$$

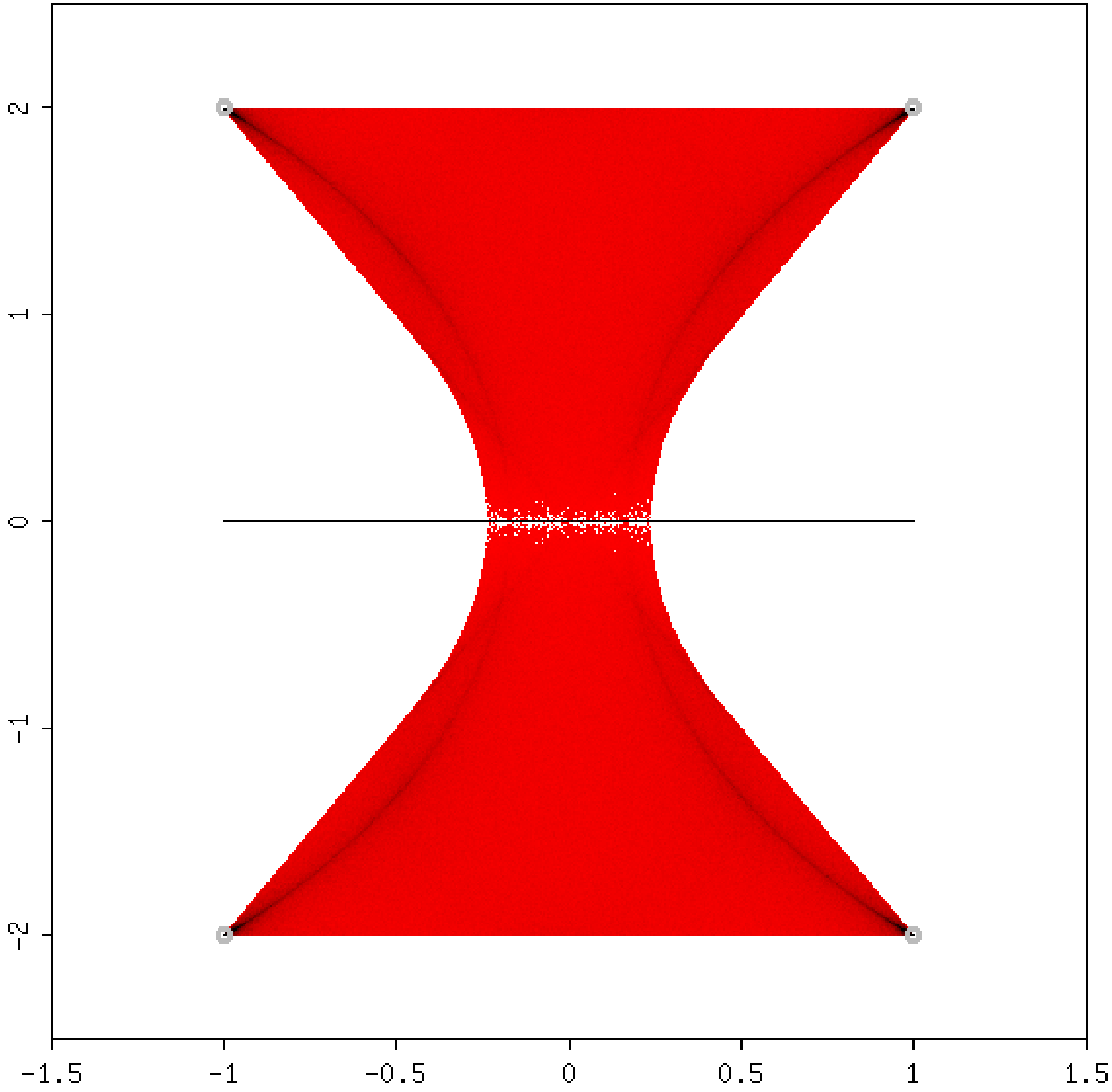


Numerical range of

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix}$$

QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right)$$



QNR of

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right)$$

3. Other properties [LMT'01, LMMT'01]

angle $< \pi$

Theorem 3. *Let* $\lambda_0 \in W^2(\mathcal{A})$ *be a corner,*

$$\det\left(\mathcal{A}_{x_1^0, x_2^0} - \lambda_0\right) = 0,$$

with $x_i^0 \in \mathcal{H}_i$, $\|x_i^0\| = 1$, *for* $i = 1, 2$. Then either

i) $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$ with eigenvector $\binom{x_1^0}{\gamma x_2^0}$ for some $\gamma \in \mathbb{C}$, or

ii) $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ with eigenvector x_1^0, or

iii) $\lambda_0 \in \sigma_{\mathrm{p}}(D)$ with eigenvector x_2^0.

3. Other properties [LMT'01, LMMT'01]

angle $< \pi$

Theorem 3. *Let* $\lambda_0 \in W^2(\mathcal{A})$ *be a corner,*

$$\det\left(\mathcal{A}_{x_1^0,x_2^0} - \lambda_0\right) = 0,$$

with $x_i^0 \in \mathcal{H}_i$, $\|x_i^0\| = 1$, *for* $i = 1, 2$. Then either

i) $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$ with eigenvector $\binom{x_1^0}{\gamma x_2^0}$ for some $\gamma \in \mathbb{C}$, or

ii) $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ with eigenvector x_1^0, or

iii) $\lambda_0 \in \sigma_{\mathrm{p}}(D)$ with eigenvector x_2^0.

Theorem 3'. $\lambda_0 \in \overline{W^2(\mathcal{A})}$ *corner* $\implies \lambda_0 \in \sigma(\mathcal{A}) \cup \sigma(A) \cup \sigma(D)$.

3. Other properties [LMT'01, LMMT'01]

angle $< \pi$

Theorem 3. *Let* $\lambda_0 \in W^2(\mathcal{A})$ *be a corner,*

$$\det\left(\mathcal{A}_{x_1^0,x_2^0} - \lambda_0\right) = 0,$$

with $x_i^0 \in \mathcal{H}_i$, $\|x_i^0\| = 1$, *for* $i = 1, 2$. Then either

i) $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{A})$ with eigenvector $\binom{x_1^0}{\gamma x_2^0}$ for some $\gamma \in \mathbb{C}$, or

ii) $\lambda_0 \in \sigma_{\mathrm{p}}(A)$ with eigenvector x_1^0, or

iii) $\lambda_0 \in \sigma_{\mathrm{p}}(D)$ with eigenvector x_2^0.

Theorem 3'. $\lambda_0 \in \overline{W^2(\mathcal{A})}$ *corner* $\implies \lambda_0 \in \sigma(\mathcal{A}) \cup \sigma(A) \cup \sigma(D)$.

Theorem 4. $$\left\|(\mathcal{A} - \lambda)^{-1}\right\| \le \frac{\|\mathcal{A}\| + |\lambda|}{\operatorname{dist}\left(\lambda, W^2(\mathcal{A})\right)^2}, \quad \lambda \notin \overline{W^2(\mathcal{A})}.$$

3. Other properties [LMT'01, LMMT'01]

angle $< \pi$

Theorem 3. *Let* $\lambda_0 \in W^2(\mathcal{A})$ *be a corner,*

$$\det\left(\mathcal{A}_{x_1^0,x_2^0} - \lambda_0\right) = 0,$$

with $x_i^0 \in \mathcal{H}_i$, $\|x_i^0\| = 1$, *for* $i = 1, 2$. Then either

i) $\lambda_0 \in \sigma_{\rm p}(\mathcal{A})$ with eigenvector $\binom{x_1^0}{\gamma x_2^0}$ for some $\gamma \in \mathbb{C}$, or

ii) $\lambda_0 \in \sigma_{\rm p}(A)$ with eigenvector x_1^0, or

iii) $\lambda_0 \in \sigma_{\rm p}(D)$ with eigenvector x_2^0.

Theorem 3'. $\lambda_0 \in \overline{W^2(\mathcal{A})}$ *corner* $\implies \lambda_0 \in \sigma(\mathcal{A}) \cup \sigma(A) \cup \sigma(D)$.

Theorem 4. $$\left\|\left(\mathcal{A} - \lambda\right)^{-1}\right\| \leq \frac{\|\mathcal{A}\| + |\lambda|}{\text{dist}\left(\lambda, W^2(\mathcal{A})\right)^2}, \quad \lambda \notin \overline{W^2(\mathcal{A})}.$$

Theorem 5. $$\dim \mathcal{H}_2 \geq 2 \implies W(A) \subset W^2(\mathcal{A}),$$
$$\dim \mathcal{H}_1 \geq 2 \implies W(D) \subset W^2(\mathcal{A}).$$

4. Block diagonalization **[LMMT'01]**

Ass. dim $\mathcal{H}_1$, dim $\mathcal{H}_2 \geq 2$, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$ (w.l.o.g. $W(A) \subset \mathcal{F}_1$).

4. Block diagonalization [LMMT'01]

Ass. $\dim \mathcal{H}_1, \dim \mathcal{H}_2 \geq 2$, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$ (w.l.o.g. $W(A) \subset \mathcal{F}_1$).

$\Longrightarrow$ $\sigma(\mathcal{A}) = \sigma_1 \dot{\cup} \sigma_2$, $\mathcal{L}_1, \mathcal{L}_2$ corr. spectral subspaces.

4. Block diagonalization **[LMMT'01]**

Ass. dim $\mathcal{H}_1, \dim \mathcal{H}_2 \geq 2$, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$ (w.l.o.g. $W(A) \subset \mathcal{F}_1$).

$\Longrightarrow$ $\sigma(\mathcal{A}) = \sigma_1 \dot{\cup} \sigma_2$, $\mathcal{L}_1$, $\mathcal{L}_2$ corr. spectral subspaces.

Theorem 6.

i) *There exist* $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$, $K_2 \in L(\mathcal{H}_2, \mathcal{H}_1)$*:*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_1 \\ K_1 x_1 \end{pmatrix} : x_1 \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 x_2 \\ x_2 \end{pmatrix} : x_2 \in \mathcal{H}_2 \right\};$$

4. Block diagonalization [LMMT'01]

Ass. $\dim \mathcal{H}_1, \dim \mathcal{H}_2 \geq 2$, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot{\cup} \mathcal{F}_2$ (w.l.o.g. $W(A) \subset \mathcal{F}_1$).

$\Longrightarrow$ $\sigma(\mathcal{A}) = \sigma_1 \dot{\cup} \sigma_2$, $\mathcal{L}_1, \mathcal{L}_2$ corr. spectral subspaces.

Theorem 6.

i) *There exist* $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$, $K_2 \in L(\mathcal{H}_2, \mathcal{H}_1)$*:*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_1 \\ K_1 x_1 \end{pmatrix} : x_1 \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 x_2 \\ x_2 \end{pmatrix} : x_2 \in \mathcal{H}_2 \right\};$$

ii) K_1, K_2 are solutions of Riccati equations, e.g.

$$K_1 B K_1 + K_1 A - D K_1 - C = 0;$$

4. Block diagonalization [LMMT'01]

Ass. dim $\mathcal{H}_1$, dim $\mathcal{H}_2 \geq 2$, $\overline{W^2(\mathcal{A})} = \mathcal{F}_1 \dot\cup \mathcal{F}_2$ (w.l.o.g. $W(A) \subset \mathcal{F}_1$).

$\Longrightarrow$ $\sigma(\mathcal{A}) = \sigma_1 \dot\cup \sigma_2$, $\mathcal{L}_1$, $\mathcal{L}_2$ corr. spectral subspaces.

Theorem 6.

i) *There exist* $K_1 \in L(\mathcal{H}_1, \mathcal{H}_2)$, $K_2 \in L(\mathcal{H}_2, \mathcal{H}_1)$*:*

$$\mathcal{L}_1 = \left\{ \begin{pmatrix} x_1 \\ K_1 x_1 \end{pmatrix} : x_1 \in \mathcal{H}_1 \right\}, \quad \mathcal{L}_2 = \left\{ \begin{pmatrix} K_2 x_2 \\ x_2 \end{pmatrix} : x_2 \in \mathcal{H}_2 \right\};$$

ii) K_1, K_2 are solutions of Riccati equations, e.g.

$$K_1 B K_1 + K_1 A - D K_1 - C = 0;$$

iii) $\mathcal{A}$ is block diagonalizable:

$$\begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & K_2 \\ K_1 & I \end{pmatrix} = \begin{pmatrix} A + BK_1 & 0 \\ 0 & D + CK_2 \end{pmatrix}.$$

5. Block numerical range (BNR) [WT'03]

Now: $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$,

$$\mathcal{A} = \left(A_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad A_{ij} \in L\left(\mathcal{H}_j, \mathcal{H}_i\right).$$

5. Block numerical range (BNR) [WT'03]

Now: $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$,

$$\mathcal{A} = \left(A_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i).$$

Definition.

$$W^n(\mathcal{A}) := \Bigg\{ \lambda \in \mathbb{C} : \det \overbrace{\begin{pmatrix} (A_{11}x_1, x_1) - \lambda & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) - \lambda \end{pmatrix}}^{=: \mathcal{A}_{x_1,\ldots,x_n} - \lambda} = 0 \text{ for some } x_i \in \mathcal{H}_i,\ \|x_i\| = 1,\ i = 1, \ldots, n \Bigg\}$$

$$= \bigcup_{\|x_i\|=1} \sigma_{\mathrm{p}}\left(\mathcal{A}_{x_1,\ldots,x_n}\right)$$

is called (n-) block numerical range (BNR) of $\mathcal{A}$.

5. Block numerical range (BNR) [WT'03]

Now: $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ with Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$,

$$\mathcal{A} = \left(A_{ij}\right)_{i,j=1}^{n} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}, \quad A_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i).$$

Definition.

$$W^n(\mathcal{A}) := \Bigg\{ \lambda \in \mathbb{C} : \det \overbrace{\begin{pmatrix} (A_{11}x_1, x_1) - \lambda & \cdots & (A_{1n}x_n, x_1) \\ \vdots & \ddots & \vdots \\ (A_{n1}x_1, x_n) & \cdots & (A_{nn}x_n, x_n) - \lambda \end{pmatrix}}^{=: \mathcal{A}_{x_1,\ldots,x_n} - \lambda} = 0 \text{ for some } x_i \in \mathcal{H}_i,\ \|x_i\| = 1,\ i = 1, \ldots, n \Bigg\}$$

$$= \bigcup_{\|x_i\|=1} \sigma_{\mathrm{p}}\left(\mathcal{A}_{x_1,\ldots,x_n}\right)$$

is called (n-) block numerical range (BNR) of $\mathcal{A}$.

Note: $\mathcal{A} \in M_n(\mathbb{C}) \implies \sigma(\mathcal{A}) = W^n(\mathcal{A})$!

Theorem 7. *If* $\widehat{n} \geq n$ *and* $\mathcal{H} = \widehat{\mathcal{H}}_1 \times \cdots \times \widehat{\mathcal{H}}_{\widehat{n}}$ *is a "refinement" of* $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, *then*

$$W^{\widehat{n}}(\mathcal{A}) \subset W^n(\mathcal{A}).$$

Theorem 7. *If* $\widehat{n} \geq n$ *and* $\mathcal{H} = \widehat{\mathcal{H}}_1 \times \cdots \times \widehat{\mathcal{H}}_{\widehat{n}}$ *is a "refinement" of* $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, *then*

$$W^{\widehat{n}}(\mathcal{A}) \subset W^n(\mathcal{A}).$$

Corollary.
$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset \cdots \subset W^n(\mathcal{A}) \subset \cdots \subset W^2(\mathcal{A}) \subset W(\mathcal{A}),$$
$$\sigma(\mathcal{A}) \subset \cdots \subset \overline{W^n(\mathcal{A})} \subset \cdots \subset \overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})}.$$

Theorem 7. *If* $\widehat{n} \geq n$ *and* $\mathcal{H} = \widehat{\mathcal{H}}_1 \times \cdots \times \widehat{\mathcal{H}}_{\widehat{n}}$ *is a "refinement" of* $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$*, then*

$$W^{\widehat{n}}(\mathcal{A}) \subset W^n(\mathcal{A}).$$

Corollary.

$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset \cdots \subset W^n(\mathcal{A}) \subset \cdots \subset W^2(\mathcal{A}) \subset W(\mathcal{A}),$$
$$\sigma(\mathcal{A}) \subset \cdots \subset \overline{W^n(\mathcal{A})} \subset \cdots \subset \overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})}.$$

Theorem 4'. $\left\|\left(\mathcal{A} - \lambda\right)^{-1}\right\| \leq \dfrac{\left(\|\mathcal{A}\| + |\lambda|\right)^{n-1}}{\text{dist}\left(\lambda, W^n(\mathcal{A})\right)^n}, \quad \lambda \notin \overline{W^n(\mathcal{A})}.$

Theorem 7. *If* $\widehat{n} \geq n$ *and* $\mathcal{H} = \widehat{\mathcal{H}}_1 \times \cdots \times \widehat{\mathcal{H}}_{\widehat{n}}$ *is a "refinement" of* $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, *then*

$$W^{\widehat{n}}(\mathcal{A}) \subset W^n(\mathcal{A}).$$

Corollary.

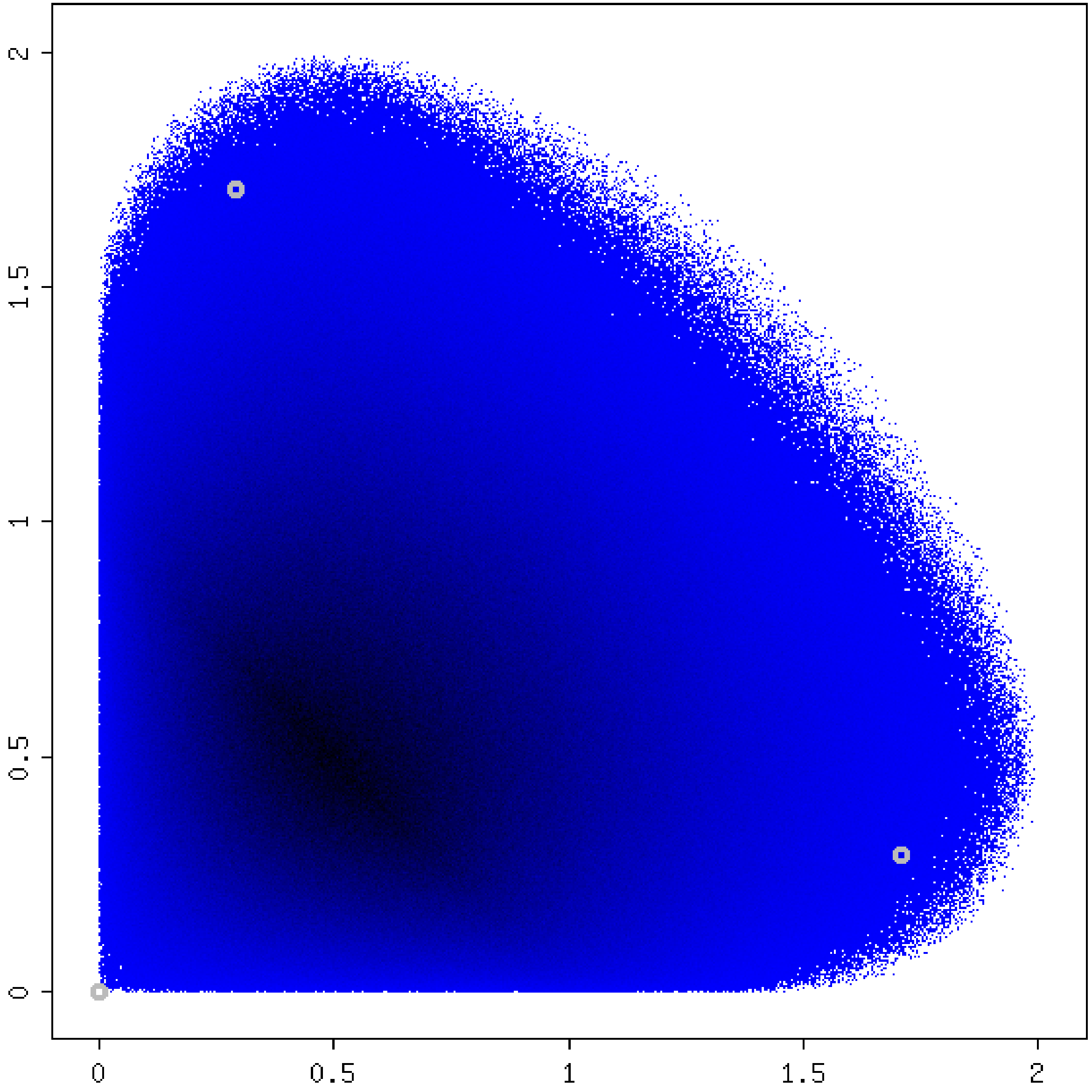
$$\sigma_{\mathrm{p}}(\mathcal{A}) \subset \cdots \subset W^n(\mathcal{A}) \subset \cdots \subset W^2(\mathcal{A}) \subset W(\mathcal{A}),$$
$$\sigma(\mathcal{A}) \subset \cdots \subset \overline{W^n(\mathcal{A})} \subset \cdots \subset \overline{W^2(\mathcal{A})} \subset \overline{W(\mathcal{A})}.$$

Theorem 4'.

$$\left\|(\mathcal{A}-\lambda)^{-1}\right\| \leq \frac{\left(\|\mathcal{A}\| + |\lambda|\right)^{n-1}}{\mathrm{dist}\left(\lambda, W^n(\mathcal{A})\right)^n}, \qquad \lambda \notin \overline{W^n(\mathcal{A})}.$$

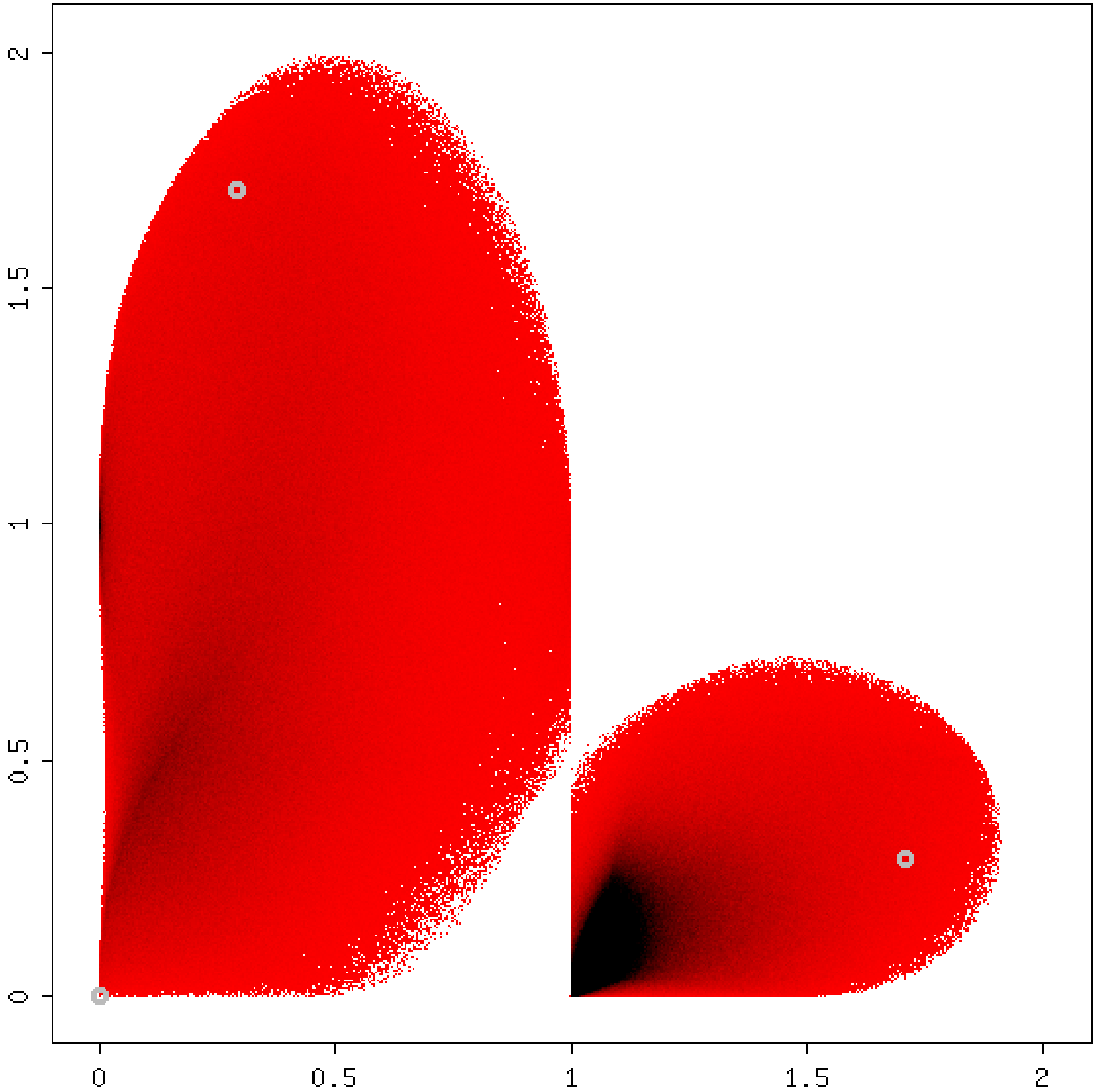
Theorem 6'. *If* $\dim \mathcal{H}_i \geq n$ *and* $\overline{W^n(\mathcal{A})} = \mathcal{F}_1 \,\dot\cup\, \ldots \,\dot\cup\, \mathcal{F}_n$ *with "strongly separated"* $\mathcal{F}_i$, *then there are* $K_{ij} \in L(\mathcal{H}_j, \mathcal{H}_i)$ *and* $Z_i \in L(\mathcal{H}_i)$ *with*

$$\begin{pmatrix} I & \ldots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{1n} & \cdots & I \end{pmatrix}^{-1} \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} I & \ldots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{1n} & \cdots & I \end{pmatrix} = \begin{pmatrix} Z_1 & & 0 \\ & \ddots & \\ 0 & & Z_n \end{pmatrix}.$$



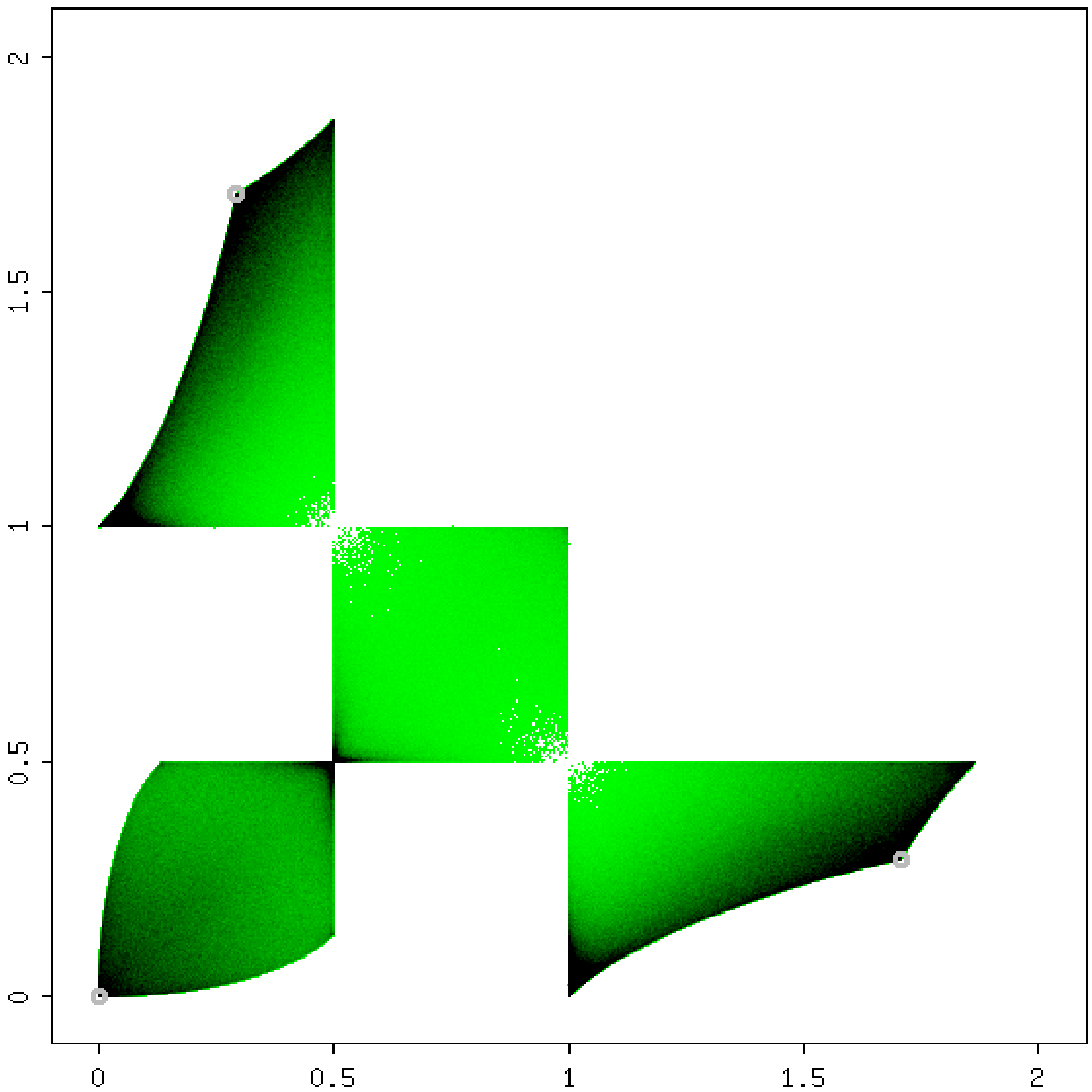
Numerical range of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 & 0 & i \\ 0 & 0 & 0 & i & i & 0 \\ 1 & 0 & 0 & i & 1+i & 0 \\ 0 & 1 & i & 0 & 0 & 1+i \end{pmatrix}$$



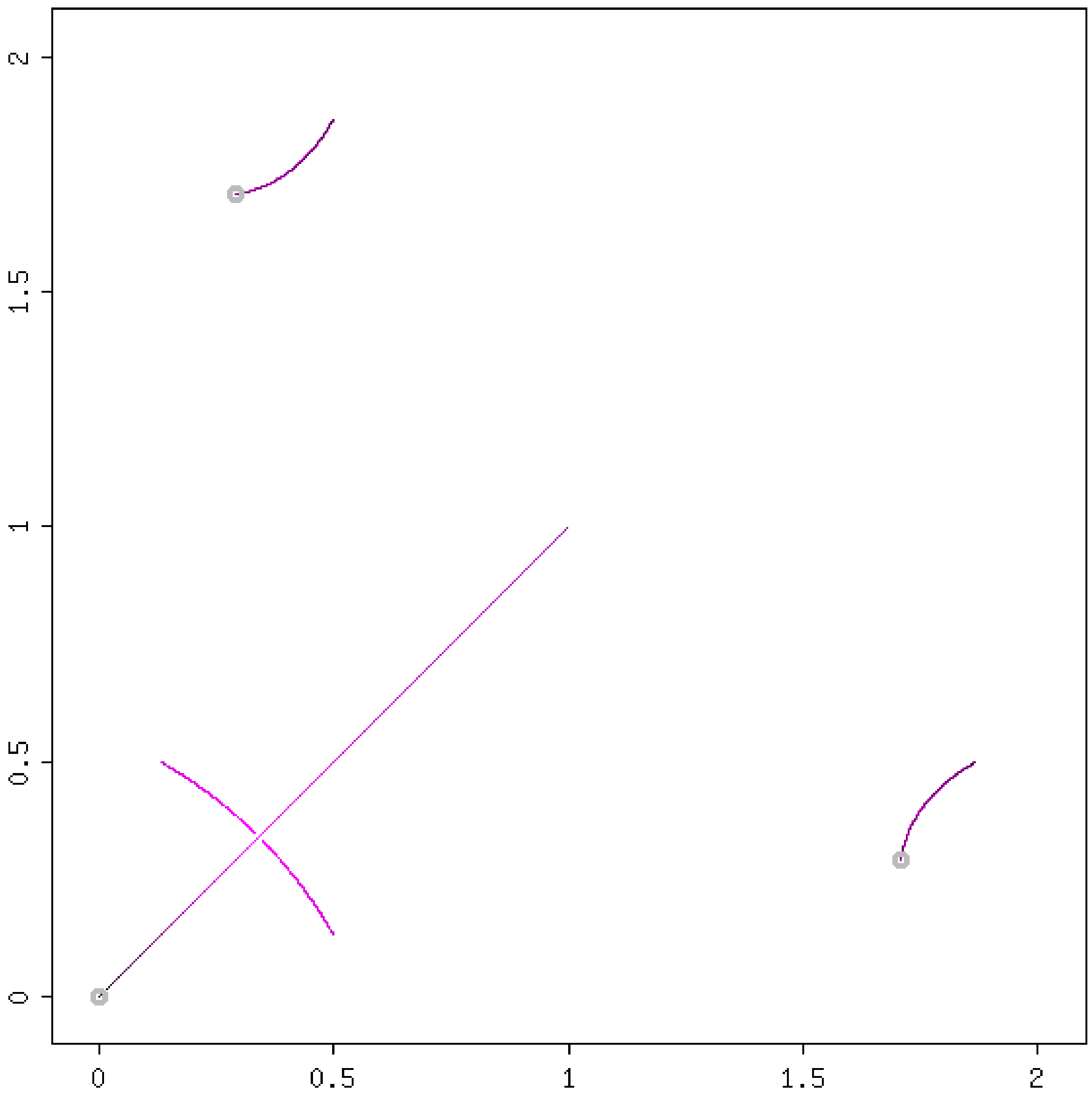
QNR of

$$\left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & i & 0 & 0 & i \\ 0 & 0 & 0 & i & i & 0 \\ 1 & 0 & 0 & i & 1+i & 0 \\ 0 & 1 & i & 0 & 0 & 1+i \end{array}\right)$$



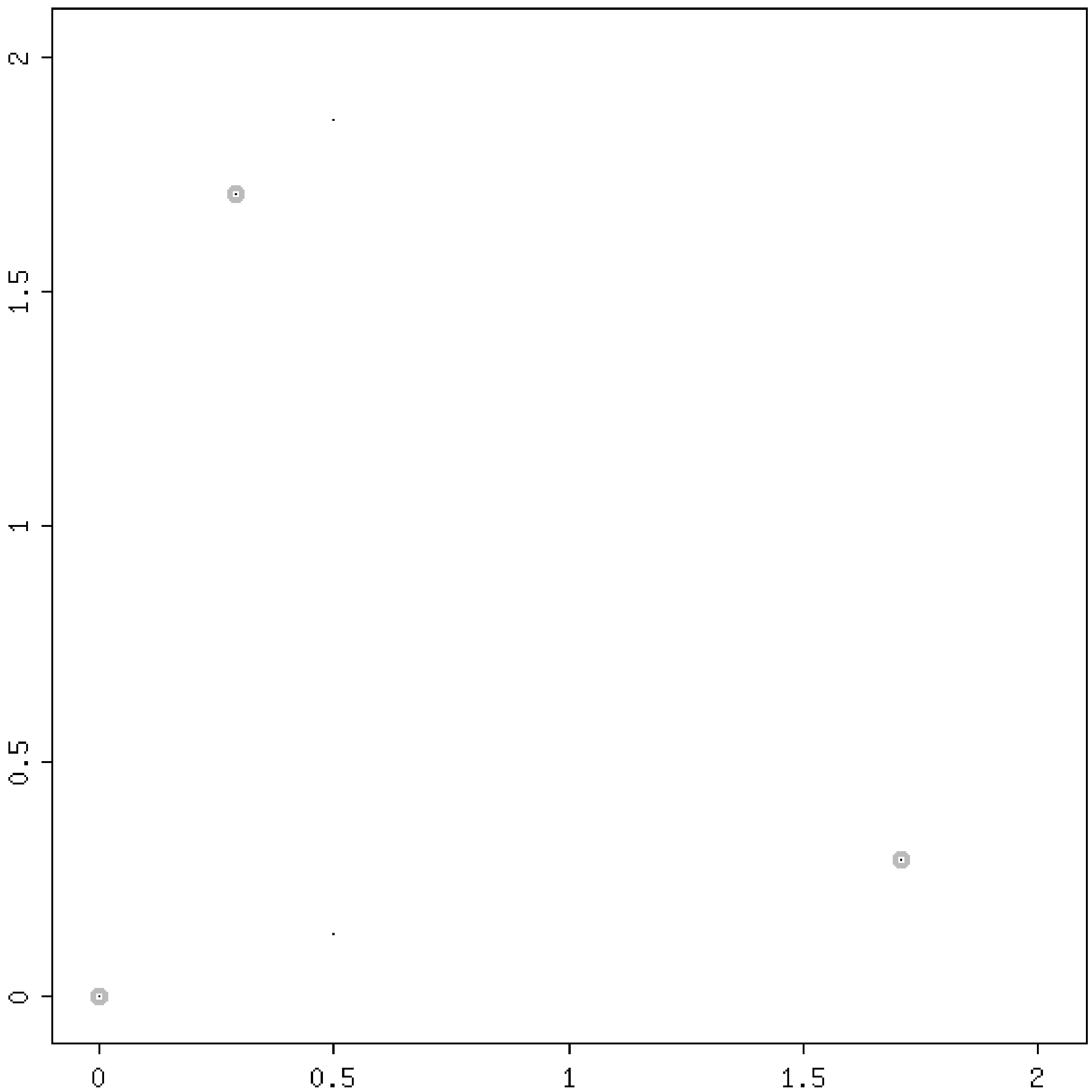
CNR of

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & i & 0 & 0 & i \\ 0 & 0 & 0 & i & i & 0 \\ \hline 1 & 0 & 0 & i & 1+i & 0 \\ 0 & 1 & i & 0 & 0 & 1+i \end{array}\right)$$



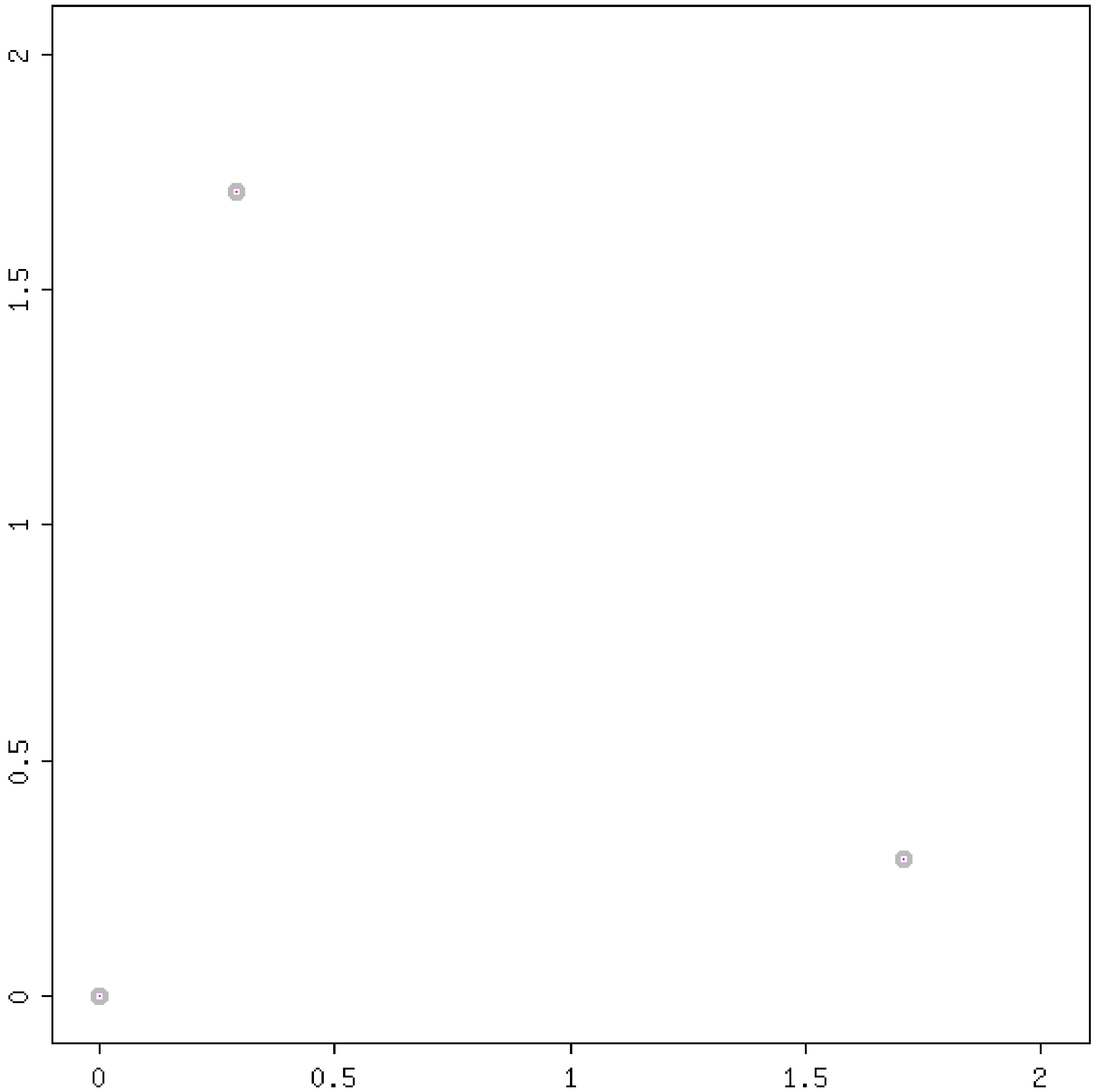
4-BNR of

$$\left(\begin{array}{cc|cc|c|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & \mathrm{i} & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 \\ \hline 1 & 0 & 0 & \mathrm{i} & 1+\mathrm{i} & 0 \\ \hline 0 & 1 & \mathrm{i} & 0 & 0 & 1+\mathrm{i} \end{array}\right)$$



5-BNR of

$$
\left(\begin{array}{cc|c|c|c|c}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & \mathrm{i} & 0 & 0 & \mathrm{i} \\
\hline
0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 \\
\hline
1 & 0 & 0 & \mathrm{i} & 1+\mathrm{i} & 0 \\
\hline
0 & 1 & \mathrm{i} & 0 & 0 & 1+\mathrm{i}
\end{array}\right)
$$

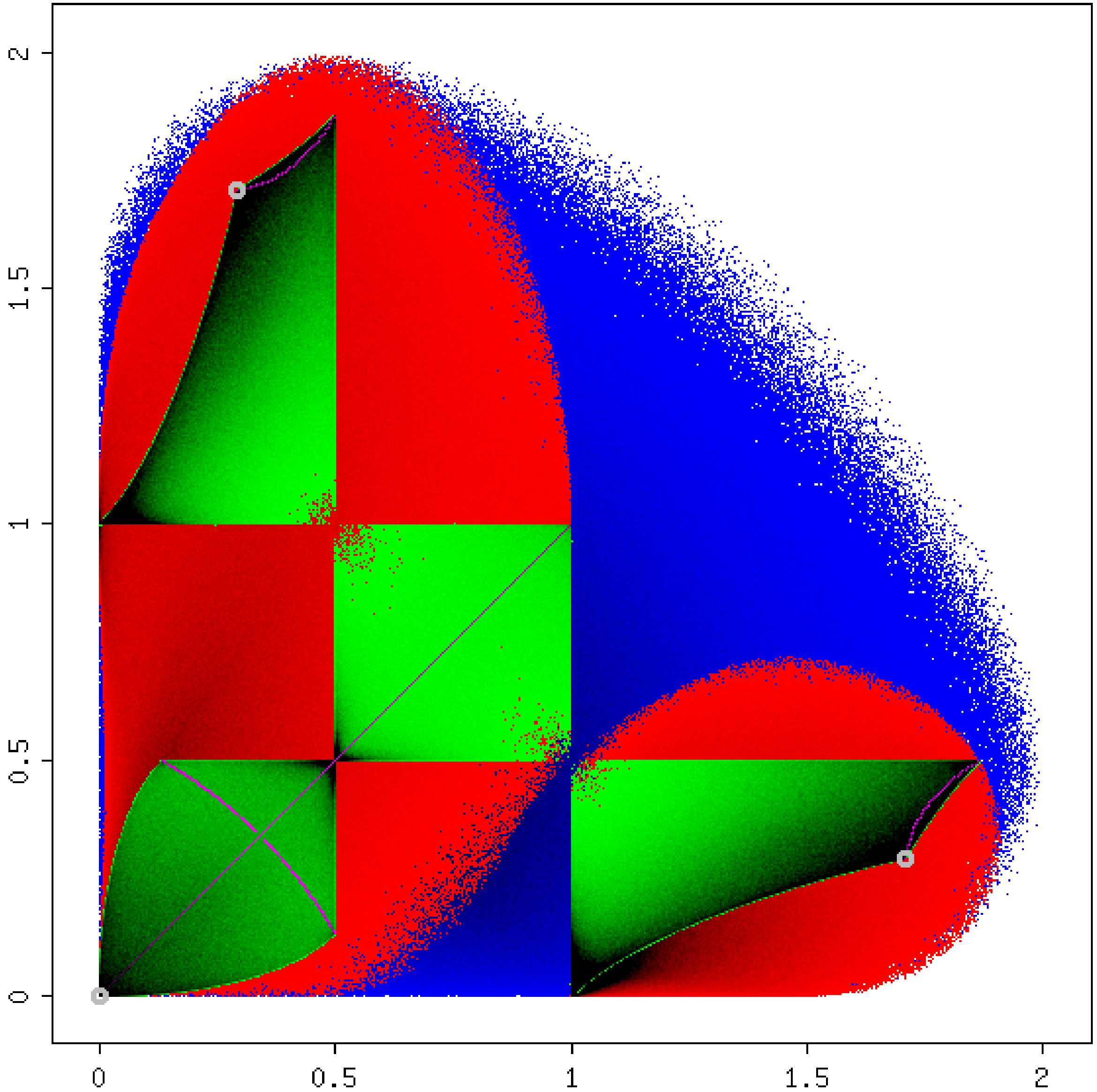


6-BNR of

$$\left(\begin{array}{c|c|c|c|c|c}
1 & 0 & 0 & 0 & 1 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 1 \\ \hline
0 & 0 & \mathrm{i} & 0 & 0 & \mathrm{i} \\ \hline
0 & 0 & 0 & \mathrm{i} & \mathrm{i} & 0 \\ \hline
1 & 0 & 0 & \mathrm{i} & 1+\mathrm{i} & 0 \\ \hline
0 & 1 & \mathrm{i} & 0 & 0 & 1+\mathrm{i}
\end{array}\right)$$

Numerical range, QNR, CNR, 4-BNR, . . . of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 & 0 & i \\ 0 & 0 & 0 & i & i & 0 \\ 1 & 0 & 0 & i & 1+i & 0 \\ 0 & 1 & i & 0 & 0 & 1+i \end{pmatrix}$$



Further results and work in progress [T'08]

- BNR of companion operators and NR of operator polynomials;
- Variational principles for eigenvalues in spectral gaps;
- QNR for unbounded block operator matrices;
- QNR and BNR for analytic block operator matrix functions like

$$\mathcal{A}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \times \mathcal{H}_2;$$

Open problems:

- Can a component of $W^2(\mathcal{A})$ have a hole?
- Does dimension condition imply "strongly separated"?
- ......

Further results and work in progress **[T'08]**

- BNR of companion operators and NR of operator polynomials;
- Variational principles for eigenvalues in spectral gaps;
- QNR for unbounded block operator matrices;
- QNR and BNR for analytic block operator matrix functions like

$$\mathcal{A}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \times \mathcal{H}_2;$$

Open problems:

- Can a component of $W^2(\mathcal{A})$ have a hole?
- Does dimension condition imply "strongly separated"?
- . . . . . .

Further results and work in progress **[T'08]**

- BNR of companion operators and NR of operator polynomials;
- Variational principles for eigenvalues in spectral gaps;
- QNR for unbounded block operator matrices;
- QNR and BNR for analytic block operator matrix functions like

$$\mathcal{A}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \times \mathcal{H}_2;$$

Open problems:

- Can a component of $W^2(\mathcal{A})$ have a hole?
- Does dimension condition imply "strongly separated"?
- ......