The Painlevé Paradox

slip and stick with impact as a hybrid system

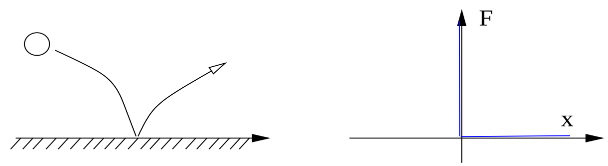
Alan Champneys, Arne Nordmark (KTH) & Harry Dankowicz (UIUC)

Engineering Mathematics Dept, University of Bristol

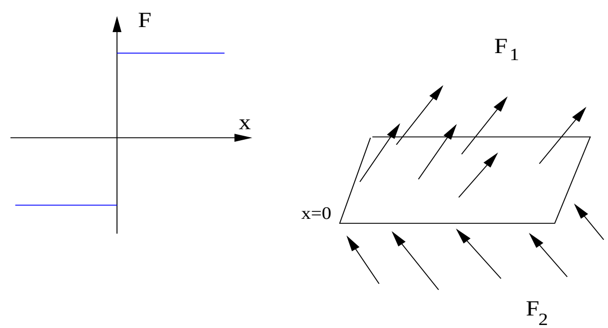
Motivation

di Bernardo, Budd, C. & Kowalczyk (2008) Springer book
General theory of piecewise smooth hybrid systems, e.g.

- Impacting systems e.g. bouncing ball



- Filippov systems e.g. Coulomb friction oscillators



and study via discontinuity-induced bifurcation
but what if have Filippov + Impact?

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- 1. Lagrangian mechanics with impact and friction

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- 2. Unfolding the impact phase
  - impact law via new time-scaling
  - discontinuity induced bifurcation

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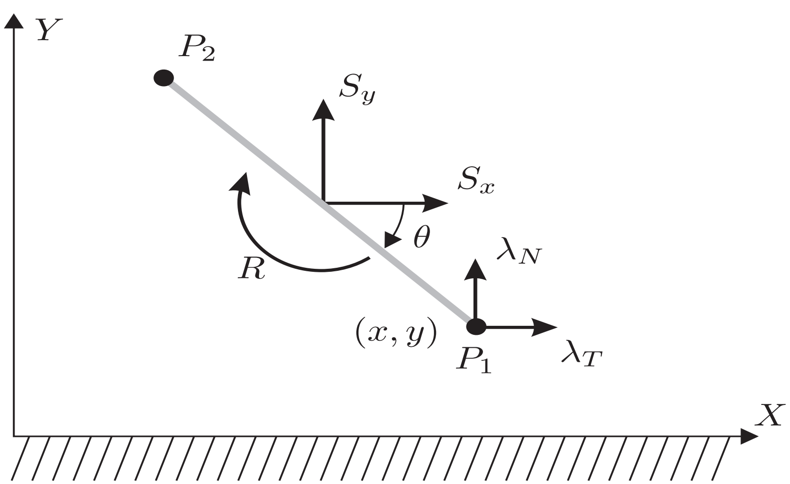
- 1. Lagrangian mechanics with impact and friction
- 2. Unfolding the impact phase
  - impact law via new time-scaling
  - discontinuity induced bifurcation
- 3. Painlevé paradoxes
  - unfolding via smoothing and taking the limit
  - a. to impact or lift off at end of stick?
  - b. to transition to stick or to reverse chatter?
- 4. Conclusion

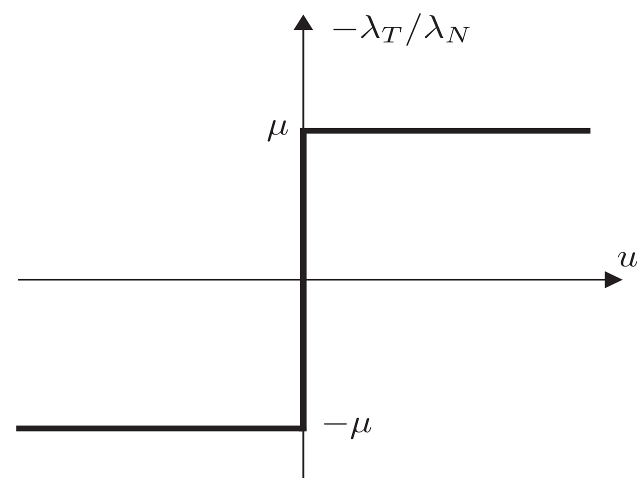
1. A hybrid Lagrangian systems

- $q \in \mathbb{R}^n$, with rigid contact in 2D + Coulomb friction

$$M(q,t)\,\ddot{q} = f(q,\dot{q},t) + \lambda_T c_u^T(q,t) + \lambda_N c_v^T(q,t),$$

- Scalar constraint $y \geq 0$, $y \in \mathbb{R}$ normal distance; $\lambda_N \geq 0$, $\lambda_T \in \mathbb{R}$ normal and tangential forces;
- Coulomb friction, $|\lambda_T| \leq \mu\lambda_N$, $\lambda_T = -\text{sign}(u)\mu\lambda_N$ if $u \neq 0$
- e.g. rod & table Painlevé 1905, Brogliato et al.





- Project Lagrangian onto u and v directions:

$$\dot{u} = a(q,\dot{q},t) + \lambda_T A(q,t) + \lambda_N B(q,t)\,,$$
$$\dot{v} = b(q,\dot{q},t) + \lambda_T B(q,t) + \lambda_N C(q,t)\,,$$

$A = c_u \cdot M^{-1} \cdot c_u^T,\ B = c_u \cdot M^{-1} \cdot c_v^T, \quad C = c_v \cdot M^{-1} \cdot c_v^T,$

- positive definite $M \Rightarrow A > 0\ C > 0$, $AC - B^2 > 0$
- special case $B = 0 \Rightarrow$ “independent” normal and tangential motion $\Rightarrow$ can use Newtonian restitution $v \to -rv$ at impact (well posed)
- what if $B \neq 0$?, e.g. for rod example ($l = 2$, $m = 2$): $A = 1 + 3\sin^2\theta, B = 3\sin 2\theta, C = 1 + 3\cos^2\theta$

4 modes of sustained motion

free flight: $y > 0$. No contact forces:

$$(\lambda_T, \lambda_N) = (0, 0).$$

positive/negative slip: $y = 0$, $v = 0$, $\lambda_N > 0$, $u \neq 0$. Full friction $\lambda_T = -\text{sign}(u)\mu\lambda_N$.

$$(\lambda_T, \lambda_N) = \frac{b}{C - \text{sign}(u)\mu B}(\text{sign}(u)\mu, -1).$$

stick: $y = 0$, $v = 0$, $\lambda_N > 0$, $u = 0$, $|\lambda_T| < \mu\lambda_N$.

$$(\lambda_T, \lambda_N) = \frac{1}{AC - B^2}(bB - aC, aB - bA)$$

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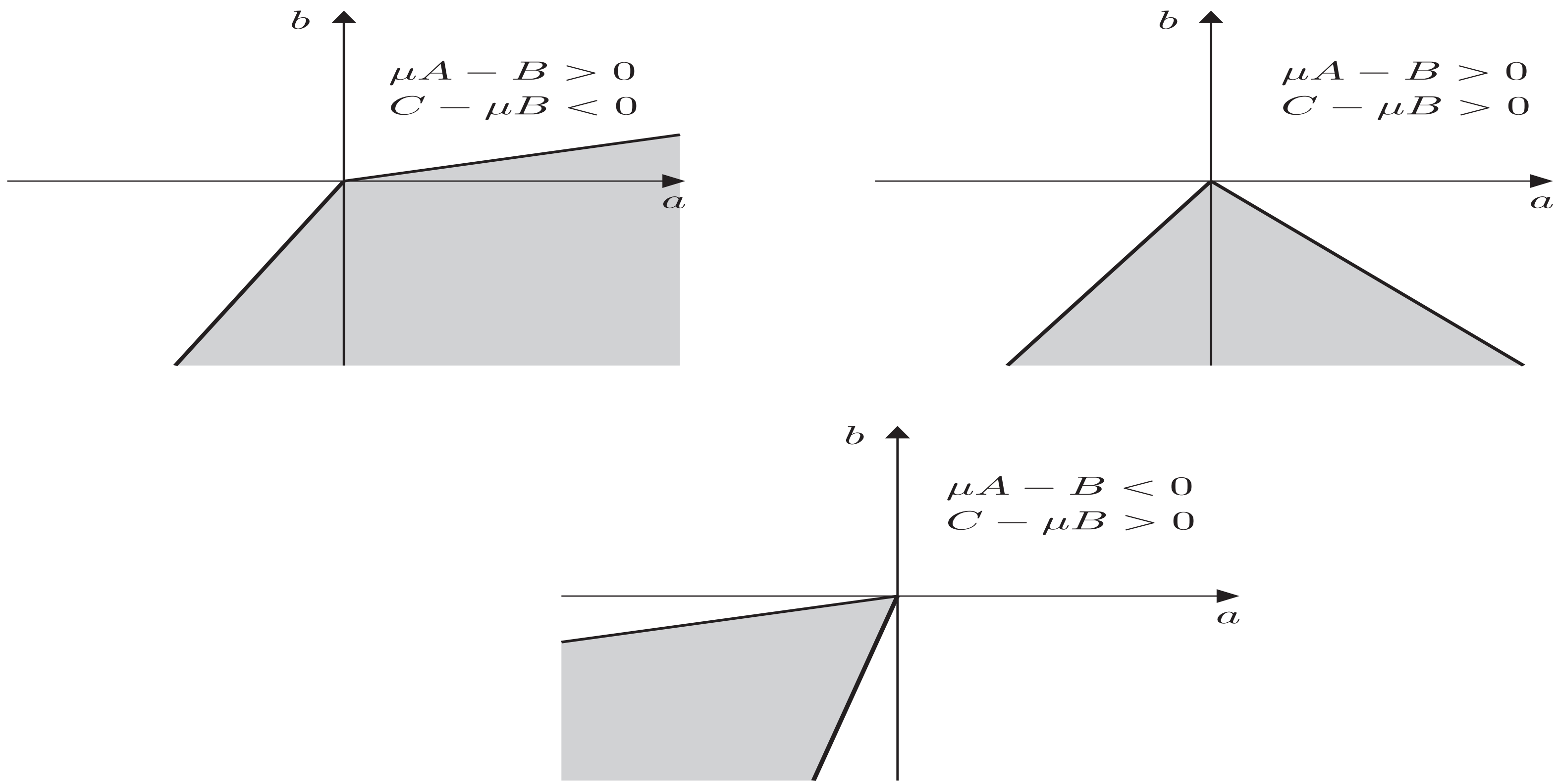
Friction cone for stick

$|\lambda_T| < \mu\lambda_N$ implies
$a(C - \mu B) + b(\mu A - B) < 0$ and $-a(C + \mu B) + b(\mu A + B) < 0$



Note: most one of $C - \mu B$, $C + \mu B$, $\mu A - B$, $\mu A + B \leq 0$.
[$C \pm \mu B < 0 \Rightarrow$ “Painlevé paradox”]

2. Impacts

- **Def:** impact phase infinitesimal time intervals in which λ_N and λ_T are impulses (distributions)
- key idea: rescale $\tau = t/\varepsilon, \quad \Lambda_{N,T} = \varepsilon\lambda_{N,T} = O(1)$ and let $\varepsilon \to 0$.
- impact-phase dynamics: $\quad q' = 0$ and

$$u' = A\Lambda_T + B\Lambda_N, \qquad v' = B\Lambda_T + C\Lambda_N$$

(A, B, C are constant during impact since $q' = 0$.

- integrating $I_{N,T} = \int_{\text{impact}} \Lambda_{N,T} d\tau$ gives:
$(I_T, I_N) = \frac{1}{AC-B^2}(C\Delta u - B\Delta v, A\Delta v - B\Delta u)$.
Change in $\dot{q}$ is then: $\Delta\dot{q} = M^{-1}(c_u^T I_T + c_v^T I_N)$

But how to compute Δu, Δv?

$$u' = A\Lambda_T + B\Lambda_N, \quad v' = B\Lambda_T + C\Lambda_N,$$

$\Rightarrow$ 3 modes of impulsive motion:

impulsive positive slip: $u > 0$. Full friction $\lambda_T = -\mu\lambda_N$.

impulsive negative slip: $u < 0$. Full friction $\lambda_T = \mu\lambda_N$.

impulsive stick: $u = 0$, $|\lambda_T| < \mu\lambda_N$. Only possible if $|B| < \mu A$.

$\Rightarrow$ For all modes: $u' = k_u\lambda_N$, $v' = k_v\lambda_N$ where

$$(k_u, k_v) = (k_u^+, k_v^+) = (B - \mu A, C - \mu B) \quad \text{for pos. slip}$$

$$(k_u, k_v) = (k_u^-, k_v^-) = (B + \mu A, C + \mu B) \quad \text{for neg. slip}$$

$$(k_u, k_v) = (k_u^0, k_v^0) = (0, \frac{AC - B^2}{A}) \quad \text{for stick}$$

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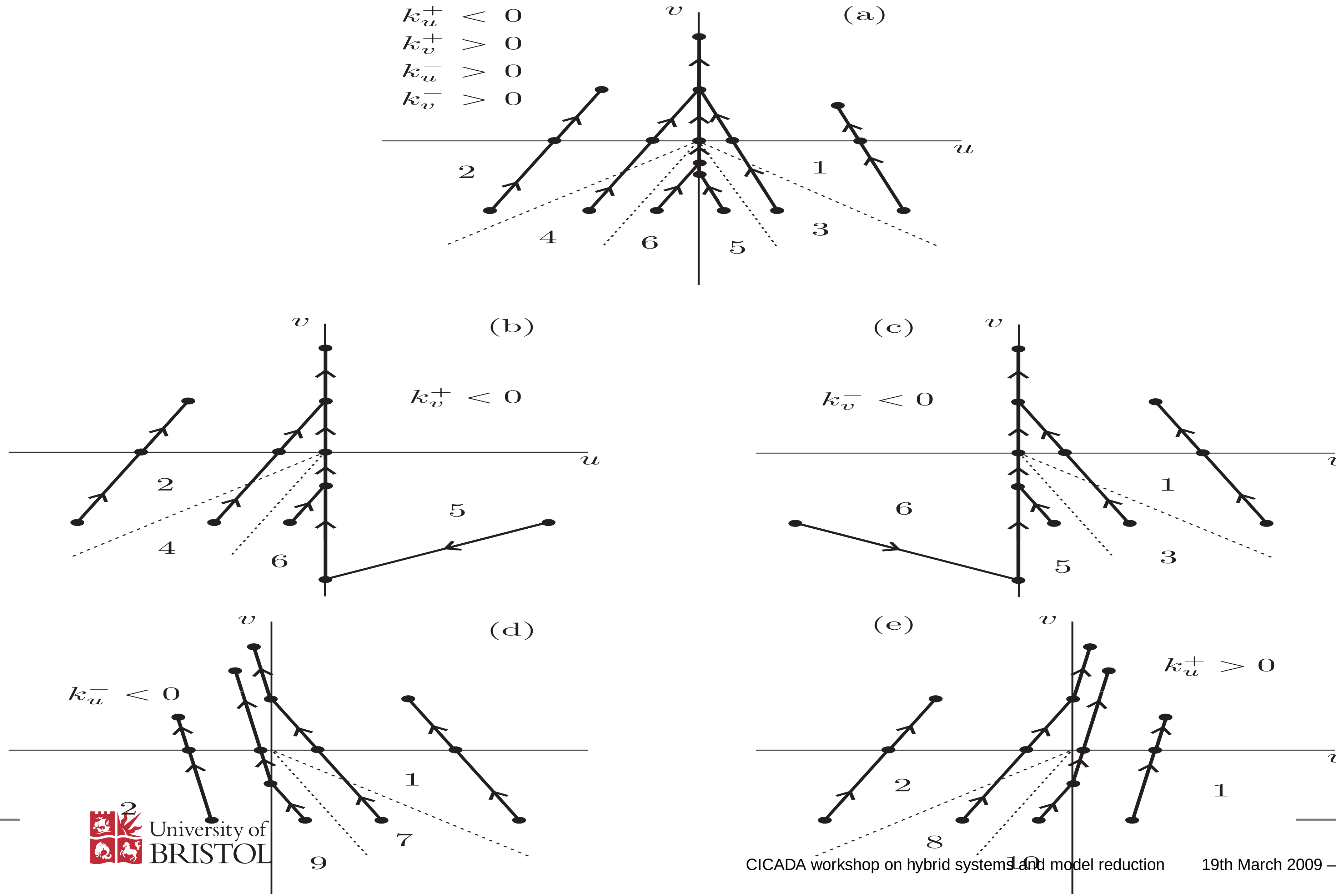
When is the impact finished?

3 possibilities :

1. **Newtonian coefficient of restitution** Relate post-impact velocities to pre-impact: $v_1 = -rv_0$
2. **Poisson coefficient of restitution (Glocker)** Relate normal impulses during compression and restitution: $I_r = -rI_c$
3. **Energetic coefficient of restitution (Stronge)** Relate normal-force work during compression and restitution: $W_r = -r^2 W_c$

If impact phase has a sinlge mode $\Rightarrow$ all 3 agree.
But (Stewart) 1 & 2 may **increase** kinetic energy for $r < 1$.
Hence we use 3 & derive explicit formulae (cf. Stronge)

Impulsive motion follows straight lines



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Expressions for the impact law

Mapping g_I Two segments, one in compression and one in restitution

$$u_1 = u_0 - (1+r)\frac{k_u}{k_v}v_0$$
$$v_1 = -rv_0$$

Mapping g_{II} Three segments, one in compression and two in restitution

$$u_1 = \frac{k_u'}{k_v'}\left(\frac{k_v}{k_u}u_0 - v_0 + \sqrt{\left(1-\frac{k_v'}{k_v}\right)\left(\frac{k_v}{k_u}u_0 - v_0\right)^2 + r^2\frac{k_v'}{k_v}v_0^2}\right)$$

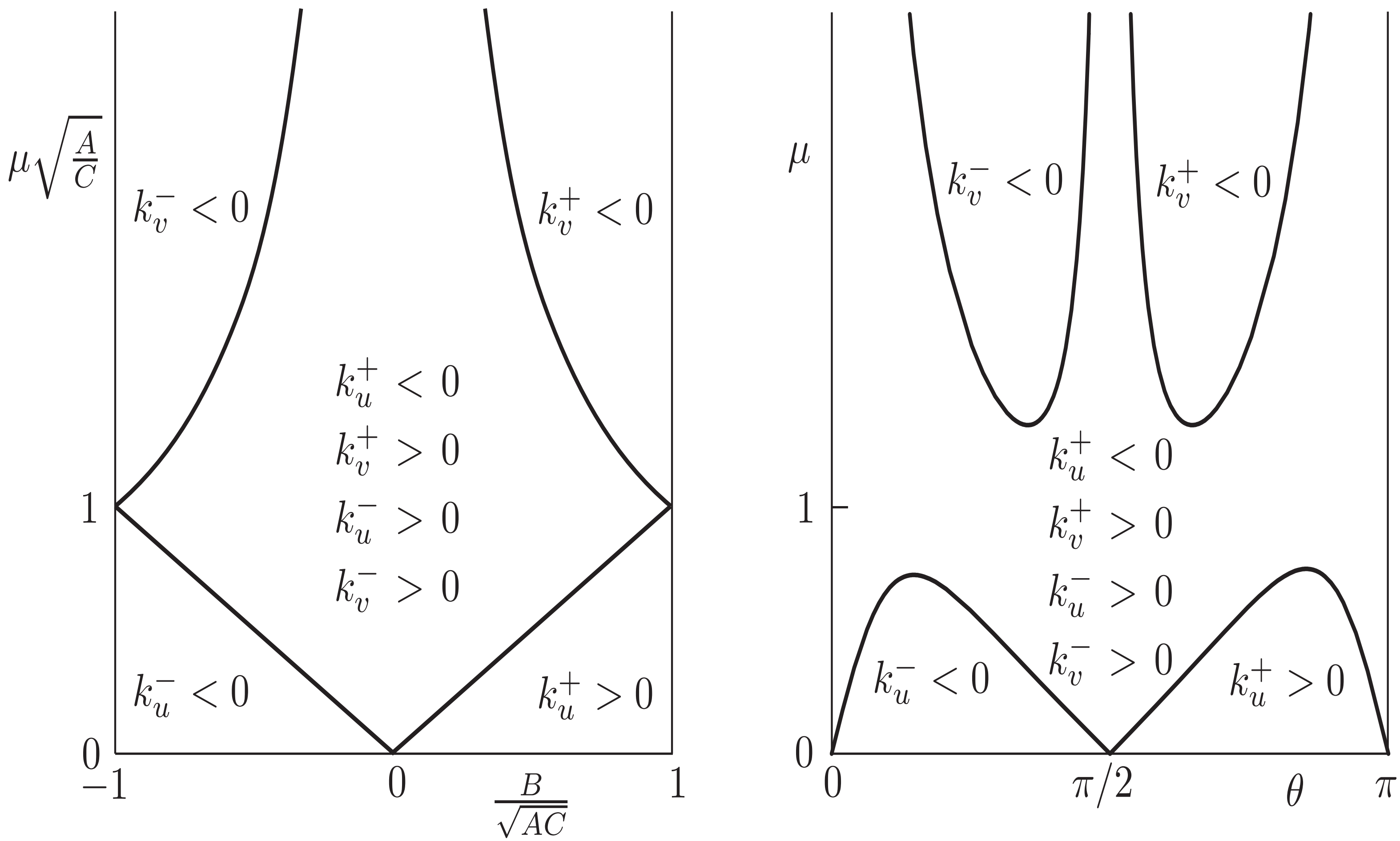
$$v_1 = \sqrt{\left(1-\frac{k_v'}{k_v}\right)\left(\frac{k_v}{k_u}u_0 - v_0\right)^2 + r^2\frac{k_v'}{k_v}v_0^2}$$

Mapping g_{III} Three segments, two in compression and one in restitution

$$u_1 = \frac{k'_u}{k'_v}\left(\frac{k_v}{k_u}u_0 - v_0 + r\sqrt{\left(1-\frac{k'_v}{k_v}\right)\left(\frac{k_v}{k_u}u_0 - v_0\right)^2 + \frac{k'_v}{k_v}v_0^2}\right)$$

$$v_1 = r\sqrt{\left(1-\frac{k'_v}{k_v}\right)\left(\frac{k_v}{k_u}u_0 - v_0\right)^2 + \frac{k'_v}{k_v}v_0^2}$$

Dependence on A, B, C and μ



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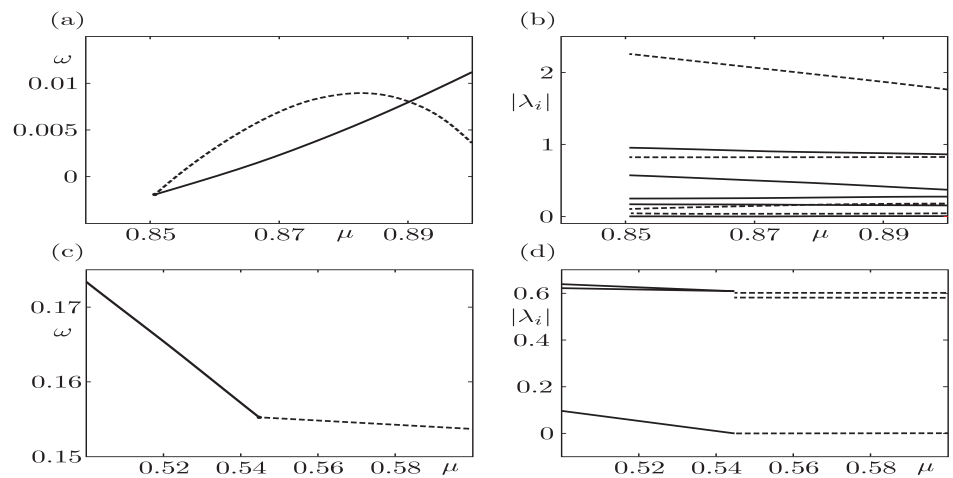
discontinuity-induced bifurcation

- dynamics cross region boundary as parameters vary
- $\Rightarrow$ hybrid flow map can be C^1 (no bifurcation) or C^0 (jump in multipliers)
- e.g. loss of period-one impacting periodic orbit



rod example with Van-der-pol type forcing: $S_x = -k_1(x - u_{\text{dr}}t) - c_1(u - u_{\text{dr}})$

$S_y = -k_2(y - y_0) - c_2(y - y_0)^2 - y_1^2)v$ $R = -k_3(\theta - \theta_0) - c_3\dot{\theta}$

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3. Ambiguities during sustained motion

To to simulate as a hybrid system, need to resolve:

A. Painlevé paradox for slip If $y = 0$, $v = 0$, $b > 0$ and $C - \mu B < 0$, $u > 0$ (or $C + \mu B < 0$, $u < 0$), then motion could continue with

- Sustained free flight
- Sustained positive (negative) slip
- An impact with zero initial normal velocity

B. Painlevé paradox for stick If $y = 0$, $v = 0$, $u = 0$, $b > 0$, $|bB - aC| < \mu(aB - bA)$ and $C - \mu B < 0$, (or $C + \mu B < 0$), then motion could continue with

- Sustained free flight
- Sustained stick

Show consistency via smoothing

- Introduce constitutive relation $\lambda_N(y,v)$ that is "stiff", "restoring", and "dissipative".
- Case A slip (WLOG positive slip),

$$\dot{y} = v, \quad \dot{v} = b + (C - \mu B)\lambda_N(y,v).$$

$b > 0$, $C - \mu B < 0 \Rightarrow$ large negative stiffness,
$\Rightarrow$ slipping will never occur, must immediately lift off ($y > 0$) or take impact ($y < 0$)

- Case B stick $\dot{v} = \frac{(bA - aB) + (AC - B^2)\lambda_N(y,v)}{A}$
$\Rightarrow$ always large positive "stiffness" hence vertical motion is asymptotically stable $(even if b > 0)$

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Ambiguities at mode transitions

Sustained motion is consistent BUT what about transitions

- Case a. approach to the Painlevé boundary $(C - \mu B = 0)$ during (positive) slip.

  previous analysis shows: can't actually reach $C - \mu B = 0$, so what happens instead?

- Case b. transitions into stick or chatter

  Def: chattering (also known as zeno-ness) is accumulation of impacts. No contradiction if accumulate in forwards time. But can get reverse chatter.

a. Unfolding $C - \mu B \to 0$ while slipping

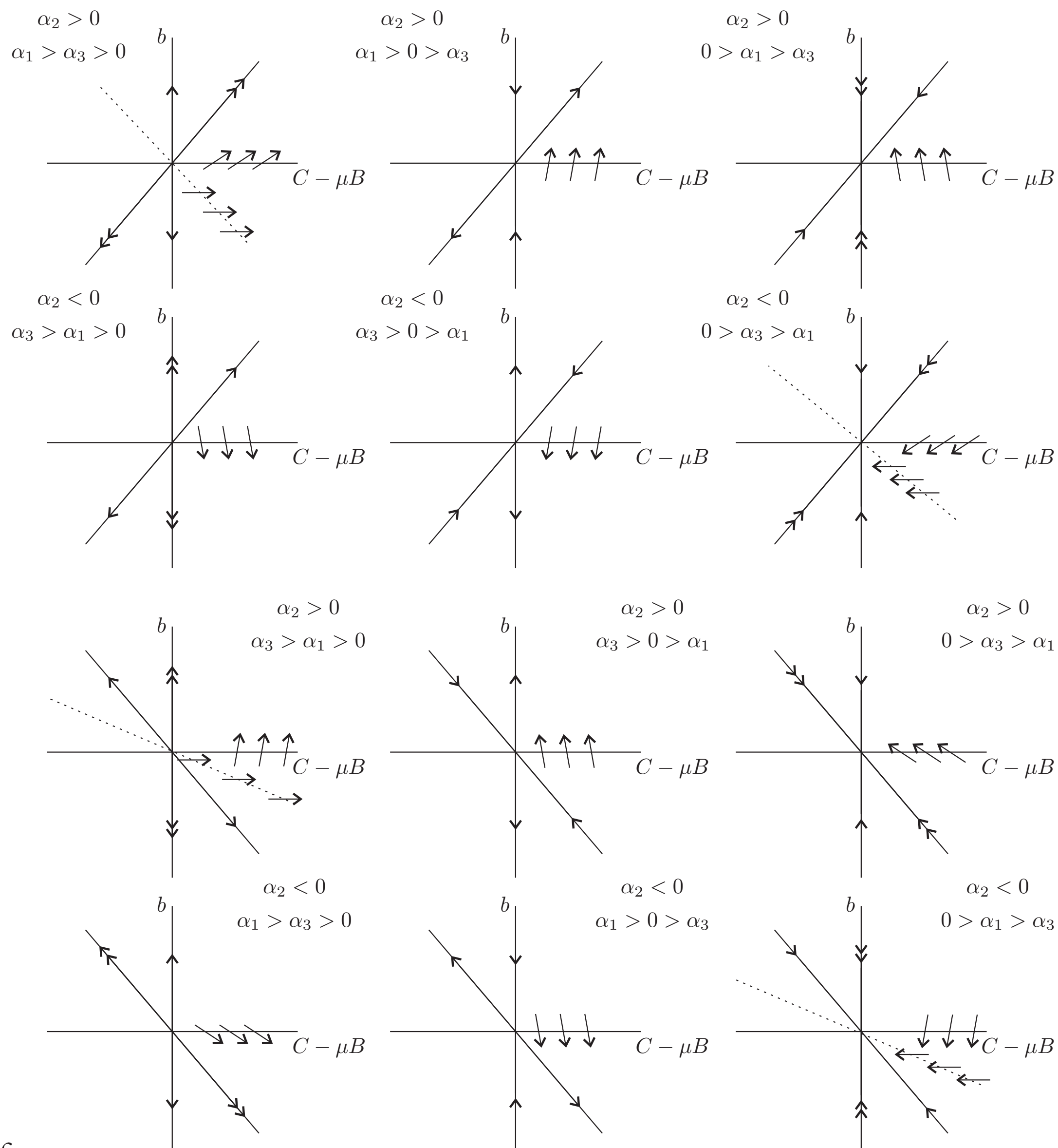
cf. Genôt & Brogliato

- Rescale time $t = (C - \mu B)s \Rightarrow$

$$\frac{d}{ds}\begin{pmatrix} C - \mu B \\ b \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \alpha_3 \end{pmatrix}\begin{pmatrix} C - \mu B \\ b \end{pmatrix}$$

- Eigenvector $(0, 1)^T \Rightarrow$ trajectory tend to $C - \mu B = 0$, only if $b = 0$

Approaching the singular point



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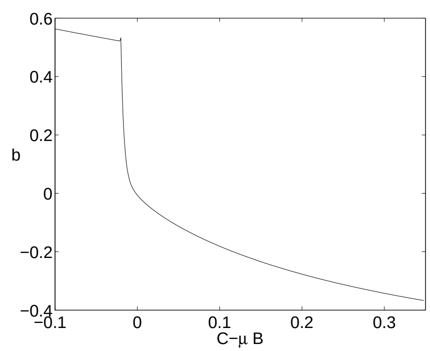
What happens after singular point?

- could lift off, or take a (zero-velocity) impact.
- e.g. simulate example for stiff, compliant contact force

$$\lambda_N(y,v) = \frac{(1+r^2) - (1-r^2)\tanh\left(\frac{v}{\delta}\right)}{2}\left(-\frac{y}{\varepsilon}\right)$$

for small δ, ε



- resolvable (ongoing work) $\Rightarrow$(?) impact always occurs

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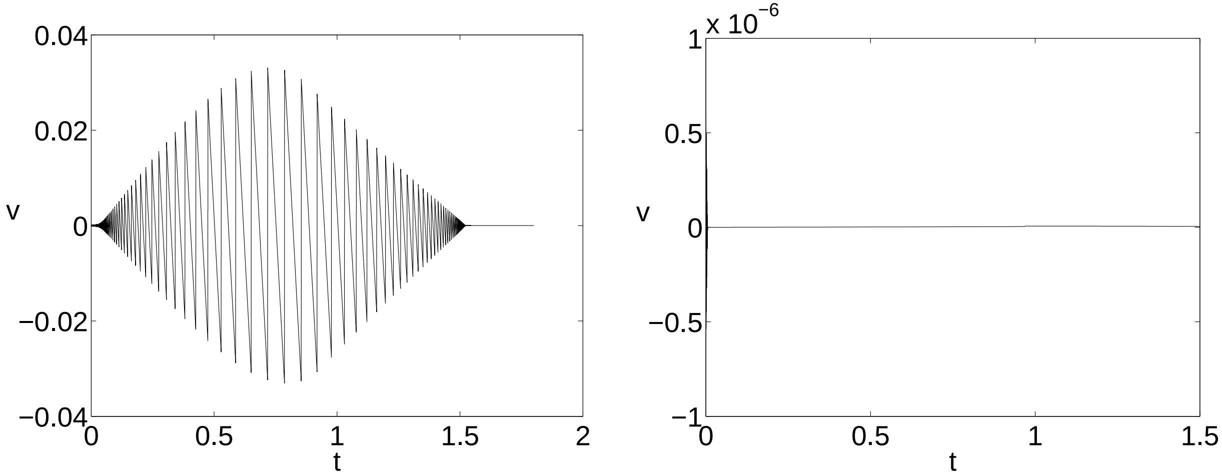
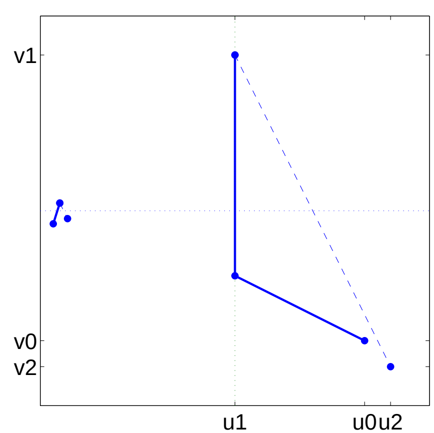
b. Transition into stick or chatter

e.g. nearby initial conditions with $b < 0$



. . . Define multiplier e: $v \to ev$ after impact + lift off.



analysis of chatter

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- ongoing work . . .

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- the possibility of reverse chattering . . .
- could this explain why it’s easy to drag chalk across a blackboard but hard to push it ?