Folded saddle node

Martin Krupa

Radboud University Nijmegen

joint work with Martin Wechselberger and Morten Brons.

My apologies for coming late.

I missed my plane last night!

Outline

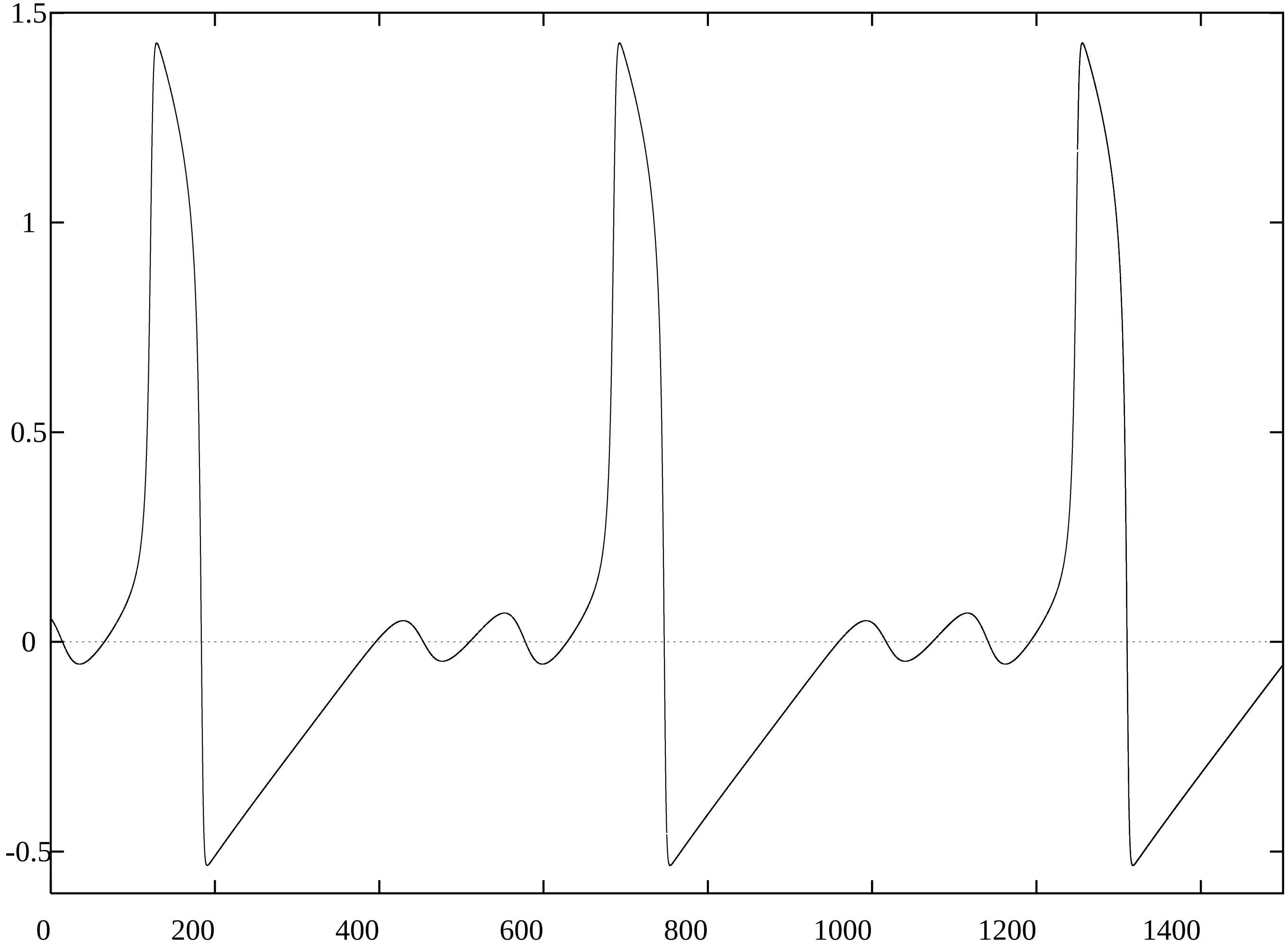
- Motivation: mixed mode oscillation, in particular in neuronal models
- Folded node vs. folded saddle node
- Analysis – slow passage through Hopf bifurcation
- Introduction/review
- Statement of main resuts
- Conclusions/implications

Mixed mode oscillations

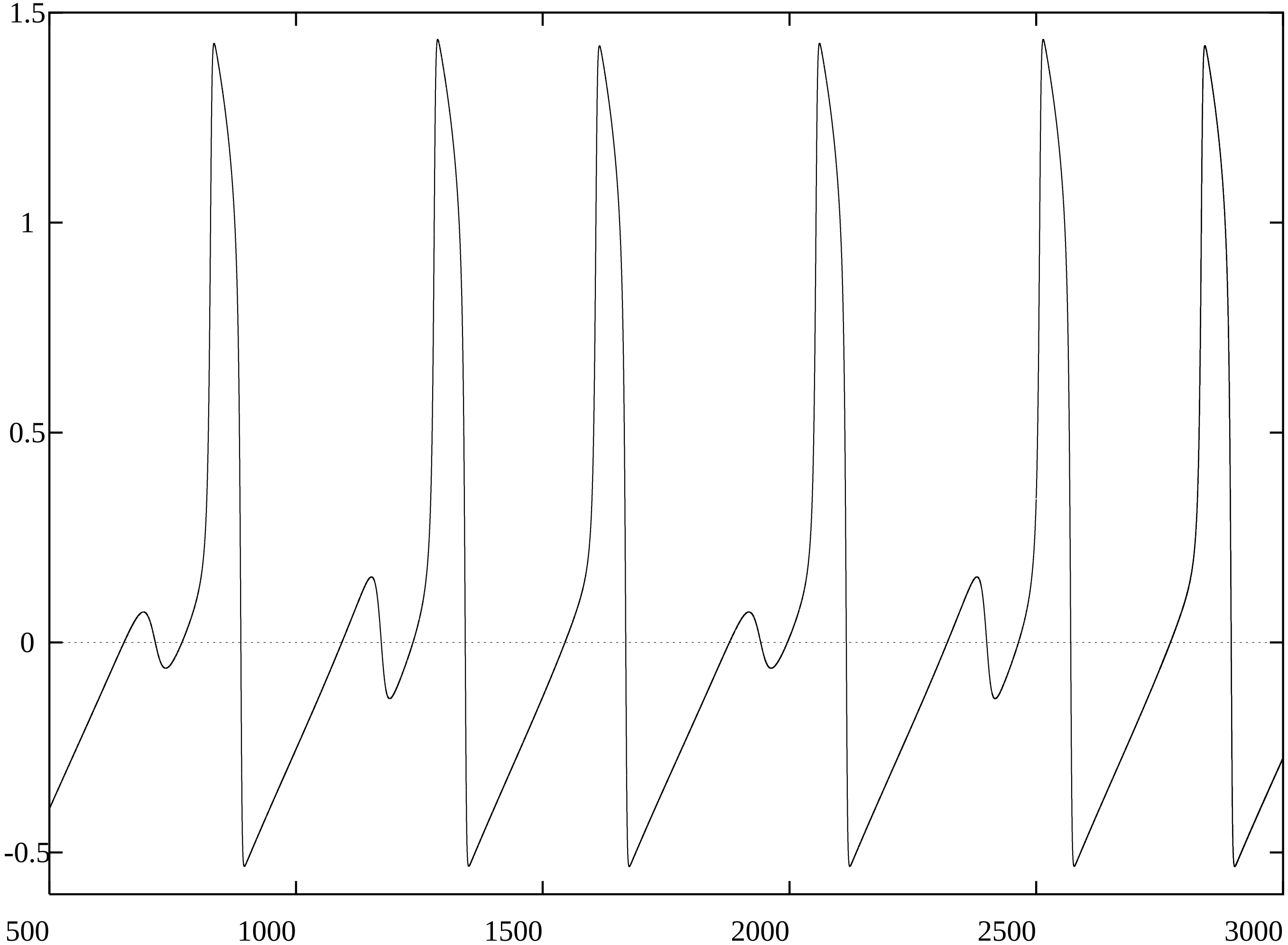
Mixed mode oscillations are a mixture of small amplitude and large amplitude oscillations.

Example: a voltage trace of a neuron with spikes interspersed with small amplitude oscillations (STOs).

Simple MMO



'Complicated MMO'



One simple mechanism: a combination of a subcritical Hopf bifurcation and a return mechanism:

J. Guckenheimer and A. Willms, *Asymptotic analysis of subcritical Hopf-homoclinic bifurcation*, Physica D 139 (2000), pp. 195–216.

In a similar way, Shilnikov phenomenon was used to explain mixed mode oscillations:

M. Koper and P. Gaspard, *The modeling of mixed-mode and chaotic oscillations in electrochemical systems*, J. Chem. Phys 96 (1992), pp. 7797–7813

Canard mechanisms

- The idea of canard mechanism was introduced by

  A. Milik, P. Szmolyan, H. Loeffelmann, E. Groeller, Geometry of mixed-mode oscillations in the 3d autocatalator, *Int. J. of Bifurcation and Chaos* **8** (1998), 505-519

  in the context of the autocatalator. Two basic ingredients: folded saddle node and a global return mechanism. The term folded saddle node was first used by these authors.
- Canard mechanism applies to the context of singularly perturbed equations.
- In simple terms: slow passage through canard explosion plus a drift.

References

Jonathan Drover, Jonathan Rubin, Jianzhong Su, and Bard Ermentrout. Analysis of a canard mechanism by which excitatory synaptic coupling can synchronize neurons at low firing frequencies. *SIAM J. Appl. Math.* **65** (2004), 65-92.

G. Medvedev, J. Cisternas, Multimodal regimes in a compartmental model of the dopamine neuron, *Physica D* **194** (2004), 333-356.

H. Rotstein, M. Wechselberger, N. Kopell, Canard induced mixed-mode oscillations in a medial entorhinal cortex layer II stellate cell model, SIAM J. Appl. Dyn. Syst. (2008), Vol. 7, No. 4, 1582-1611.

J. Rubin, M. Wechselberger, Giant Squid - Hidden Canard: the 3D Geometry of the Hodgkin Huxley Model, Biological Cybernetics (2007), Vol. 97, No. 1, 5-32.

Folded node vs folded saddle node

- Folded node is in a sense more typical than folded saddle node
- Folded node has been used in some of the literature to explain mixed mode oscillations.
- Both provide a mechanism for small oscillations
- In this talk we attempt to make a comparison between the two.
- Recall: either of these singularities lives in the context of singularly perturbed systems.

Singular Perturbation Theory

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \qquad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m \qquad \begin{aligned} x' &= f(x,y) \\ y' &= \varepsilon g(x,y), \end{aligned}$$

slow equation — fast equation

0th order approximations are given by:

$$\begin{aligned} f(x,y) &= 0 \\ \dot{y} &= g(x,y) \end{aligned} \qquad \begin{aligned} x' &= f(x,y) \\ y' &= 0, \end{aligned}$$

reduced equation — layer equation

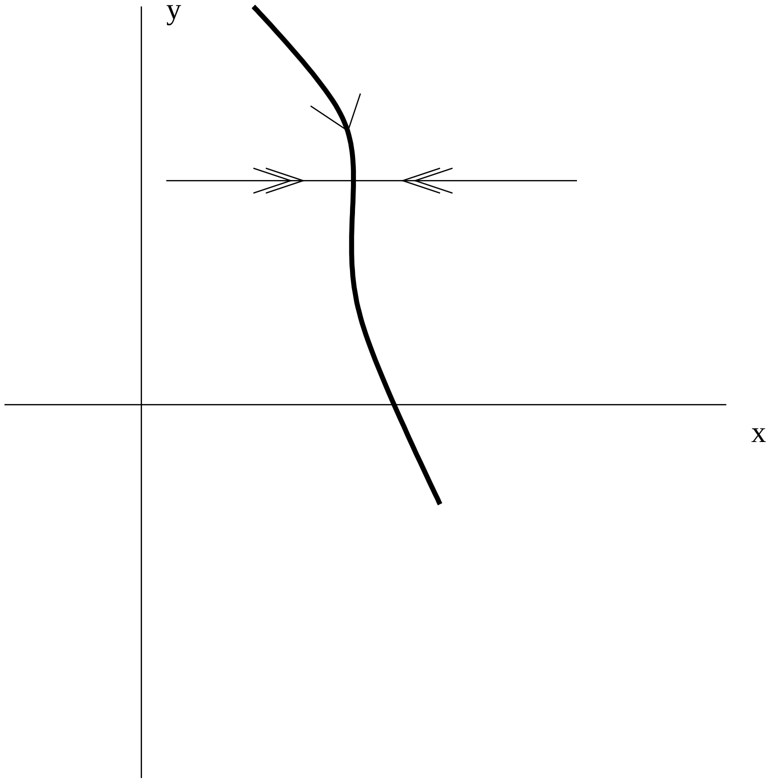
- The set $S_0 = \{(x,y)\ :\ f(x,y) = 0\}$ is called the critical manifold (fast nullsurface).
- S_0 is the phase space for the reduced problem and the set of equilibria for the layer problem.

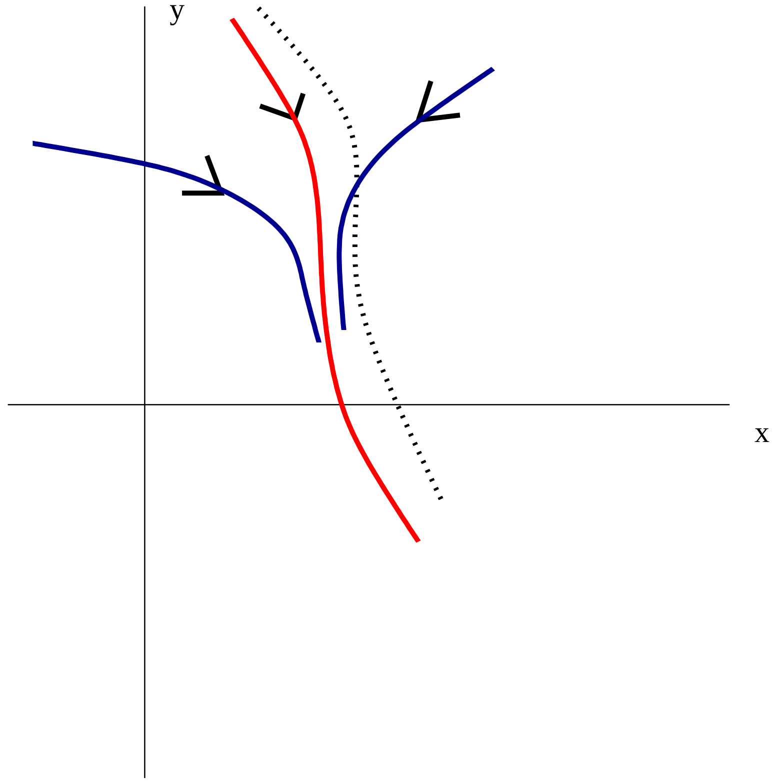
Fenichel Theorem

$$x' = f(x, y)$$
$$y' = \varepsilon g(x, y).$$

Normal hyperbolicity Suppose $\tilde{S}_0$ is an open subset of S_0. Then $\tilde{S}_0$ is normally hyperbolic if for every $(x, y) \in \mathrm{cl}\,(\tilde{S}_0)$ the matrix $D_x f$ has no eigenvalues on the imaginary axis.

Theorem If $\tilde{S}_0$ is normally hyperbolic then, for $\varepsilon > 0$, there exists a locally invariant manifold S_ε close to $\tilde{S}_0$ and the flow on S_ε is close to the flow of the reduced equation on S_0.


y
x

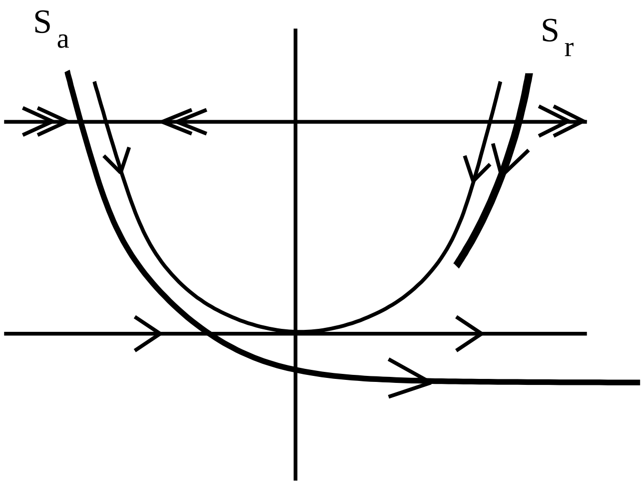

y
x

Non-hyperbolic points

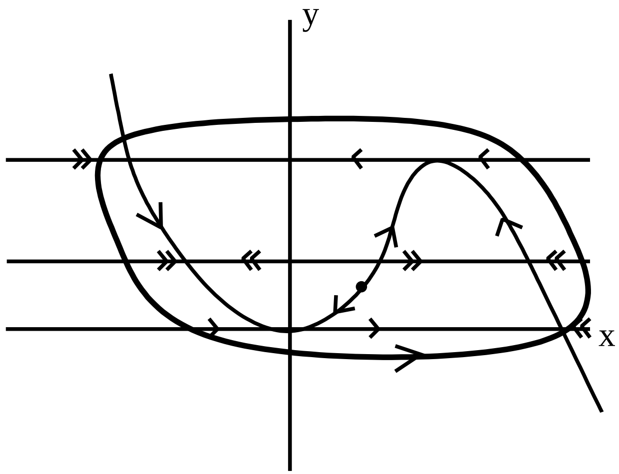
Simplest example: *fold*. The following equations give an example:

$$\varepsilon \dot{x} = -y + x^2$$

$$\dot{y} = g(x, y), \qquad g(0,0) < 0.$$



(a) simple fold



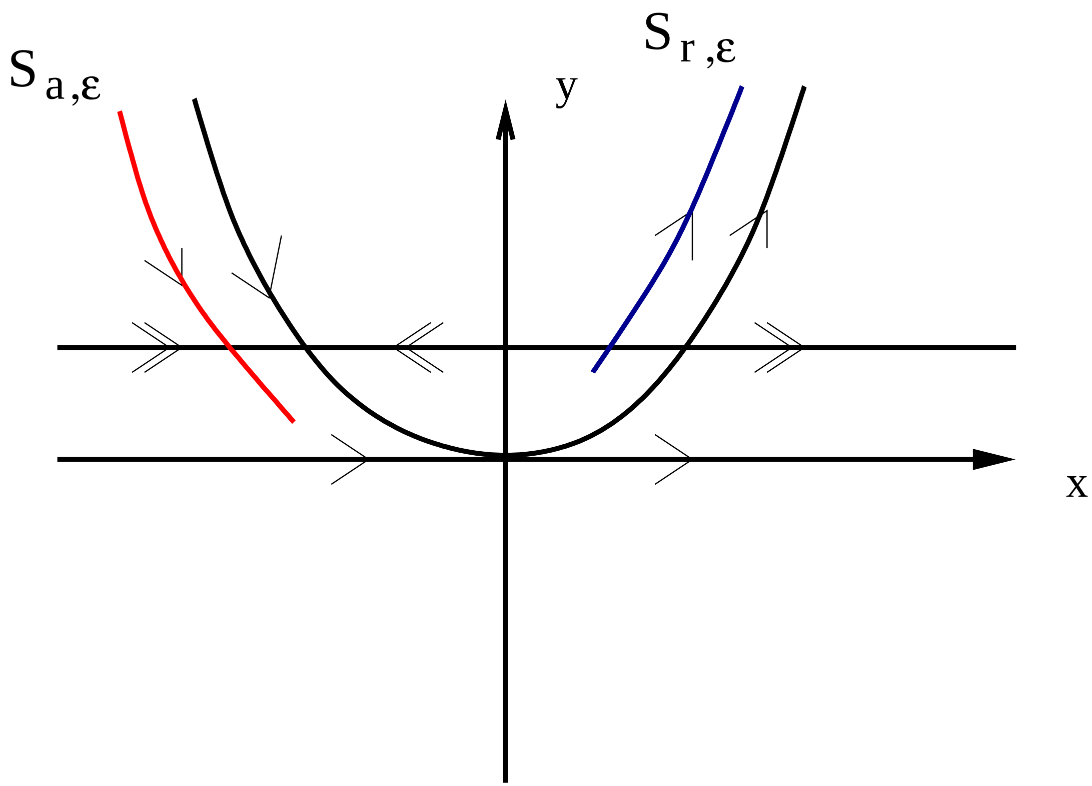
(a) relaxation oscillation

Canard point

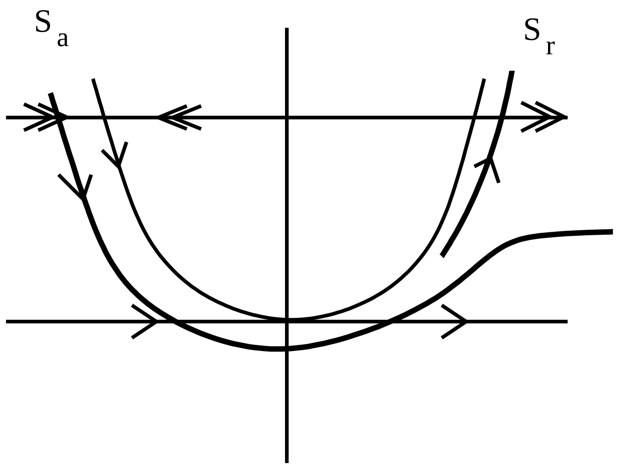
Canard point is a degenerate fold defined by the condition $g(0,0) = 0$ The following equations give an example:

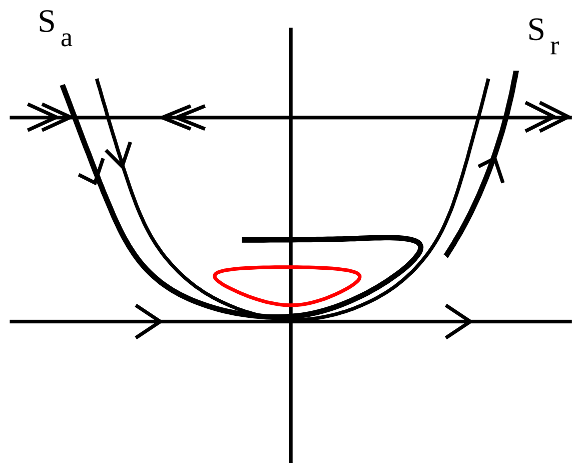
$$\varepsilon \dot{x} = -y + x^2$$

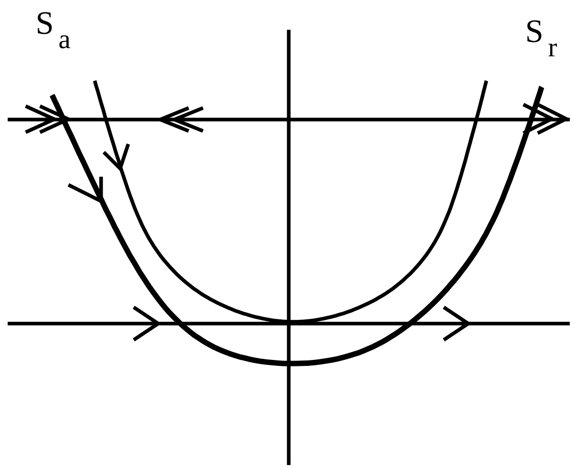
$$\dot{y} = x - \lambda \qquad \lambda \approx 0$$



Unfoldings of a canard point





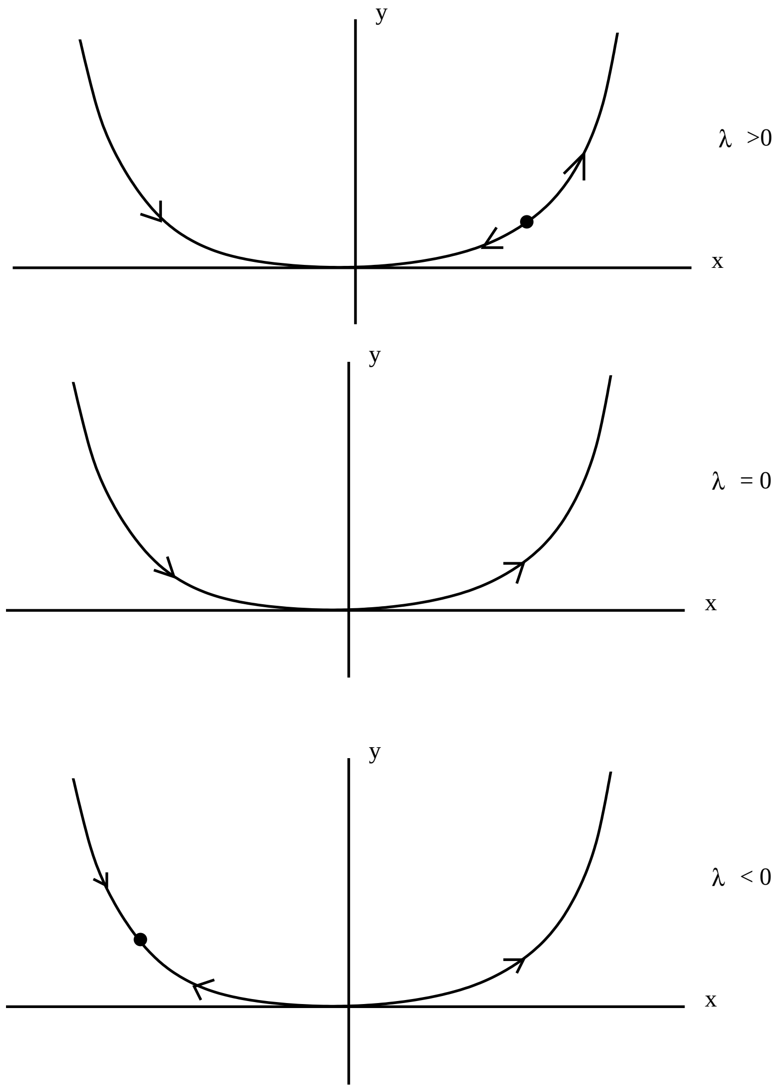


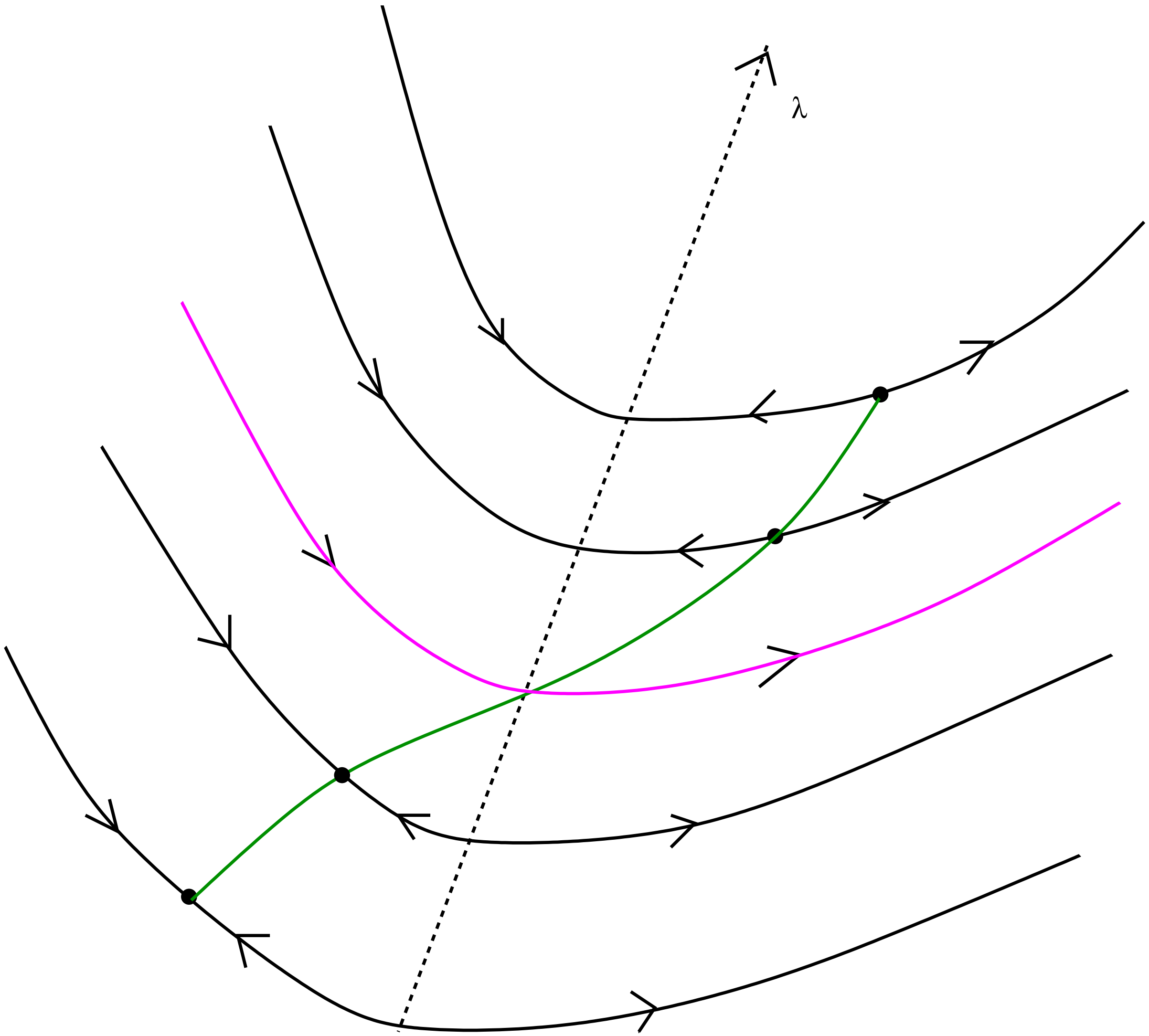
canard solution

Folded node/folded saddle node

- Folded node is an example of a folded singularity
- Folded node can be understood as canard point with a drift
- Fsn is the limiting case corresponding to the speed of the slow drift going to 0

λ unfoldings of a canard point



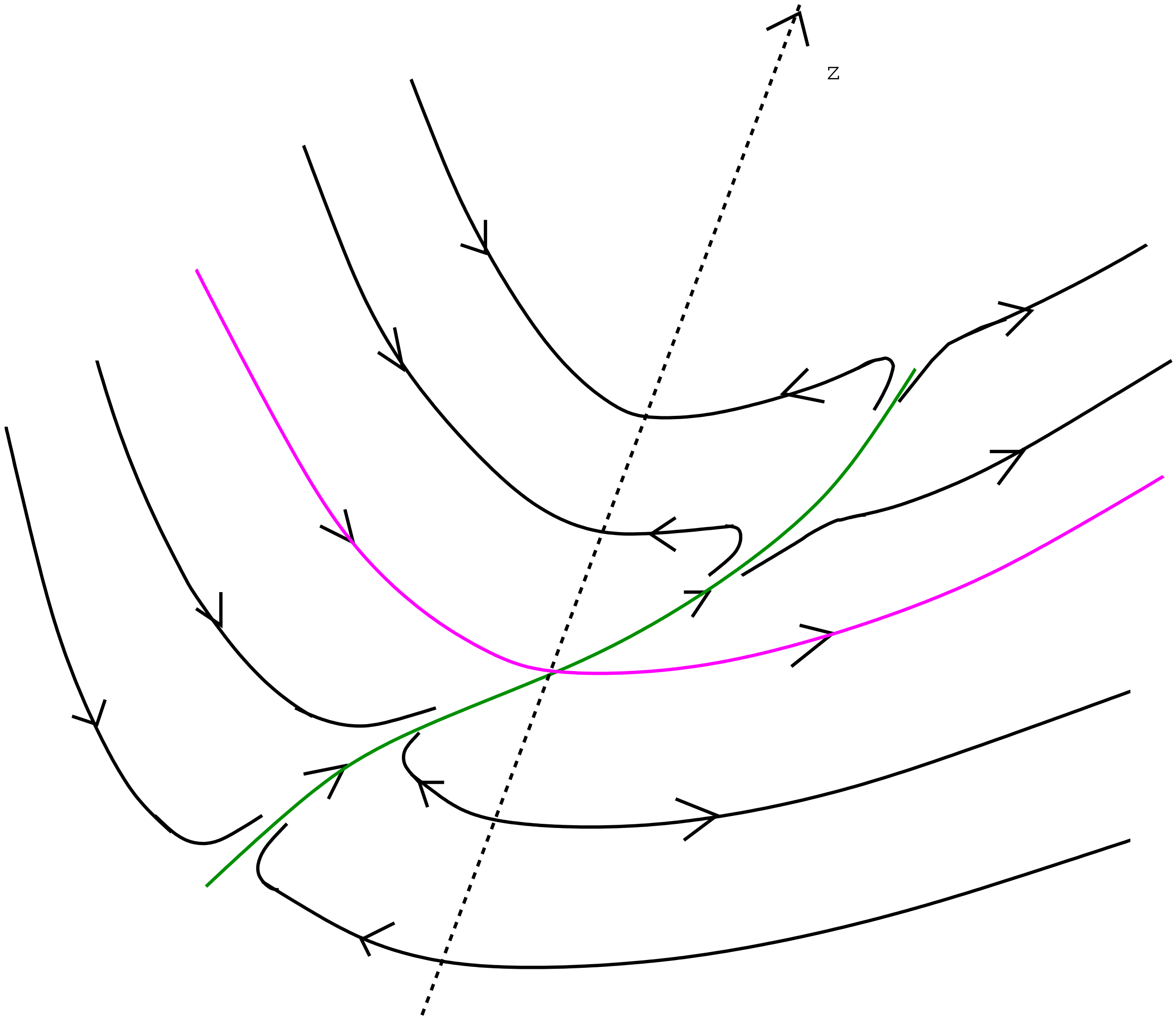


$$y' = x - \lambda$$
$$y = x^2$$
$$\lambda' = 0$$

Folded node

'Local normal form':

$$x' = -y + x^2$$
$$y' = \varepsilon(x - z)$$
$$z' = \varepsilon\mu$$


z

Folded saddle node

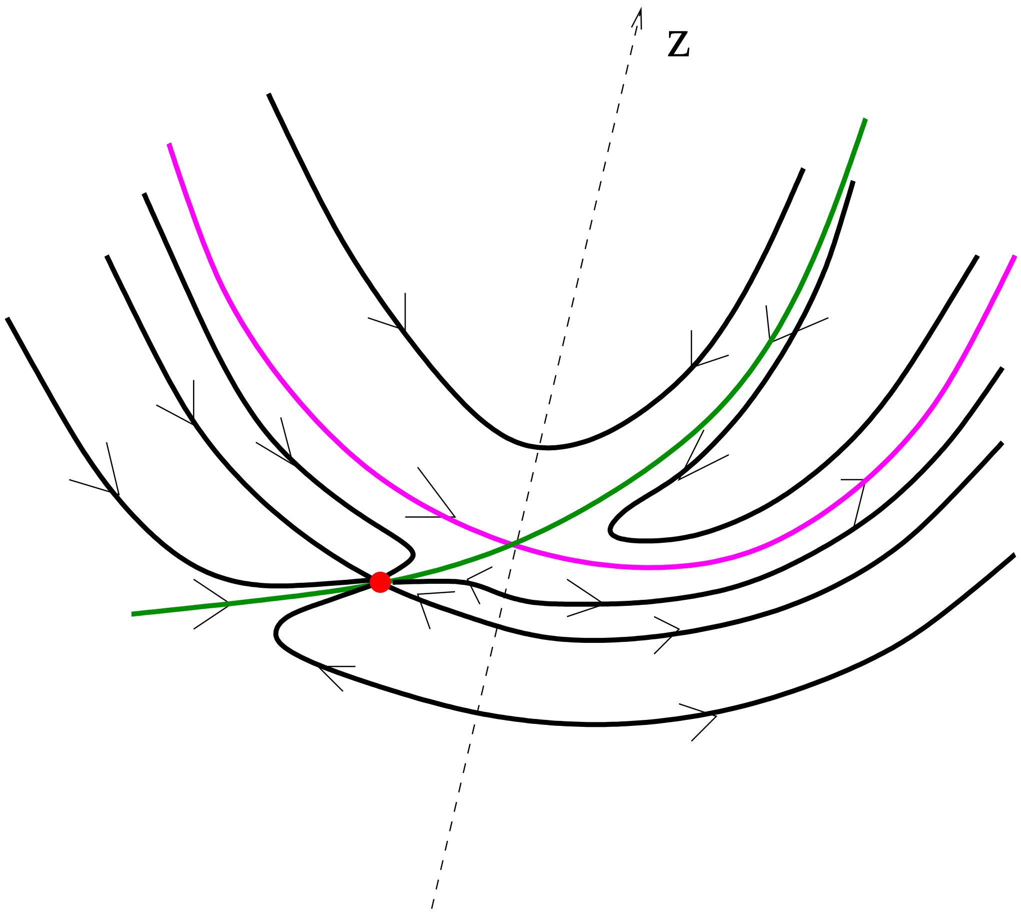
'Local normal form':

$$x' = -y + x^2$$

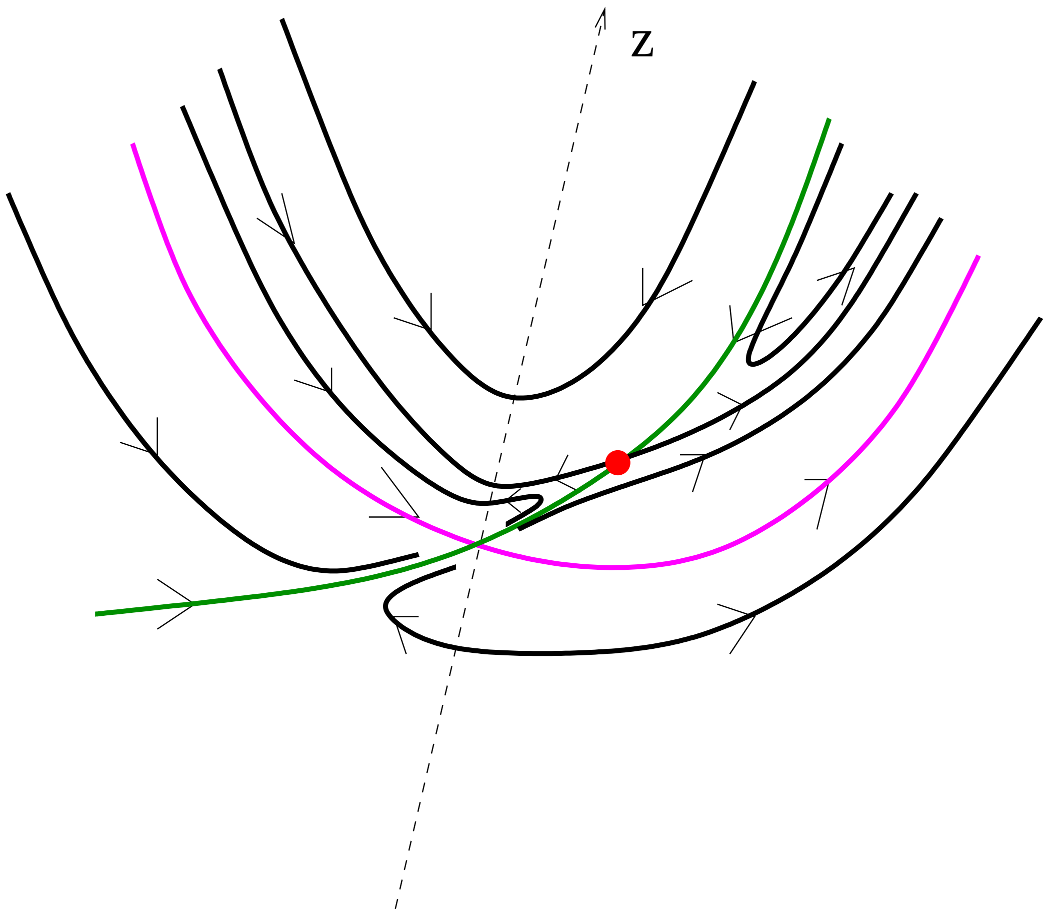
$$y' = \varepsilon(x - z)$$

$$z' = \varepsilon(\mu + ax + bz), \qquad \mu \approx 0$$

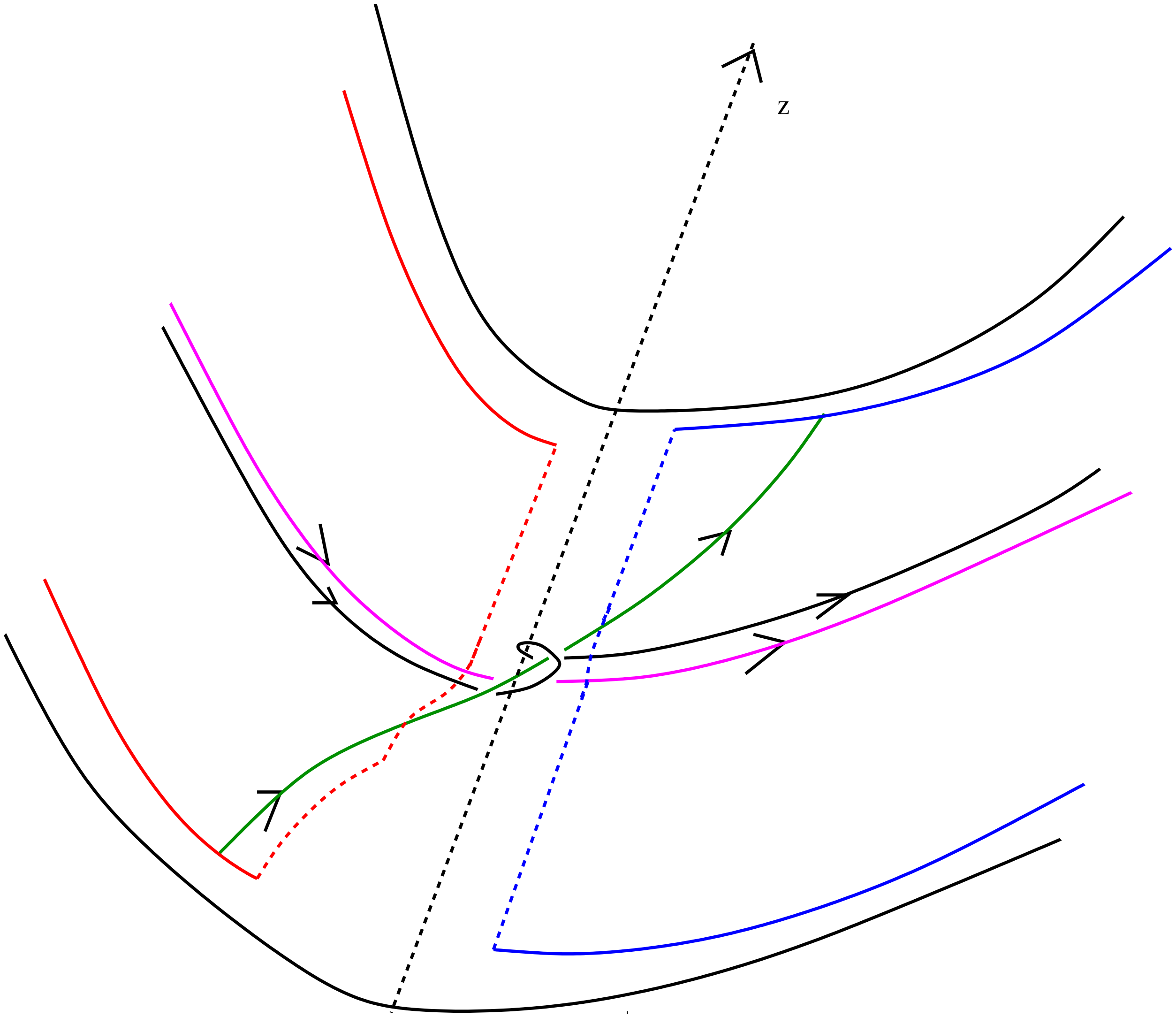
μ unfoldings of fsn, $\mu < 0$



μ unfoldings of fsn, $\mu > 0$



Folded node, $\varepsilon > 0$



The green trajectory and the magenta trajectory are called *primary canards*
The black trajectory is a *secondary canard*

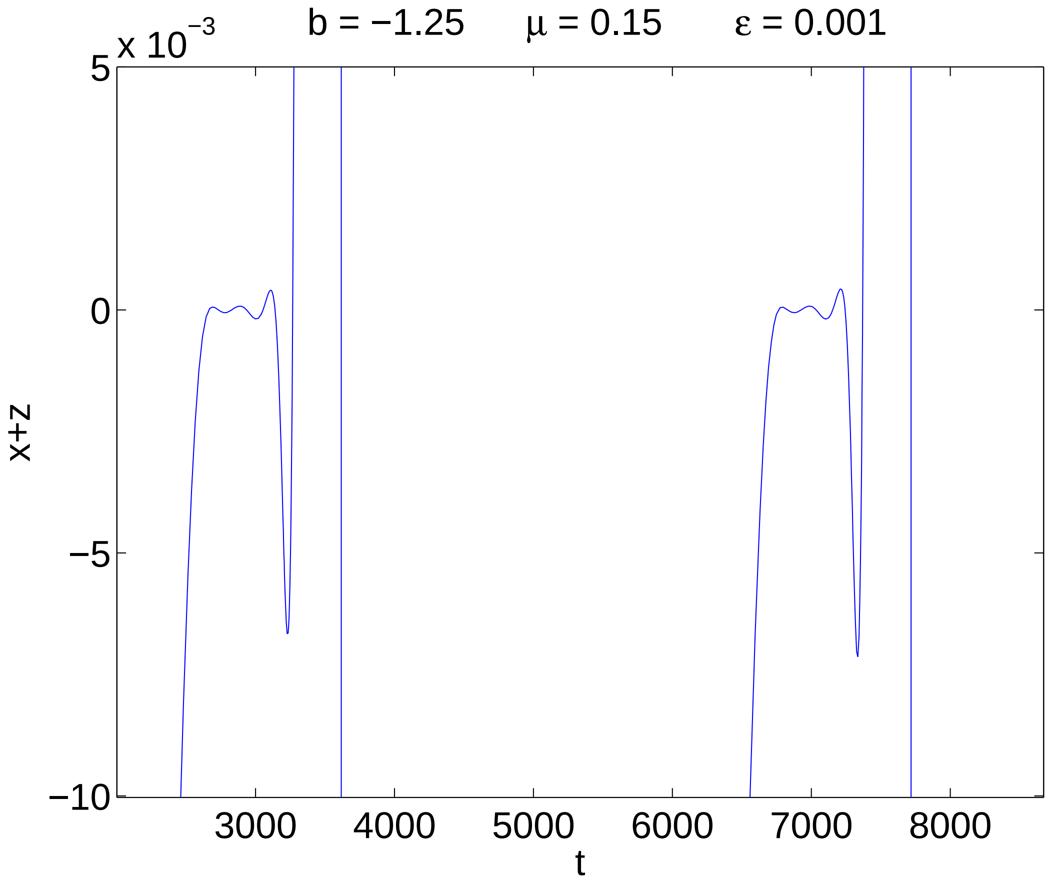
Folded node versus folded saddle node

- Near folded saddle node trajectories move slower through the fold region, so that small oscillations play a more prominent role
- The maximal number of small oscillations is bounded for folded node and not bounded for folded saddle node
- an equilibrium is always present near the fold. This is the case in many (all?) of the examples coming from the applications.

Argument against folded node

M. Brons, M. Krupa, and M. Wechselberger, *Mixed mode oscillations due to the generalized canard phenonenon*, Fields Institute Communications 49 (2006), pp. 39–63.

In this paper we show simulations for a 'real' folded node, i.e. $\mu \gg \sqrt{\varepsilon}$ (will explain this estimate later). The simulations show that most of the small oscillations are so small that they cannot be seen.


b = −1.25 μ = 0.15 ε = 0.001
x 10^−3
5
0
−5
−10
x+z
3000
4000
5000
6000
7000
8000
t

Folded saddle-node

Some advances on folded saddle node:

M. Krupa, M. Wechselberger, Local analysis near a folded saddle-node singularity, submitted to *JDE*, (2008)
In the remainder of the talk we discuss this work.

Rescaling

$$x = r_2 x_2,\ y = r_2^2 y_2,\ z = r_2 z_2,\ \varepsilon = r_2^2,\ \mu = r_2 \mu_2.$$

$$x_2' = r_2(-y_2 + x_2^2)$$

$$y_2' = r_2(x_2 - z_2)$$

$$z_2' = r_2^2(\mu_2 + a x_2 + b z_2)$$

'cancel' r_2 (time rescaling)

$$x_2' = -y_2 + x_2^2$$
$$y_2' = x_2 - z_2$$
$$z_2' = r_2(\mu_2 + a x_2 + b z_2)$$

- slow/fast problem
- 1D slow manifold defined by $x_2 = z_2$, $y_2 = z_2^2$

Assume $(a+b) < 0$. Then, for $\mu_2 > 0$,

- there is an equilibrium on the slow manifold with $z_2 > 0$
- there is a delayed Hopf bifurcation

Delayed Hopf – references

Neishtadt A I (1987), *Persistence of stability loss for dynamic bifurcations. I*, Differential Equations **23**: 1385-1390.

Neishtadt A I (1988), *Persistence of stability loss for dynamic bifurcations. II*, Differential Equations **24**: 171-176.

Wallet G (1986), *Entrée-sorte dans un tourbillon*, Annales de l'institut Fourier **36**, 157-184

Delayed Hopf – review

Given is a problem of the form:

$$z' = \lambda(\sigma)z + f(\sigma, z)$$

$$\text{c.c.}$$

$$\sigma' = \varepsilon \qquad\qquad z \in \mathbb{C}, \sigma \in \mathbb{R}$$

with $\lambda(\sigma) \in \mathbb{C}$ and satisfying

$$\mathrm{Re}\,\lambda < 0 \text{ for } \sigma < 0$$

$$\mathrm{Re}\,\lambda = 0 \text{ for } \sigma = 0$$

$$\mathrm{Re}\,\lambda > 0 \text{ for } \sigma > 0$$

λ, f analytic

Way in/way out function

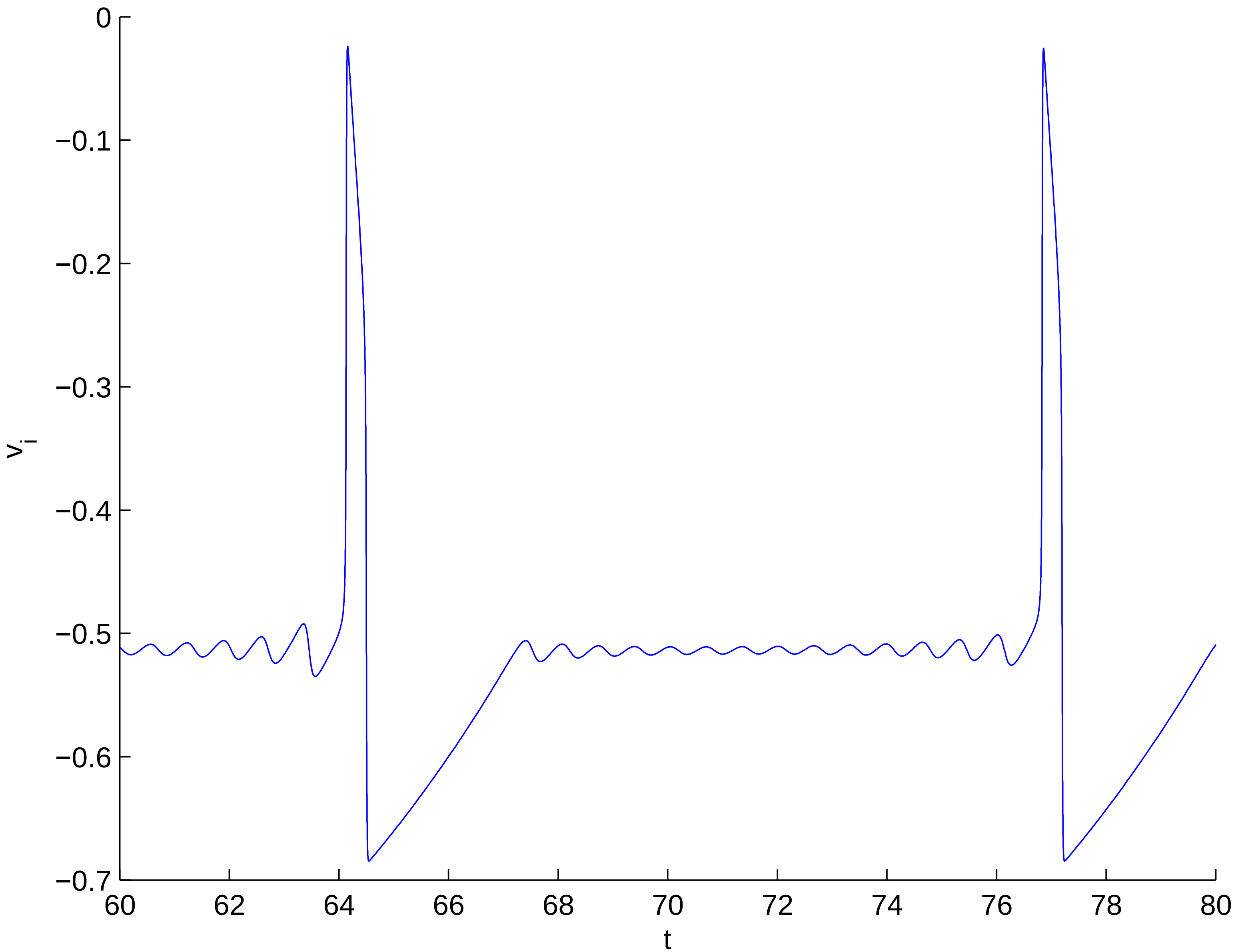
Given is an initial condition (σ_0, z_0), $\sigma_0 < 0$, $|z_0| = 1$ and the corresponding trajectory $z(\sigma)$.

The way in/way out function assigns to σ_0 the value σ^*, such that, asymptotically in ε, $z(\sigma)$ is repelled from the vicinity of 0 near σ^*.

For $\sigma_0 < 0$ let $\sigma_* = \Phi_0(\sigma_0)$ be defined by

$$\int_{\sigma_0}^{\sigma_*} \mathrm{Re}\ \lambda(s) ds = 0$$

$\Phi_0(\sigma)$ is the way in/way out function for σ small.


0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
v_i
60
62
64
66
68
70
72
74
76
78
80
t

Method of analysis

Fundamental problem: exponential contraction and then expansion along real time.

Solution:

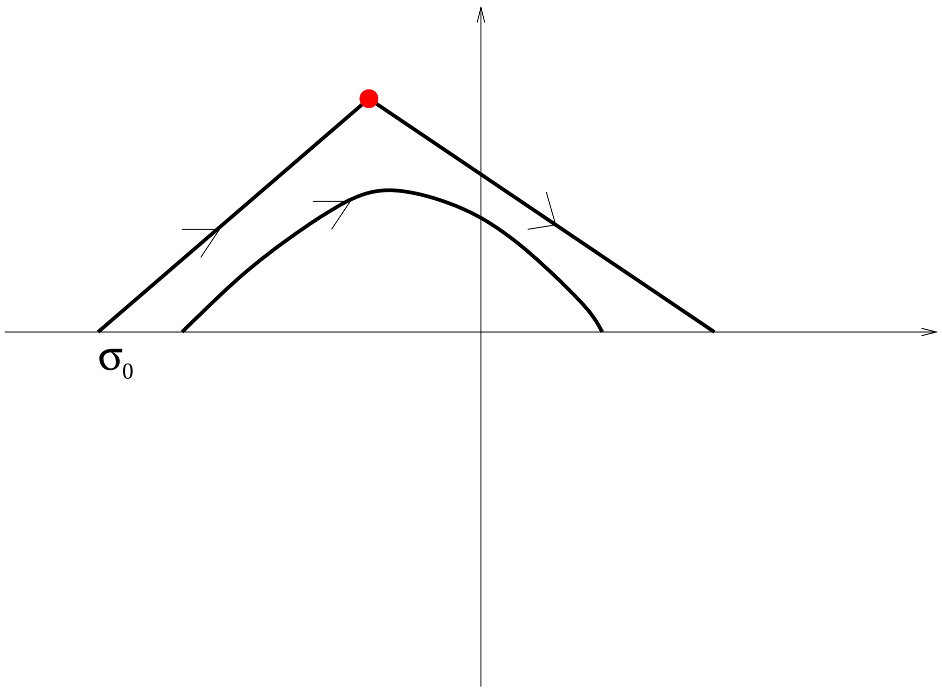
- Complexify time σ
- Track solutions along eliptic paths in the complex plane. Eliptic paths are integral curves for the Hamiltonian system with Hamiltonian

$$H(\sigma) = \int^{\sigma} \mathrm{Re}\ \lambda(s) ds.$$

- Along the eliptic paths there is no contraction/expansion, just very fast rotation.

Buffer points

Buffer points correspond to saddle type equilibria of the Hamiltonian system for elliptic paths.



If there is a buffer point the way in/way out function is given by

$$\Phi(\sigma) = \Phi_0(\sigma) \qquad \text{for } \sigma_0 < \sigma < 0$$

$$\Phi(\sigma) = \Phi_0(\sigma_0) \qquad \text{for } \sigma < \sigma_0.$$

Way in/ way out for fsn – why?

We wish to compute the way in/ way out function for fsn, but why?

- If we know the relation between the incoming and the outgoing trajectories for the passage near the fold, and we know the return mechanism, then we can analyze the global dynamics (in particular MMOs). In fact, knowing the way in/way out function we can also determine the phase (angle) of the outgoing trajectory.
- Maybe even the local problem is interesting by itself? Recently I have heard of work claiming functional significance of STOs.

Important assumptions for Neistadt theory:

- $\lambda(\sigma)$ must not be real for $\sigma \in \mathbb{R}$
- the flow in the σ direction is non-singular.

Features of fsn delayed Hopf

- there are no buffer points
- eigenvalues become real
- there is a singularity due to the equilibrium on the slow manifold

Eigenvalue sequence

Fsn

Real negative $\longrightarrow$ Complex with negative real part $\longrightarrow$ Complex with positive real part $\longrightarrow$ Real positive

In Neistadt's theory only the shorter sequence

Complex with negative real part $\longrightarrow$ Complex with positive real part

is considered.

For fsn there exist $\sigma_- < 0$ and $\sigma_+ > 0$ such that

- For $\sigma < \sigma_-$ there are two real negative eigenvalues $\lambda_{ss}(\sigma) < \lambda_s(\sigma) < 0$
- For $\sigma_{-1} < \sigma < \sigma_{+1}$ the eigenvalues are complex and the Neistadt case occurs.
- For $\sigma > \sigma_+$ there are two real positive eigenvalues $\lambda_{uu}(\sigma) > \lambda_u(\sigma) > 0$.

Remark The behavior of the system has characterics of a node equilibrium.

Important feature

$$\int_{-\infty}^{\sigma_-} \lambda_s(\sigma) d\sigma$$

is finite. This means that there is a ’finite amount of contraction’ on the stable side!

On the repelling side the ’amount of expansion’ goes to infinity, because of the equilibrium.

New way in/way out function, 3 cases

Case 1 If $\sigma_0 > \sigma_-$ we can still use Φ_0, defined by

$$\int_{\sigma_0}^{\sigma_*} \mathrm{Re}\ \lambda(s) ds = 0$$

Case II If $\sigma_0 < \sigma_-$ we define Φ implicitly using the equation

$$\int_{\sigma_0}^{\sigma_-} \lambda_s(\sigma) d\sigma + \int_{\sigma_-}^{\sigma_*} \mathrm{Re}\ \lambda(s) ds = 0$$

This works as long as $\sigma^* < \sigma_+$.

Case III For σ_0 so small that Case II fails we use the equation

$$\int_{\sigma_0}^{\sigma_-} \lambda_s(\sigma) d\sigma + \int_{\sigma_-}^{\sigma_+} \mathrm{Re}\ \lambda(s) ds + \int_{\sigma_+}^{\sigma^*} \lambda_{uu}(s) ds = 0$$

Gist of the results

- Way in/way out function works well for cases I and II
- For case III the way in/way out function works well only on a 'good set' of μ values. This set is a complement of the union of a finite set of intervals. However, as $\varepsilon \to 0$ the number of 'bad intervals' goes to infinity. The total length of the 'bad set' is uniformly small in comparison to the size of the 'good set'.
- We have proved the existence of a family of secondary canards. The number of canards in this family increases to ∞ as $\varepsilon \to 0$ (and $\mu \to 0$)

Conclusions/Conjectures

- For cases I and II we have good control over the transition through the fold region and there is good hope for analyzing global dynamics.
- Case III corresponds to a transition to folded node. The problem with the way in/way out function corresponds to resonances in foded-node (the maximal number of small oscillations jumps by 1). This kind of resonance must cause a complication in the global dynamics. We have reasons to believe that case III is not very significant.
- Complications in global dynamics arise due to the passage of returning trajectories near secondary canards (but there are many secondary canards).