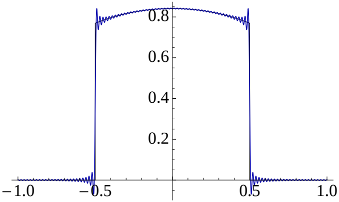
Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon

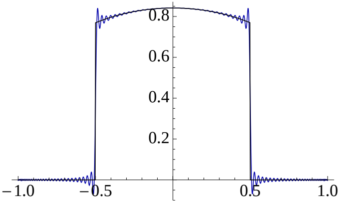
Ben Adcock
Department of Mathematics
Simon Fraser University
Canada

The Gibbs Phenomenon

Occurs in the expansion of a piecewise smooth function in an orthogonal series of smooth functions.



Fourier series



Chebyshev series

Graphs of $f(x) - f_n(x)$, where $n = 200$, $f(x) = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(\cos x)$ and $f_n(x)$ is the truncated Fourier/Chebyshev series of f.

Both poor local and poor global approximation:

- $\mathcal{O}(1)$ oscillations near each discontinuity.
- No uniform convergence of the approximation.

Notable Examples

Spectral methods for PDEs

- Spectral methods converge spectrally (or even exponentially) fast whenever the PDE has smooth (analytic) solution.
- Far less efficient for PDEs that develop discontinuities (shocks), e.g. hyperbolic conservation laws.

Image and signal processing

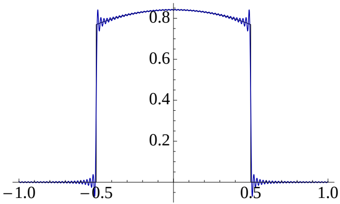
- Known as the ringing artifact.
- In particular, Magnetic Resonance Imaging (MRI).

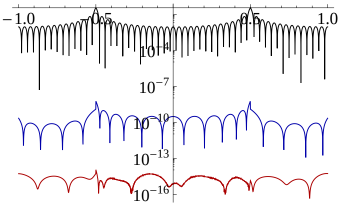
This leads naturally to the following question:

How can one recover high accuracy from the given expansion?

A New Method

A fundamentally new approach. Based on the interpretation of the Gibbs phenomenon as the result of a poor basis in which to represent the function f.





Left: Fourier series of $f(x) = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(\cos x)$. Right: Fourier series (black) and the reconstruction using $m = 25$ (blue) and $m = 50$ (red) Fourier samples.

- Using only 50 Fourier samples, we obtain $\approx$ 14 digits of accuracy.

In fact, this method is just one example of a general framework for solving the so-called sampling and reconstruction problem.

The Sampling and Reconstruction Problem

Suppose that we have access to the fixed samples of an object f, (e.g. a signal/image), with respect to some orthonormal basis:

$$\hat{f}_j = \langle f, \psi_j \rangle, \quad j = 1, 2, \ldots .$$

- The sampling scheme is typically specified by some physical device.
- E.g. Fourier samples in MRI.

Many physical signals/images are poorly represented in terms of the sampling basis $\{\psi_j\}_{j=1}^{\infty}$.

- i.e. $\hat{f}_j \to 0$ very slowly.

However, suppose we know that f can be better represented in a new basis $\{\phi_j\}_{j=1}^{\infty}$.

- i.e. $f = \sum_{j=1}^{\infty} \alpha_j \phi_j$ with $\alpha_j \to 0$ rapidly.

This leads to the sampling and reconstruction problem:

How can one recover f in terms of $\{\phi_j\}_{j=1}^{\infty}$ from its given samples?

The Sampling and Reconstruction Problem

Suppose that we have access to the fixed samples of an object f, (e.g. a signal/image), with respect to some orthonormal basis:

$$\hat{f}_j = \langle f, \psi_j \rangle, \quad j = 1, 2, \ldots.$$

- The sampling scheme is typically specified by some physical device.
- E.g. Fourier samples in MRI.

Many physical signals/images are poorly represented in terms of the sampling basis $\{\psi_j\}_{j=1}^{\infty}$.

- i.e. $\hat{f}_j \to 0$ very slowly.

However, suppose we know that f can be better represented in a new basis $\{\phi_j\}_{j=1}^{\infty}$.

- i.e. $f = \sum_{j=1}^{\infty} \alpha_j \phi_j$ with $\alpha_j \to 0$ rapidly.

This leads to the sampling and reconstruction problem:

How can one recover f in terms of $\{\phi_j\}_{j=1}^{\infty}$ from its given samples?

Generalised Sampling

A new method for the sampling and reconstruction problem.

Benefits include:

- Numerical stability.
- Linear, and easy to implement.
- Optimal in the sense that the accuracy of the reconstruction is predominantly determined by the reconstruction basis and not by the nature of the sampling.

Resolution of the Gibbs Phenomenon

The resolution of the Gibbs phenomenon is just one example of this general framework.

The resulting method possesses

- numerical stability.
- root-exponential convergence in the number of given samples m.
- exponential convergence in the number of degrees of freedom $n = \mathcal{O}\left(\sqrt{m}\right)$ in the final approximation.
- a computational complexity of $\mathcal{O}\left(nm\right)$.

Moreover, the method

- is optimally stable for this problem.
- often outperforms other methods in numerical examples.

The Importance of Numerical Stability

Numerical stability is vital to avoid large output errors, due to

- round-off error.
- noise in the samples: $\hat{f}_j \to \hat{f}_j + \epsilon_j$.
- sampling errors: e.g. jitter in MRI machines.
- model error: in practice, we compute with some perturbation $\tilde{f}$ of f, which may not be well represented in $\{\phi_j\}_{j=1}^{\infty}$, e.g. shock capturing.

Outline

Generalised Sampling

Resolution of the Gibbs Phenomenon

Operator-Theoretic Techniques

Hilbert Space Formulation

Let H be a separable Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

- Let $\psi_1, \psi_2, \ldots$ be sampling vectors that form an orthonormal basis of H.
- Let $\mathrm{T}_n \subseteq \mathrm{H}$ be a subspace of dimension n, the reconstruction space, and $\phi_1, \ldots, \phi_n$ a basis for T_n.

The Sampling and Reconstruction Problem

Given a subspace $\mathrm{T}_n \subseteq \mathrm{H}$ and the first m samples $\hat{f}_j = \langle f, \psi_j \rangle$, $j = 1, \ldots, m$, of $f \in \mathrm{H}$, compute a reconstruction $f_{n,m} \in \mathrm{T}_n$.

- Naturally we want $f_{n,m} \approx f$ to high accuracy.
- We also want numerical stability.

Key idea: allow the number of samples m to differ from the number of degrees of freedom n in $f_{n,m}$.

Hilbert Space Formulation

Let H be a separable Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

- Let $\psi_1, \psi_2, \ldots$ be sampling vectors that form an orthonormal basis of H.
- Let $\mathrm{T}_n \subseteq \mathrm{H}$ be a subspace of dimension n, the reconstruction space, and $\phi_1, \ldots, \phi_n$ a basis for T_n.

The Sampling and Reconstruction Problem

Given a subspace $\mathrm{T}_n \subseteq \mathrm{H}$ and the first m samples $\hat{f}_j = \langle f, \psi_j \rangle$, $j = 1, \ldots, m$, of $f \in \mathrm{H}$, compute a reconstruction $f_{n,m} \in \mathrm{T}_n$.

- Naturally we want $f_{n,m} \approx f$ to high accuracy.
- We also want numerical stability.

Key idea: allow the number of samples m to differ from the number of degrees of freedom n in $f_{n,m}$.

Best Possible Reconstruction and Generalised Sampling

The best (error minimising) approximation to f from T_n is the orthogonal projection $\mathcal{Q}_n f$.

- $\mathcal{Q}_n f$ is defined by the equations

$$\langle \mathcal{Q}_n f, \phi \rangle = \langle f, \phi \rangle, \ \forall \phi \in \mathrm{T}_n, \qquad \mathcal{Q}_n f \in \mathrm{T}_n. \tag{1}$$

- If we knew $\langle f, \phi_j \rangle$, $j = 1, \ldots, n$, then we could compute $\mathcal{Q}_n f$.
- However, we only have access to $\hat{f}_j$, $j = 1, \ldots, m$.

Instead, we let $\mathcal{P}_m : \mathrm{H} \to \mathrm{S}_m := \mathrm{span}\{\psi_1, \ldots, \psi_m\}$ by

$$\mathcal{P}_m g = \sum_{j=1}^{m} \langle g, \psi_j \rangle \psi_j, \quad g \in \mathrm{H},$$

and define $f_{n,m}$ by

$$\langle \mathcal{P}_m f_{n,m}, \phi \rangle = \langle \mathcal{P}_m f, \phi \rangle, \ \forall \phi \in \mathrm{T}_n, \qquad f_{n,m} \in \mathrm{T}_n. \tag{2}$$

Intuitive explanation: $\mathcal{P}_m \to \mathcal{I}$ strongly on H. Thus, for large m, (2) resembles (1), and hence $f_{n,m} \approx \mathcal{Q}_n f$.

Best Possible Reconstruction and Generalised Sampling

The best (error minimising) approximation to f from T_n is the orthogonal projection $\mathcal{Q}_n f$.

- $\mathcal{Q}_n f$ is defined by the equations

$$\langle \mathcal{Q}_n f, \phi \rangle = \langle f, \phi \rangle, \ \forall \phi \in \mathrm{T}_n, \qquad \mathcal{Q}_n f \in \mathrm{T}_n. \tag{1}$$

- If we knew $\langle f, \phi_j \rangle$, $j = 1, \ldots, n$, then we could compute $\mathcal{Q}_n f$.
- However, we only have access to $\hat{f}_j$, $j = 1, \ldots, m$.

Instead, we let $\mathcal{P}_m : \mathrm{H} \to \mathrm{S}_m := \mathrm{span}\{\psi_1, \ldots, \psi_m\}$ by

$$\mathcal{P}_m g = \sum_{j=1}^{m} \langle g, \psi_j \rangle \psi_j, \quad g \in \mathrm{H},$$

and define $f_{n,m}$ by

$$\langle \mathcal{P}_m f_{n,m}, \phi \rangle = \langle \mathcal{P}_m f, \phi \rangle, \ \forall \phi \in \mathrm{T}_n, \qquad f_{n,m} \in \mathrm{T}_n. \tag{2}$$

Intuitive explanation: $\mathcal{P}_m \to \mathcal{I}$ strongly on H. Thus, for large m, (2) resembles (1), and hence $f_{n,m} \approx \mathcal{Q}_n f$.

Analysis of Generalised Sampling

Let $C_{n,m} = \inf \{ \|\mathcal{P}_m \phi\| : \phi \in \mathrm{T}_n,\ \|\phi\| = 1 \}$.

- Key point: for fixed n, $C_{n,m} \to 1$ as $m \to \infty$.

Theorem (BA, Hansen)

For each $n \in \mathbb{N}$, there exists an $m_0 \in \mathbb{N}$ such that $f_{n,m}$ exists and is unique for all $m \geq m_0$, and satisfies the sharp bounds

$$\|f - \mathcal{Q}_n f\| \leq \|f - f_{n,m}\| \leq \frac{1}{C_{n,m}} \|f - \mathcal{Q}_n f\|.$$

Specifically, m_0 is the least m such that $C_{n,m} > 0$.

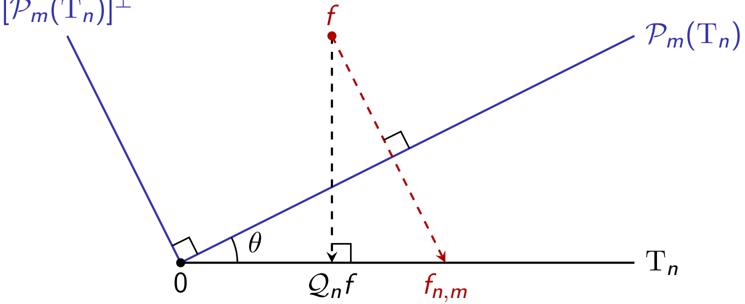
- B. Adcock and A. C. Hansen, *Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon*. Appl. Comput. Harmon. Anal. (to appear), 2011.
- B. Adcock and A. C. Hansen, *Sharp bounds and optimality for generalised sampling in Hilbert spaces*. In preparation, 2011.

Geometric Interpretation

The map $f \mapsto f_{n,m}$ is precisely the oblique projection onto T_n along $[\mathcal{P}_m(\mathrm{T}_n)]^\perp$. Moreover,

$$C_{n,m} = \cos\theta,$$

where θ is the angle between the subspaces T_n and $\mathcal{P}_m(\mathrm{T}_n)$.



- T_n and $\mathcal{P}_m(\mathrm{T}_n)$ cannot be near-perpendicular for large m. Hence $f_{n,m}$ is well-defined, and $f_{n,m} \approx \mathcal{Q}_n f$.

Numerical Implementation

If $f_{n,m} = \sum_{j=1}^{n} \alpha_j \phi_j$, then we solve the overdetermined least squares problem

$$U\alpha \approx \hat{f}, \qquad \text{where } \hat{f} = (\hat{f}_1, \ldots, \hat{f}_m),\ \alpha = (\alpha_1, \ldots, \alpha_n),$$

and $U \in \mathbb{C}^{m \times n}$ has $(j,k)^{\mathrm{th}}$ entry $\langle \phi_k, \psi_j \rangle$.

The condition number $\kappa(U)$ determines numerical stability:

Lemma (BA, Hansen)

If A is the Gram matrix for the vectors $\{\phi_1, \ldots, \phi_n\}$ then

$$\kappa(U) \leq \frac{1}{C_{n,m}} \sqrt{\kappa(A)}.$$

- If $\{\phi_1, \ldots, \phi_n\}$ are orthonormal, then $A = I$, and hence stability.
- If m is also chosen so that $(C_{n,m})^{-1}$ is bounded, then the computational cost in forming $f_{n,m}$ is $\mathcal{O}(nm)$.

Numerical Implementation

If $f_{n,m} = \sum_{j=1}^{n} \alpha_j \phi_j$, then we solve the overdetermined least squares problem

$$U\alpha \approx \hat{f}, \qquad \text{where } \hat{f} = (\hat{f}_1, \ldots, \hat{f}_m),\ \alpha = (\alpha_1, \ldots, \alpha_n),$$

and $U \in \mathbb{C}^{m \times n}$ has $(j,k)^{\mathrm{th}}$ entry $\langle \phi_k, \psi_j \rangle$.

The condition number $\kappa(U)$ determines numerical stability:

Lemma (BA, Hansen)

If A is the Gram matrix for the vectors $\{\phi_1, \ldots, \phi_n\}$ then

$$\kappa(U) \leq \frac{1}{C_{n,m}} \sqrt{\kappa(A)}.$$

- If $\{\phi_1, \ldots, \phi_n\}$ are orthonormal, then $A = I$, and hence stability.
- If m is also chosen so that $(C_{n,m})^{-1}$ is bounded, then the computational cost in forming $f_{n,m}$ is $\mathcal{O}(nm)$.

The Stable Sampling Rate

For a given n, we need m to be sufficiently large. To quantify this, we define the stable sampling rate

$$\Theta(n;\theta) = \min\left\{m \in \mathbb{N} : C_{n,m} > \theta\right\}, \quad \theta \in (0,1).$$

For given n, setting $m \geq \Theta(n;\theta)$ ensures

- Existence and uniqueness of $f_{n,m}$.
- Stability up to the choice of reconstruction basis: $\kappa(U) \leq \frac{1}{\theta}\sqrt{\kappa(A)}$.
- Quasi-optimality: $\|f - f_{n,m}\| \leq \frac{1}{\theta}\|f - \mathcal{Q}_n f\|$.

The stable sampling rate is completely computable. Indeed,

Lemma (BA, Hansen)

$C_{n,m}$ is precisely the minimum singular value of U.

- In many important cases, one can also derive analytic bounds.

This is a fundamentally new viewpoint to sampling.

The Stable Sampling Rate

For a given n, we need m to be sufficiently large. To quantify this, we define the stable sampling rate

$$\Theta(n;\theta) = \min\{m \in \mathbb{N} : C_{n,m} > \theta\}, \quad \theta \in (0,1).$$

For given n, setting $m \geq \Theta(n;\theta)$ ensures

- Existence and uniqueness of $f_{n,m}$.
- Stability up to the choice of reconstruction basis: $\kappa(U) \leq \frac{1}{\theta}\sqrt{\kappa(A)}$.
- Quasi-optimality: $\|f - f_{n,m}\| \leq \frac{1}{\theta}\|f - \mathcal{Q}_n f\|$.

The stable sampling rate is completely computable. Indeed,

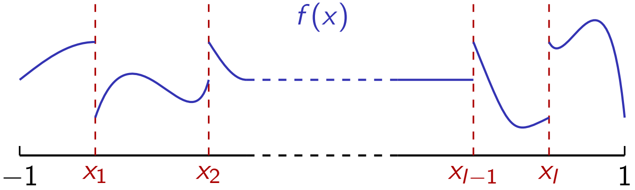
Lemma (BA, Hansen)

$C_{n,m}$ is precisely the minimum singular value of U.

- In many important cases, one can also derive analytic bounds.

This is a fundamentally new viewpoint to sampling.

Locating Discontinuities



Given $\{\hat{f}_j\}$ (Fourier/orthogonal polynomial coefficients) it is first necessary to locate $x_1, \ldots, x_I$ to high accuracy.

- Known as edge detection.
- E.g. concentration kernels (Gelb, Tadmor, Tanner,...).
- Typically nonlinear.

We consider the reconstruction step, and assume that $x_1, \ldots, x_I$ have already been computed.

- However, edge detection is an important source of errors.
- Any reconstruction method must be robust w.r.t. such errors.

Methods for Reconstruction

Filters/Mollifiers (Fejér, Vandeven, Gottlieb, Tadmor,...)

- Stable, and do not require singularity location step.
- However, high accuracy only in regions away from discontinuities.
- Based on interpreting the Gibbs phenomenon as noise polluting the coefficients $\hat{f}_j$.

Spectral reprojection (Gottlieb, Shu, Gelb, Tanner,...)

- Exponentially convergent throughout the domain (in many cases).
- Widely used, but issues with both stability and convergence. Careful selection of parameters required to avoid the Runge phenomenon.
- Based on the existence of a Gibbs complementary basis.

Inverse/Extrapolation methods (Boyd, Eckhoff, Fornberg,...)

- Exponentially convergent (in some cases), but typically exponentially ill-conditioned. Also susceptible to the Runge phenomenon.
- Based on the particular structure of the Gibbs phenomenon.

The Generalised Sampling Approach

Based on a different viewpoint: the Gibbs phenomenon is the result of a poor basis in which to represent f.

Since f is piecewise analytic, its orthogonal projection $\mathcal{Q}_n f$ onto

$$\mathrm{T}_n = \left\{ \phi : \phi|_{[x_r, x_{r+1})} \in \mathbb{P}_{n_r},\ r = 0, \ldots, l \right\}, \quad n = (n_0, \ldots, n_l) \in \mathbb{N}^{l+1},$$

converges exponentially fast as $n_0, \ldots, n_l \to \infty$.

We can now apply generalised sampling, and expect exponential convergence and stability, provided $m \geq \Theta(n; \theta)$.

Key questions:

1. How do we select a basis $\phi_1, \ldots, \phi_{n^*}$ ($n^* = n_0 + \ldots + n_l$) for T_n?
2. How does the stable sampling rate $\Theta(n; \theta)$ behave?

Choice of Piecewise Polynomial Basis

If A is the Gram matrix for $\{\phi_1, \ldots, \phi_{n^*}\}$, recall that

$$\kappa(U) \leq \frac{1}{\theta}\sqrt{\kappa(A)}.$$

Consider the Fourier case with no jumps.

- Legendre polynomials: $A = I$ – perfect conditioning.
- Conversely, Chebyshev polynomials yield $\kappa(A) = \mathcal{O}(n)$.
- In general, if $\{\phi_1, \ldots, \phi_n\}$ are Gegenbauer polynomials with parameter $\lambda > -\frac{1}{2}$, then $\kappa(A) = \mathcal{O}\left(n^{|2\lambda-1|}\right)$.

Perfect conditioning can be achieved with Gegenbauer polynomials by specifying $f_{n,m}$ as follows:

$$\langle \mathcal{P}_m f_{n,m}, \mathcal{P}_m \phi \rangle_\lambda = \langle \mathcal{P}_m f, \mathcal{P}_m \phi \rangle_\lambda, \ \forall \phi \in \mathrm{T}_n, \qquad f_{n,m} \in \mathrm{T}_n,$$

where $\langle g, h \rangle_\lambda = \int_{-1}^{1} g(x)\overline{h(x)}(1-x^2)^{\lambda - \frac{1}{2}} \, \mathrm{d}x$.

- This is based on a modification of generalised sampling that allows one to sample and reconstruct in different Hilbert spaces.

The Stable Sampling Rate

Theorem (BA, Hansen)

The stable sampling rate $\Theta(n;\theta)$ *satisfies*

$$\Theta(n;\theta) = \mathcal{O}\left(n_1^2,\dots,n_d^2\right).$$

In the case of Fourier samples, if $c_r = \frac{1}{2}(x_{r+1}-x_r)$ *then*

$$\Theta(n;\theta) \leq \left\lceil \frac{1}{2} + \frac{2(\pi-2)}{\pi^2(1-\theta)} \sum_{r=0}^{l} \frac{n_r^2}{c_r} \right\rceil,$$

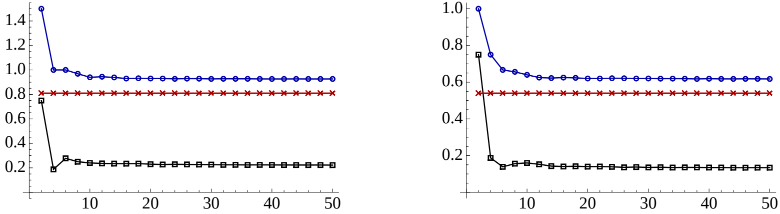
$$\Theta(n;\theta) \leq \frac{4}{\pi^2(1-\theta)} \sum_{r=0}^{l} \frac{n_r^2}{c_r} + \mathcal{O}\left(1\right), \quad n_0,\dots,n_l \to \infty.$$

Thus, $m = \mathcal{O}\left(n_0^2,\dots,n_l^2\right)$ for stable, quasi-optimal recovery.
Hence root-exponential convergence in m.

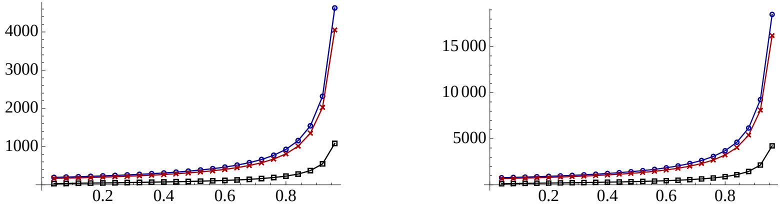
- B. Adcock and A. C. Hansen, *Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon*. Appl. Comput. Harmon. Anal. (to appear), 2010.

Comparison of Bounds

These bounds (in blue and red) are also reasonably sharp:



The quantity $n^{-2}\Theta(n;\theta)$ against n for $\theta = \frac{1}{2}$ (left) and $\theta = \frac{1}{4}$ (right).



The quantity $\Theta(n;\theta)$ against θ for $n = 20$ (left) and $n = 40$ (right).

Is the scaling $m = \mathcal{O}\left(n^2\right)$ optimal?

Theorem (Platte, Trefethen, Kuijlaars)

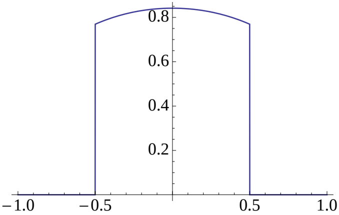
Under a number of assumptions, any stable method (linear or nonlinear) for recovering an analytic function f from its values at m equispaced nodes in $[-1,1]$ can converge at best root-exponentially fast in m. In fact, any method with a convergence rate of order ρ^{-m^τ} for some $\tau \in (\frac{1}{2}, 1]$ and $\rho > 1$ must have a condition number of order $C^{m^{2\tau-1}}$ for some $C > 1$.

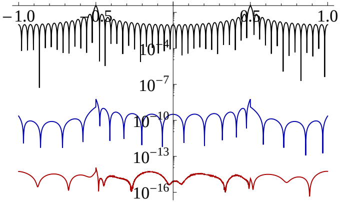
- The proof is based on certain extremal behaviour of polynomials (Schönhage, Coppersmith & Rivlin, Rakhmanov,...).

This result also extends to reconstructions from Fourier samples (a continuous analogue). Thus, generalised sampling is an optimal stable method.

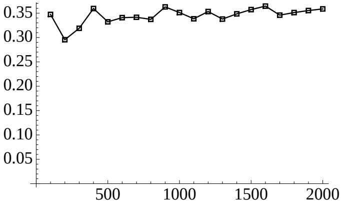
- B. Adcock and A. C. Hansen, *Sharp bounds and optimality for generalised sampling in Hilbert spaces*. In preparation, 2011.

Numerical Example I: Fourier Samples



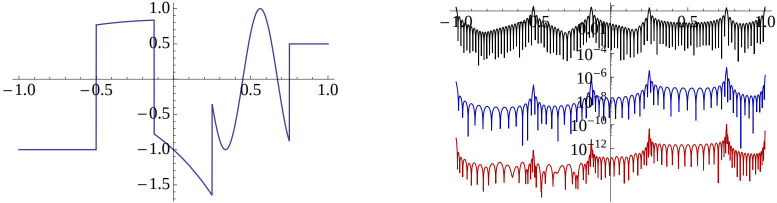


Left: $f(x) = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(\cos x)$. Right: Fourier series (black), generalised sampling with $m = 25$, $n_0 = n_2 = 5$, $n_1 = 10$ (blue) and $m = 50$, $n_0 = n_2 = 7$, $n_1 = 14$ (red).

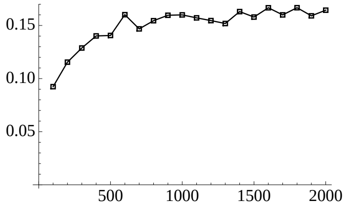


The quantity $C_{n,m}$ against m, where $n_0 = n_2 = \lceil\sqrt{m}\rceil$, $n_1 = 2n_0$.

Numerical Example II: Fourier Samples

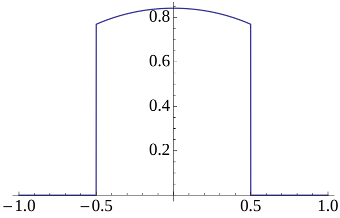


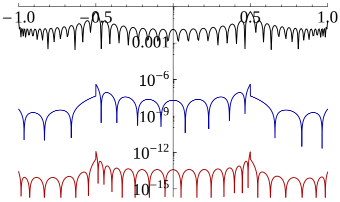
Left: $f(x)$. Right: Fourier series (black), generalised sampling (blue) with $m = 100$, $n_0 = \ldots = n_4 = 13$ (blue) and $m = 200$, $n_0 = \ldots = n_4 = 18$ (red).



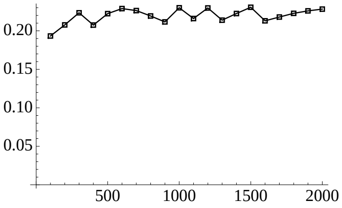
The quantity $C_{n,m}$ against m, where $n_0 = \ldots = n_4 = \lceil \sqrt{\frac{3}{2} m} \rceil$.

Numerical Example III: Legendre Polynomial Samples





Left: $f(x) = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(\cos x)$. Right: Legendre polynomial expansion (black), generalised sampling with $m = 25$, $n_0 = n_2 = 5$, $n_1 = 10$ (blue) and $m = 50$, $n_0 = n_2 = 7$, $n_1 = 14$ (red).

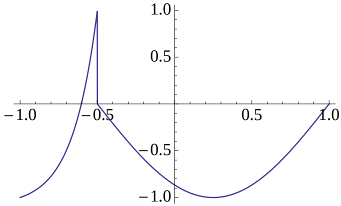


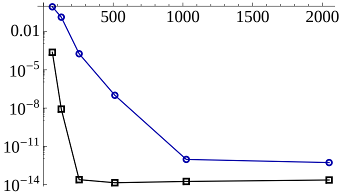
The quantity $C_{n,m}$ against m, where $n_0 = n_2 = \lceil \sqrt{m} \rceil$, $n_1 = 2n_0$.

Numerical Comparison

Consider the function

$$f(x) = \begin{cases} \frac{2e^{2\pi(x+1)} - 1 - e^{\pi}}{e^{\pi} - 1} & x \in [-1, -\frac{1}{2}) \\ -\sin(\frac{2\pi x}{3} + \frac{\pi}{3}) & x \in [-\frac{1}{2}, 1] \end{cases}$$





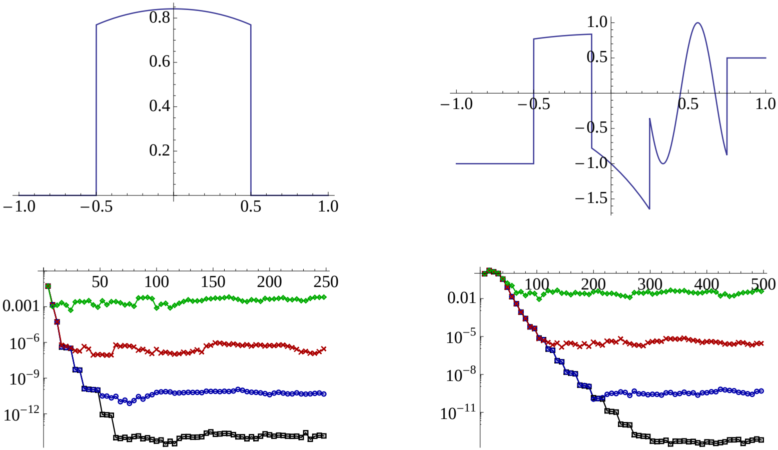
Error against m

	(a)	(b)
cost	$\mathcal{O}(m^{\frac{3}{2}})$	$\mathcal{O}(m^2)$
storage	$\mathcal{O}(m^{\frac{1}{2}})$	$\mathcal{O}(m)$

Cost and storage

Comparison of (a) generalised sampling (black) and (b) spectral reprojection (blue).

Robustness I: Noise

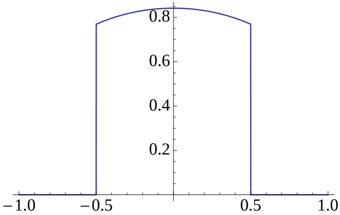


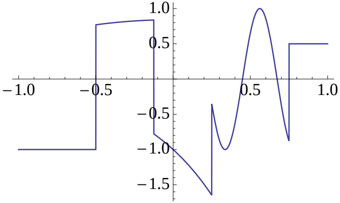
$n_0 = n_2 = \lceil \sqrt{m} \rceil, n_1 = 2n_0$

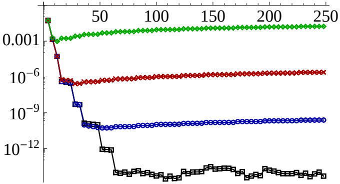
$n_0 = \ldots = n_4 = \lceil \sqrt{\frac{3}{2} m} \rceil$

Top row: $f(x)$. Bottom row: the error $\|f - f_{n,m}\|$ against m with noise at amplitudes $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

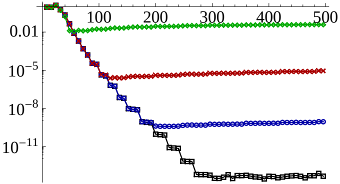
Robustness II: Shock Capturing Errors







$n_0 = n_2 = \lceil \sqrt{m} \rceil, n_1 = 2n_0$



$n_0 = \ldots = n_4 = \lceil \sqrt{\frac{3}{2} m} \rceil$

Top row: $f(x)$. Bottom row: the error $\|f - f_{n,m}\|$ against m with shock capturing errors of magnitude $\epsilon = 0, 10^{-12}, 10^{-8}, 10^{-4}$.

- It can be shown that there is at worst linear drift in $n = \sqrt{m}$.

Generalised Sampling

Resolution of the Gibbs Phenomenon

Operator-Theoretic Techniques

Infinite-dimensional Formulation of Reconstruction

Let $\hat{f} = \{\hat{f}_j\}_{j=1}^{\infty}$ be given and suppose that

$$f = \sum_{j=1}^{\infty} \alpha_j \phi_j.$$

The coefficients $\alpha = \{\alpha_j\}_{j=1}^{\infty}$ can be recovered exactly from $\hat{f}$ via

$$U\alpha = \hat{f},$$

where

$$U = \begin{pmatrix} \langle \phi_1, \psi_1 \rangle & \langle \phi_2, \psi_1 \rangle & \cdots \\ \langle \phi_1, \psi_2 \rangle & \langle \phi_2, \psi_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $U : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is bounded, invertible and unitary.

How do we discretise the equations $U\alpha = \hat{f}$?

- B. Adcock and A. C. Hansen, *A generalized sampling theorem for reconstructions in arbitrary bases*. Submitted, 2010.

Finite Sections of Infinite Matrices

Take the $n \times n$ leading submatrix of U, and solve

$$P_n U P_n \alpha^{[n]} = P_n \hat{f},$$

where $P_n : l^2(\mathbb{N}) \to \mathrm{span}\{e_1, \dots, e_n\}$ is the orthogonal projection.

Finite sections are very widely used (equivalent to consistent reconstructions in sampling applications). However,

1. $P_n U P_n$ need not be invertible for any n.
2. Even if $(P_n U P_n)^{-1}$ exists, $\|(P_n U P_n)^{-1}\|$ may blow up as $n \to \infty$.
3. If $U\alpha = \hat{f}$ and $P_n U P_n \alpha^{[n]} = P_n \hat{f}$, then $\alpha^{[n]} \nrightarrow \alpha$ in general.

E.g. let $\psi_j(x) = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i} j \pi x}$, $\phi_j(x) = (j + \frac{1}{2})^{\frac{1}{2}} P_j(x)$ and $f(x) = \frac{x}{1+16x^2}$.

n	25	50	100	200
$\|(P_n U P_n)^{-1}\|$	7.64e2	6.59e7	3.97e16	2.05e34
$\|\alpha - \alpha^{[n]}\|$	1.31e-1	4.56e0	6.15e2	8.74e3
$\|\alpha - P_n \alpha\|$	1.10e-3	2.91e-6	1.23e-11	6.73e-23

Uneven Sections

Replace the $n \times n$ square section by an $m \times n$ uneven section $P_m U P_n$, and solve

$$P_n U^* P_m U P_n \alpha^{[n,m]} = P_n U^* P_m \hat{f}.$$

Let $f_{n,m} = \sum_{j=1}^{m} \alpha_j^{[n,m]} \phi_j$. This is precisely generalised sampling.

Intuitive explanation: For large m (and fixed n) we have

$$(P_m U P_n)^* P_m U P_n \approx P_n I P_n,$$

where $I : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the identity.

- Hence $P_m U P_n$ inherits the unitary structure of U.

E.g. let ψ_j, ϕ_j and f be as before, and set $n = \lceil \sqrt{8m} \rceil$:

n	25	50	100	200	400
$\|(P_m U P_n)^\dagger\|$	4.3e0	4.86e0	4.53e0	4.63e0	4.38e0
$\|\alpha - \alpha^{[n,m]}\|$	1.01e-2	2.20e-3	2.50e-4	1.12e-5	4.12e-7

Other Applications

1. Compressed sensing
   - Suppose that f is sparse in $\{\phi_j\}_{j=1}^{\infty}$.
   - Form the uneven section $P_m U P_n$, where m represents the range from which samples are drawn, and subsample randomly from its rows.
   - Allows one to extend finite-dimensional compressed sensing techniques to infinite-dimensional problems.

2. Solving linear systems in infinite dimensions
   - Suppose that $Tx = y$, where $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is bounded, but not necessarily invertible, and consider $\inf\{\|z\|_{l^p(\mathbb{N})} : Tz = y\}$, $p \geq 1$.
   - Replace this with $\inf\{\|z\|_{l^p(\mathbb{N})} : P_n T P_k z = y\}$ where $k > n$.

3. Ill-posed problems

4. Computing spectra and pseudospectra

Conclusions

The sampling and reconstruction problem can be viewed as a question of how to discretise certain infinite-dimensional operators.

- Careful selection of two discretisation parameters leads to structure preservation and, in turn, good numerical behaviour.

The result is a fundamentally new approach to sampling and reconstruction, with numerical stability playing a central role.

- The key idea is the (completely computable) stable sampling rate.
- Both sharp bounds and a geometric interpretation for the reconstruction.

The application to orthogonal series leads to a new interpretation and method for the Gibbs phenomenon and its removal.

- Yields a simple, effective and optimally stable method.