A Model Reduction Technique for linear Model Predictive Control for Non-linear Large Scale Distributed Systems

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Overview

- Motivation
- Existing technologies
- Our proposed technique
- Case studies
- Conclusions
- Further work

Motivation

- Model Predictive Control (MPC)
  - Linear MPC is widely used in industries
  - Few nonlinear MPC applications
- Nonlinear large scale distributed system
  - Attracting more interest among researchers
  - Few well-established methods available
- Model reduction techniques
  - Great potential industrial applications
  - Relatively new

Existing technologies

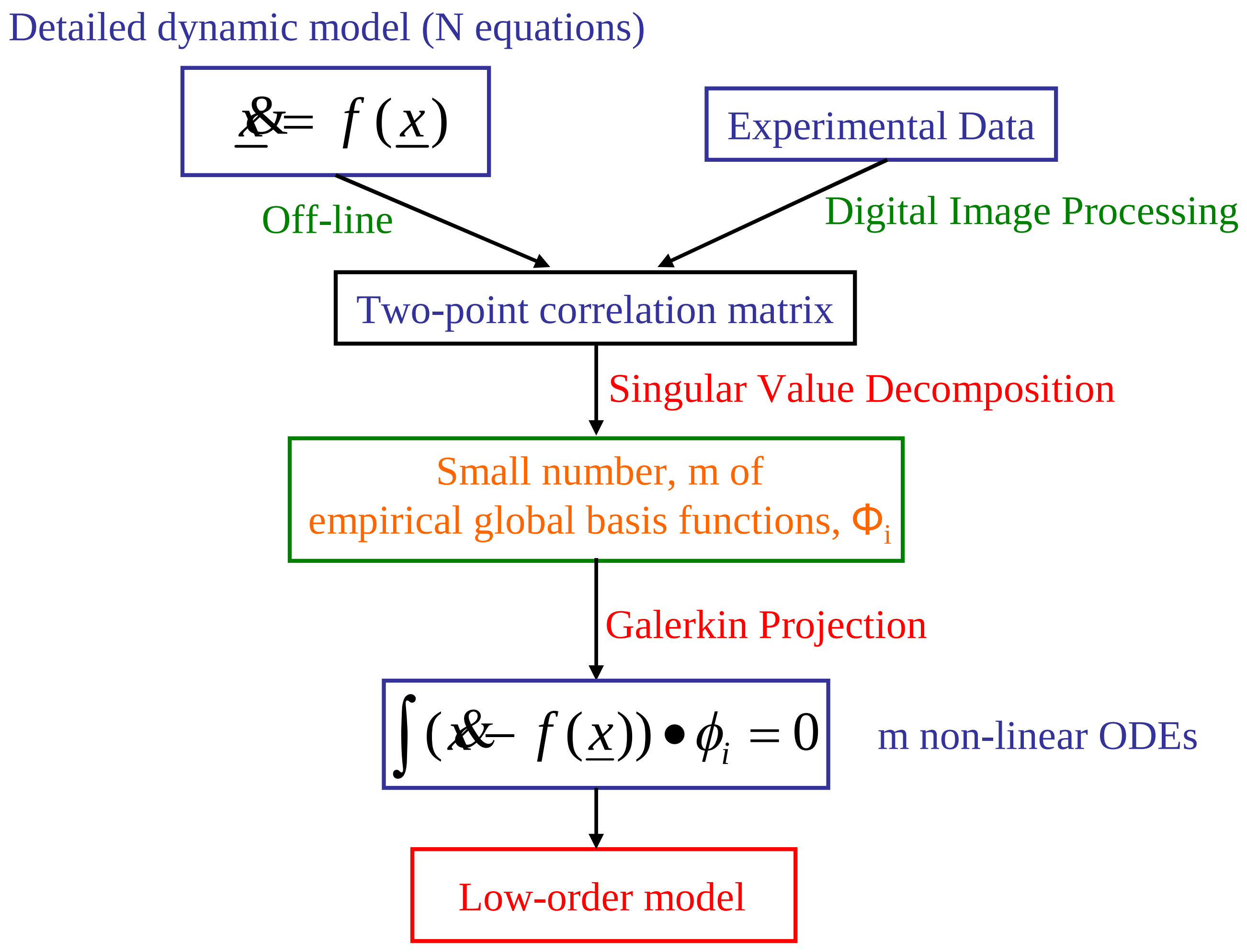
- Feedback linearisation
  - Parametric control
- Adaptive control (self-tuning control)
- Artificial neural network
- Non-parametric methods

Our proposed technique

- Aims
  - Applicable to complex dynamic systems
  - Automatic procedure
  - Good approximation of the original full-scale model
  - Explicit parametric dependence
  - High computational efficiency
- Our new method
  - 1st step: POD (proper orthogonal decomposition) -based projection
    - onto low-dimensional hyperspace
  - 2nd step: TPWL (Trajectory Piece-wise linearisation)
    - on time coefficients
  - 3rd step: QP (Quadratic programming) applied to obtain control law

Model reduction

Proper Orthogonal Decomposition (POD)



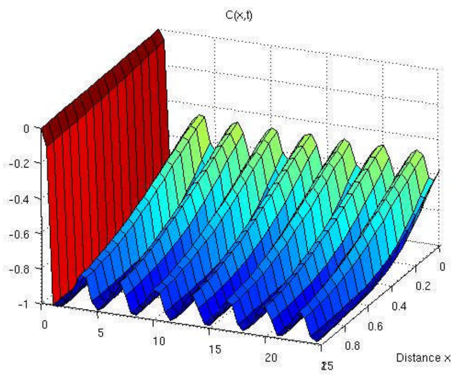
Linearisation of POD-based constraints for MPC

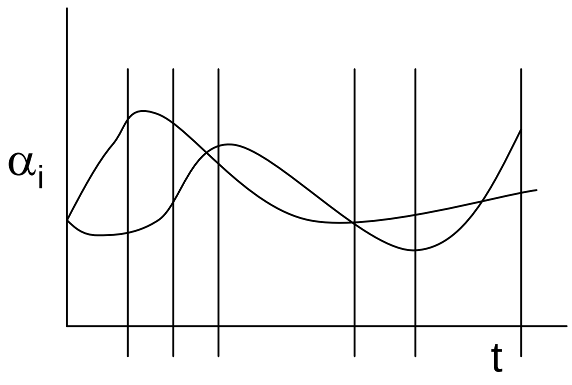
- Reduced model

$$\frac{da_i}{dt} = f(a_i, t) \qquad \text{(equation 1)}$$

$$u(x,t) = \sum_{i=1}^{m} a_i \phi_i + \bar{u} \qquad \text{(equation 2)}$$

- Idea: can linearise in terms of α_i:
  – Irrespective of (high physical) dimensionality of the problem
  – Linearisation always 1-dimensional





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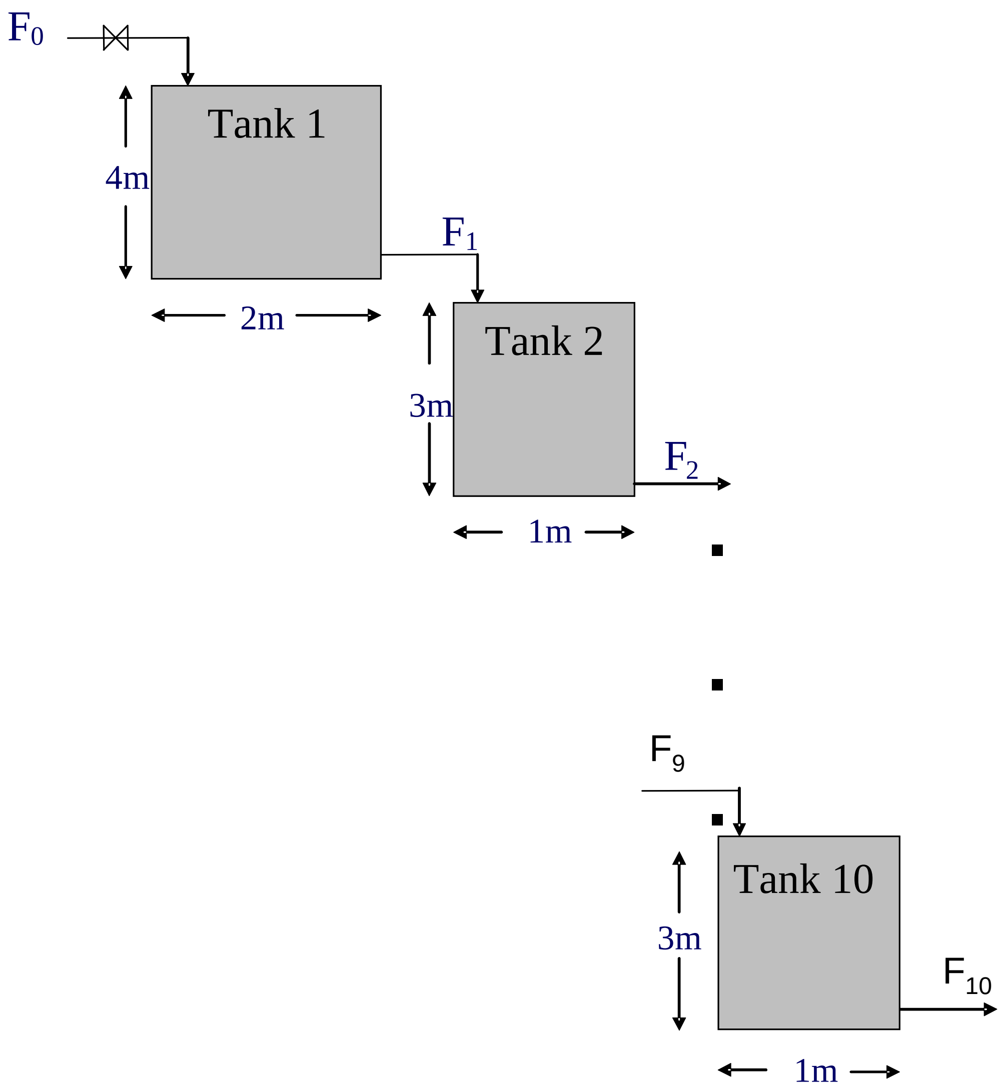
Case study 1: ten Tanks Level control



Control formulation—10 tanks

- Objective function

$$\min_{du}\left(Y - Y_{ref}\right)^T Q\left(Y - Y_{ref}\right) + DU^T RDU$$ (equation 3)

- s.t.

Mass Balance of tank 1: $$A_1 \frac{dh_1}{dt} = F_0 - F_1$$ (equation 4)

Mass balance of tank 2- tank 10: $$A_i \frac{dh_i}{dt} = F_{i-1} - F_i$$ (equation 5)

$$i = 2, \Lambda\ , 10$$

Flow rates: $$F_k = c_k h_k^{1/2}$$ (equation 6)

$$k = 1, \Lambda\ , 10$$

Non-linear model

- Non-linear model control
  - based on non-linear objective function (non-quadratic)
  - with nonlinear constraints.
- Non-linear dynamic optimisation
  - Multiple shooting (based on a set of time intervals)
    - Some kind of successive substitution
    - or better Sequential Quadratic Programming

POD model reduction

- Using $F_1 = c_1 * h_1^{1/2}$ then

$$\frac{\partial F_1}{\partial t} - \frac{F_0 \cdot c_1^2}{F_1 \cdot 2A_1} + \frac{c_1^2}{2A_1} = 0$$ (Equation 7)

- Similar equations can be obtained by the above method

$$\frac{\partial F_i}{\partial t} - \frac{F_{i-1} \cdot c_i^2}{F_i \cdot 2A_i} + \frac{c_i^2}{2A_i} = 0 \qquad i = 2,\Lambda ,10$$ (Equation 8)

- Apply method of snapshots to get basis functions $\varpi_k(x)$
- Calculate time coefficients $\alpha_k(t)$
  - using Galerkin projection on the *m* POD eigenfunctions as above
- The system dynamics are then retrieved as:

$$F_i(x,t) = v(x,t) + \overline{F_i} \qquad i = 1,\Lambda ,10$$ (Equation 9)

TPWL method

- The piecewise linear interpolation is built as follows.

$$L_i(z) = a_i + b_i(z - x_i) \qquad \text{(Equation 10)}$$

$$a_i = y_i \qquad b_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \qquad i \in [1, n-1]$$

$$L(z) = \begin{cases} L_1(z) & if \quad \alpha = x_1 \le z < x_2 \\ L_2(z) & if \quad x_2 \le z < x_3 \\ & \vdots \\ L_{n-1}(z) & if \quad x_{n-1} \le z \le x_n = \beta \end{cases} \qquad \text{(Equation 11)}$$

- Apply mean value theorem:

$$f(z) = L(z) + \frac{f^{(2)}(\eta)}{2}(z - x_i)(z - x_{i+1}) \qquad z \in [x_i, x_{i+1}] \qquad \eta \in [x_i, x_{i+1}] \qquad \text{(Equation 12)}$$

- Then,

$$|f(z) - L(z)| \le \frac{M_2 h_m^2}{8} \le \delta \qquad \text{(Equation 13)}$$

where, the second derivative of f is bounded by M_2

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Static and adaptive TPWL

- Static TPWL based on uniform partition

$$n \geq 1 + (\beta - \alpha)\sqrt{\frac{M_2}{8\delta}}$$ (Equation 14)

where δ is a given positive tolerance.

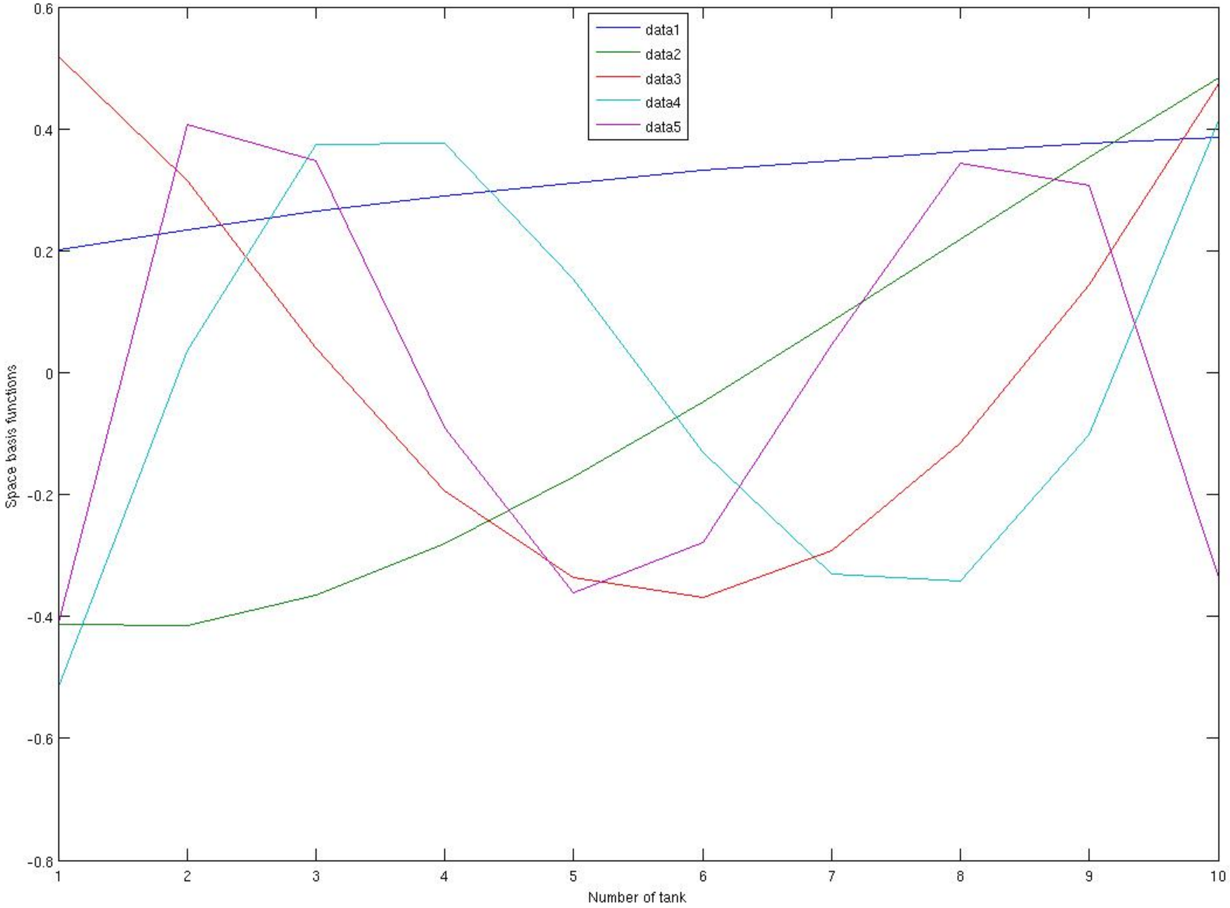
- Adaptive TPWL method
  - We propose that the subinterval $[xL, xR]$ is acceptable if

$$\left| f(\frac{xL + xR}{2}) - \frac{f(xL) + f(xR)}{2} \right| \leq \delta$$ (Equation 15)

Or, $xR - xL \leq h_{\min}$ (Equation 16)

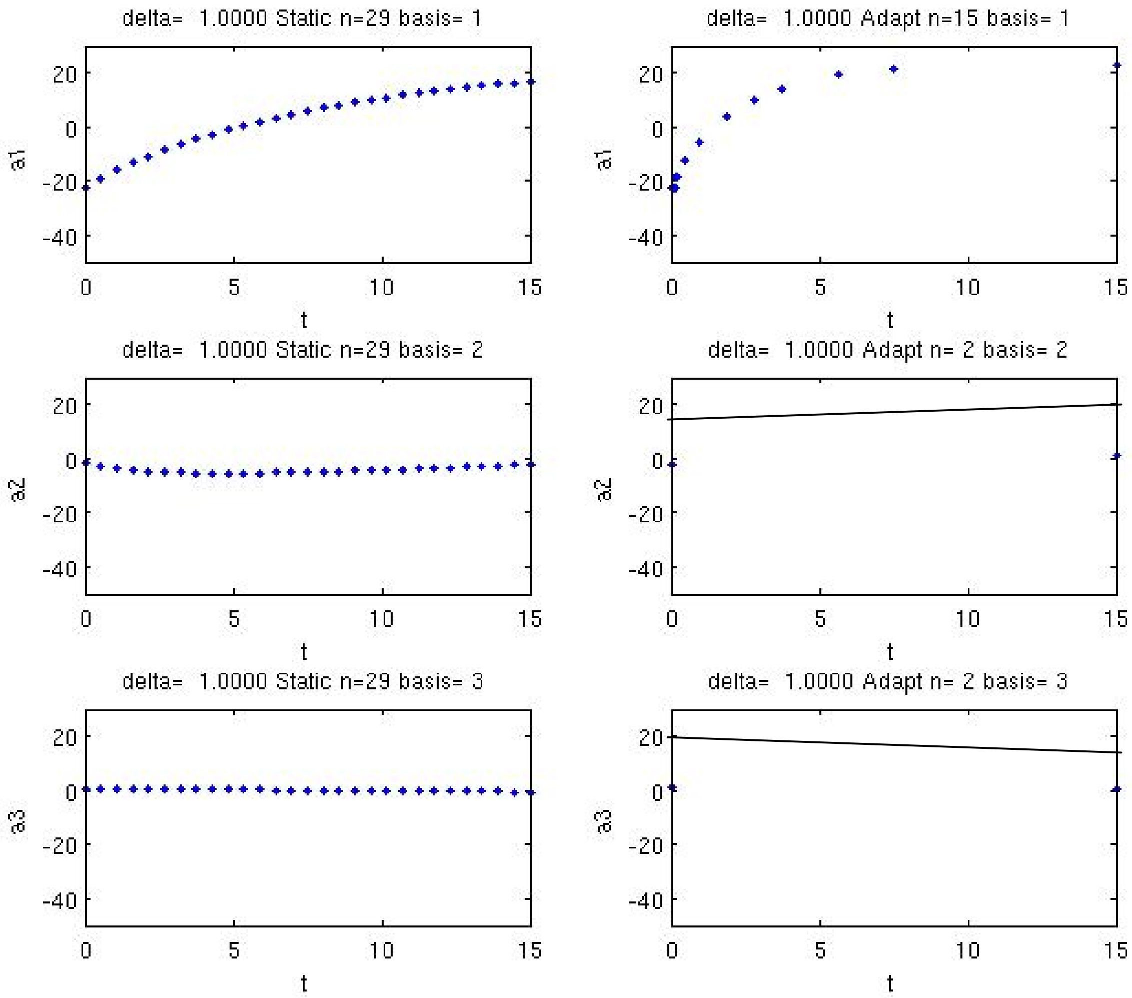
  - A partition $x_1 < \Lambda < x_n$ is acceptable if each subinterval is acceptable.

POD basis functions-ten tanks



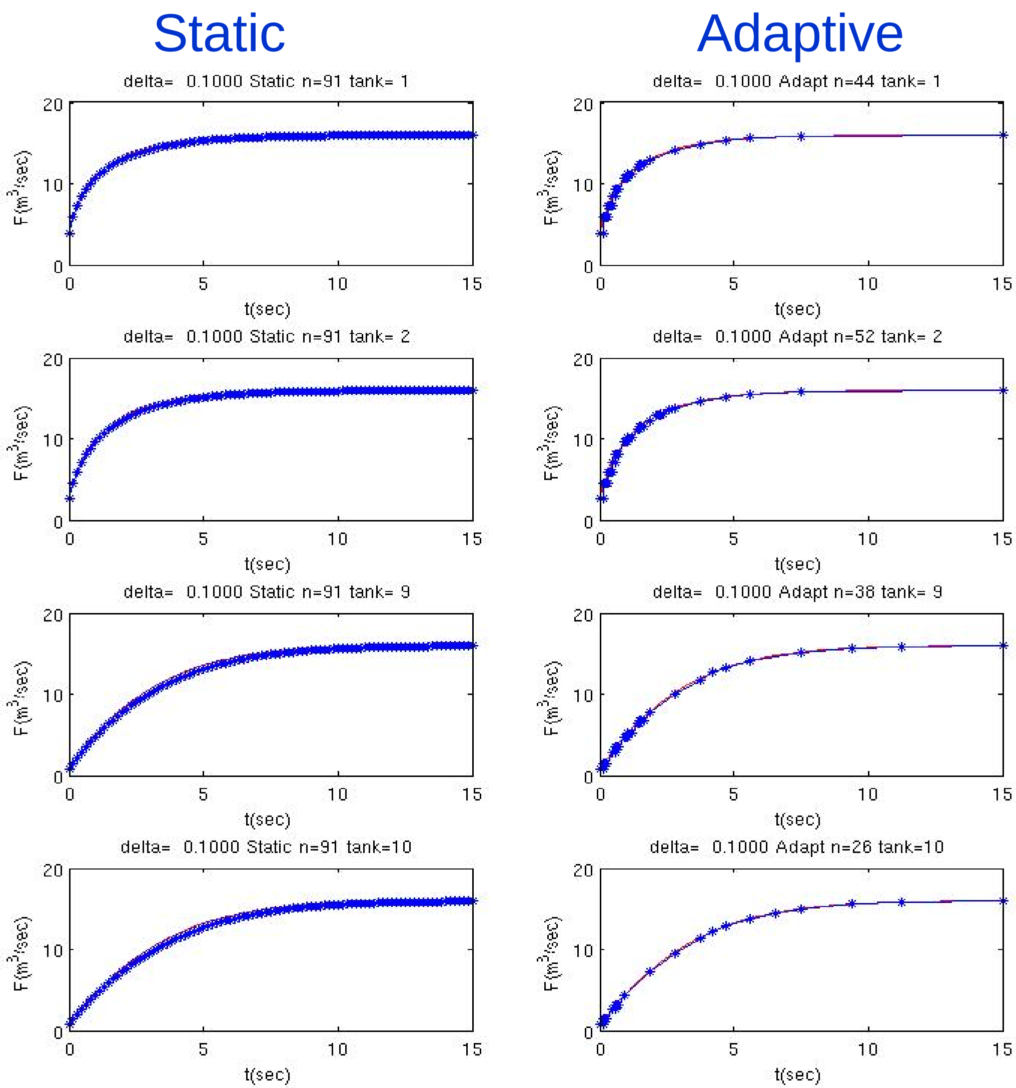
TPWL time coefficients-ten tanks

Static | Adaptive



Left-hand side are static TPWL for POD time coefficients with TPWL (29 intervals), and right-hand side are adaptive TPWL (15 intervals)

Results of TPWL with POD for 10 tanks



10 tanks (showing 1, 2, 9, and 10) w.r.t. time using PODs, F_0=16, $,\delta = 0.1$ and $dt = 0.15$

Left-hand side static TPWL, right-hand side are adaptive TPWL

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Control formulation—10 tanks

with POD method

- Objective function: quadratic due to POD formulation

$$J = \min_{du} \left((\sum_{k=1}^{m} \alpha_k(t)\varpi_k(x) + \overline{F_{10}}) - Y_{ref} \right)^T Q \left((\sum_{k=1}^{m} \alpha_k(t)\varpi_k(x) + \overline{F_{10}}) - Y_{ref} \right) + DU^T RDU$$

(Equation 17)

- s.t.

$$\int_0^{\Omega} \{\phi_i(x) \cdot [\frac{\partial(\sum_{k=1}^{m} \alpha_k(t)\varpi_k(x) + \overline{F_i})}{\partial t} - \frac{(\sum_{k=1}^{m} \alpha_k(t)\varpi_k(x) + \overline{F_{i-1}}) \cdot c_i^2}{(\sum_{k=1}^{m} \alpha_k(t)\varpi_k(x) + \overline{F_i}) \cdot 2A_i} + \frac{c_i^2}{2A_i}]\} \cdot \varpi_j(x)dx = 0$$

(Equation 18)

where, $i = 1,\Lambda\ ,10$, $\Omega = \sum_{i=1}^{10} dx_i$, and $j = 1,\Lambda\ ,m$

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Control formulation—10 tanks

with TPWL method

- Piece-wise linear form on State Space Model:

$$\alpha(t+1) = L_1\alpha(t) + B_1F_0(t)$$

$$\alpha(t+p/t_n) = L_p\alpha(t+(p-1)/t_n) + B_pF_0(t+(p-1)/t_n)$$

$$y(t) = H\alpha(t) + F_m(x_{10})$$

(Equation 19)

where,

$$H = [\varpi_1(x_{10}), \varpi_2(x_{10}), \Lambda\ \varpi_m(x_{10})]^T$$

Control law for TPWL model

Quadratic Programming applied to obtain the control law:

$$DF_0 = (G_{y1}^T Q G_{y1} + rI)^{-1} G_{y1}^T Q[Y_{ref} - G_1\alpha(t) - G_{u1}F_0(t-1)]$$ (Equation 20)

Where, $r = 1$ because only one output;

$$G_{y1} = \begin{bmatrix} HB_1 & 0 & \cdots & 0 \\ HB_2 + HL_2B_1 & HB_2 & 0 & 0 \\ \vdots & \vdots & \cdots & 0 \\ HB_n + HL_nB_{n-1} + \cdots + HL_nL_{n-1}\cdots L_2B_1 & HB_n + HL_nB_{n-1} + \cdots + HL_nL_{n-1}\cdots L_3B_2 & \cdots & HB_n \end{bmatrix}$$

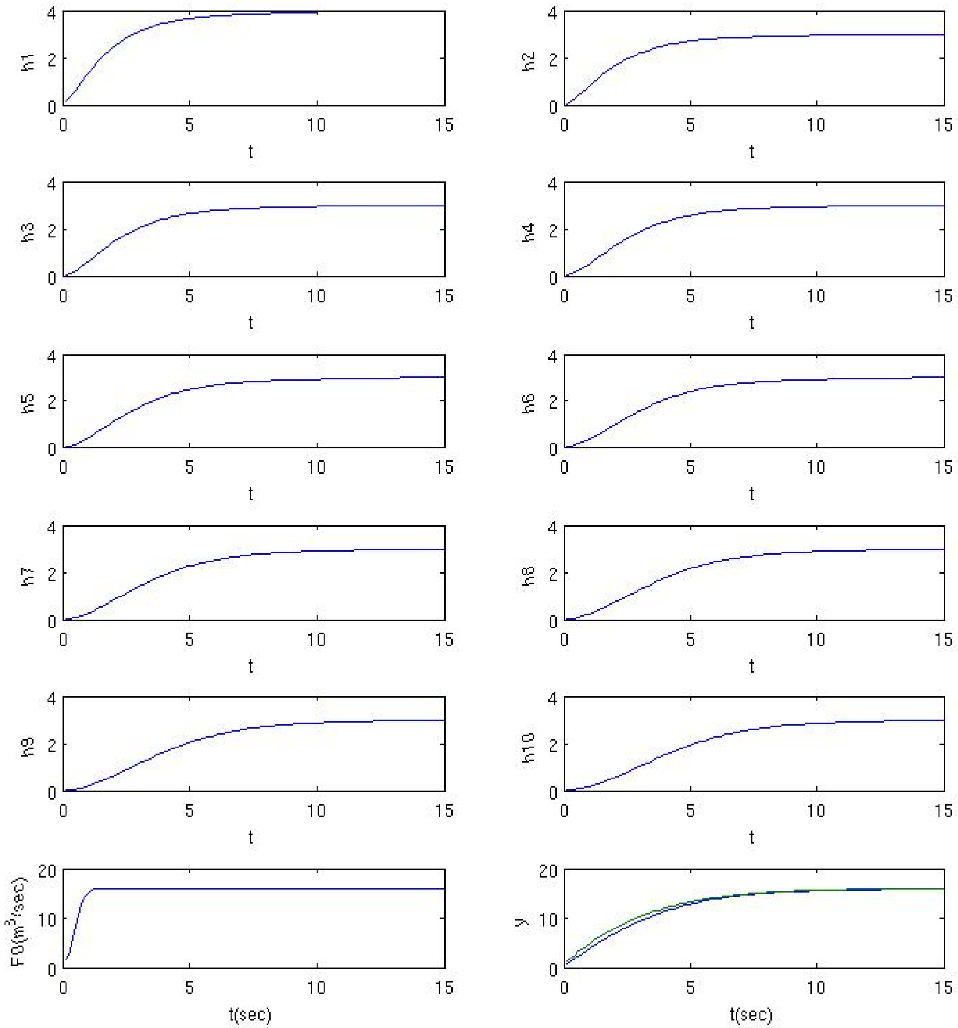
$$G_{u1} = \begin{bmatrix} HB_1 \\ HB_2 + HL_2B_1 \\ \vdots \\ HB_n + HL_nB_{n-1} + \cdots + HL_nL_{n-1}\cdots L_2B_1 \end{bmatrix}$$ and $$G_1 = \begin{bmatrix} HL_1 \\ HL_2L_1 \\ \vdots \\ HL_nL_{n-1}\cdots L_1 \end{bmatrix}$$

Then the output variables can be calculated using

$$Y = G_1\alpha(t) + G_{y1}DF_0(t) + G_{u1}F_0(t-1) + F_m(x_{10})$$ (Equation 21)

SQP results of nonlinear model—10 tanks

using direct ODE solver, dt=0.15sec



Results of nonlinear case using direct ODE's solver and

(1-10) liquid level of tank 1-10;

(11) control input;

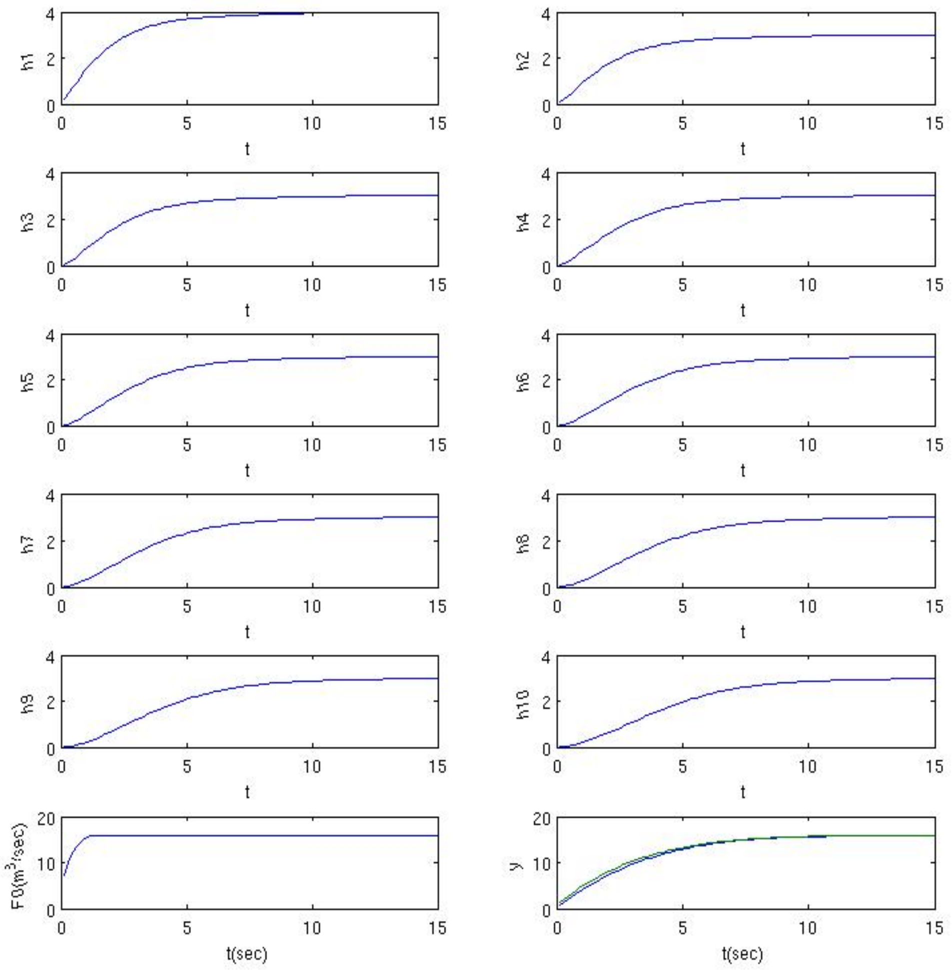
(12) output of tank ten compared to reference output

Size of problem: 10*100

Number of ODEs: 10

SQP results of nonlinear model—10 tanks

(3 basis functions) using direct ODE solver, dt=0.15sec

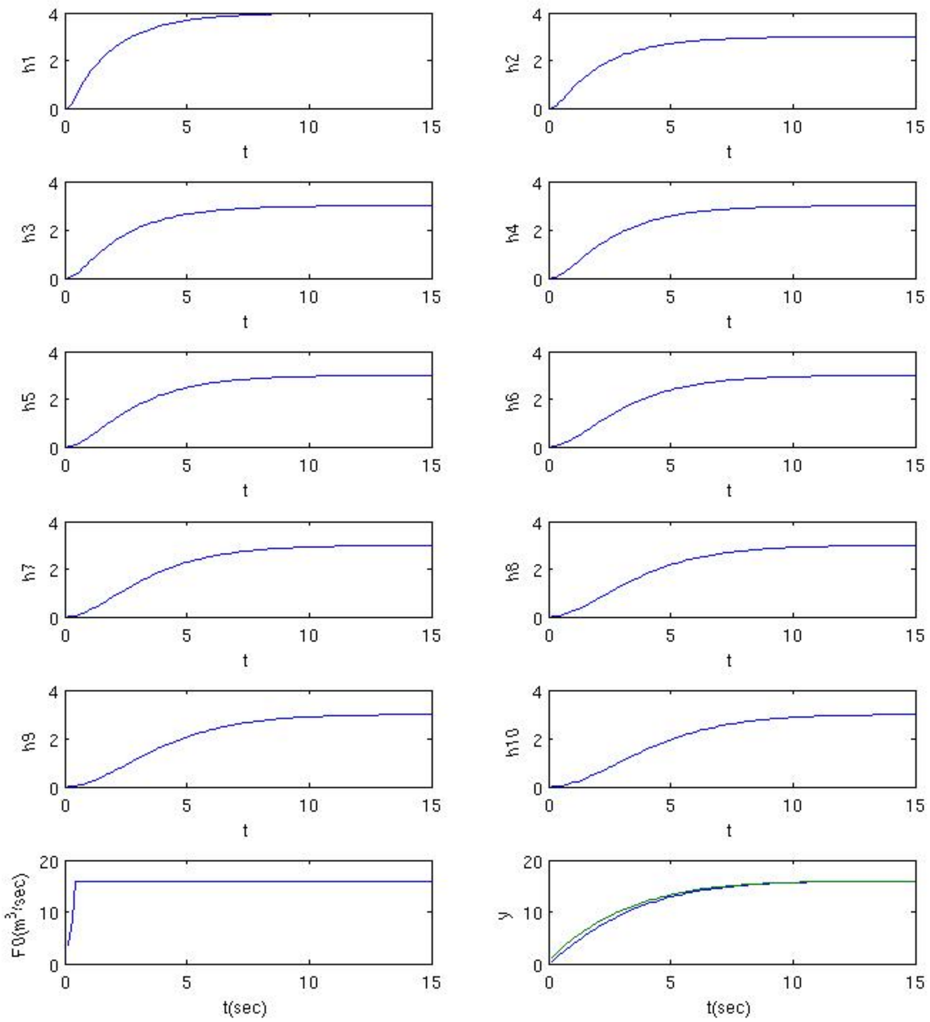


Results of nonlinear reduced model using direct ODE's solver and
(1-10) liquid level of tank 1-10;
(11) control input;
(12) output of tank ten compared to reference output

Size of problem: 3*100
Number of ODEs: 3

TPWL results of nonlinear model—10 tanks

using 3 basis functions, dt=0.15sec



Results of nonlinear case using PWL POD solver and

(1-10) liquid level of tank 1-10;

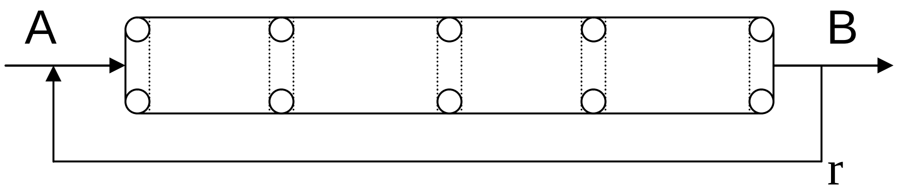
(11) control input;

(12) output of tank ten compared to reference output

Size of problem: 3*100

Number of ODEs: 3

Case study 2: Tubular reactor



r : recycling ratio

$$C_t = -\frac{\partial C}{\partial z} + \frac{1}{Pe_C}\frac{\partial^2 C}{\partial z^2} - f(C,T)$$ (equation 22)

$$T_t = -\frac{\partial T}{\partial z} + \frac{1}{Pe_T}\frac{\partial^2 T}{\partial z^2} + B_T f(C,T) + \beta_T (T_c - T)$$ (equation 23)

$$f(C,T) = B_C C \exp\left(\frac{\gamma T}{1+T}\right)$$

$$z = 0$$

$$\frac{\partial C}{\partial z} = -Pe_C[(1-r)(1+C_0) + rC(t,1) - C(t,0)] \qquad \frac{\partial T}{\partial z} = -Pe_T[(1-r)(1+T_0) + rT(t,1) - T(t,0)]$$

$$z = 1$$

$$\frac{\partial C}{\partial z} = \frac{\partial T}{\partial z} = 0$$

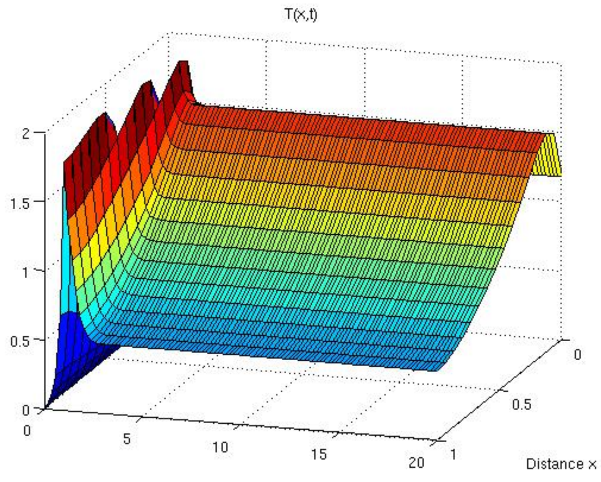
$Pe_C = 7.0$ $\quad Pe_T = 7.0$ $\quad B_C = 0.1$ $\quad B_T = 2.5$ $\quad \gamma = 10.0$ $\quad \beta_T = 2.0$ $\quad C_0 = T_0 = T_c = 0$

Tubular reactor

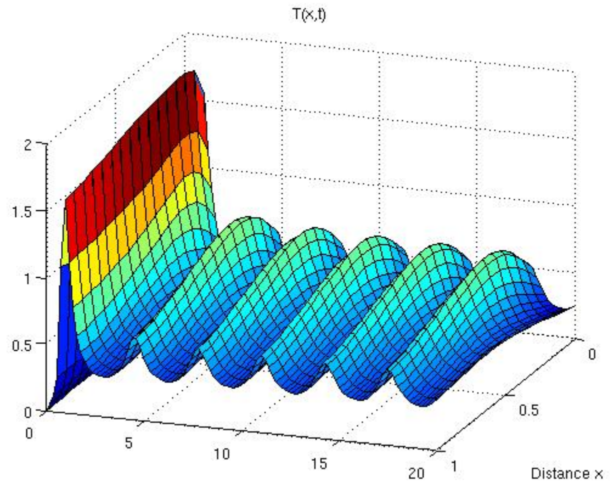
- PDE-based model
- Complex dynamics
  - Rich parametric space, bifurcations
    - Saddle nodes
    - Sustained oscillations
- Appropriate control problem
  - Through a number of system parameters
    - Recycle
    - Jacket temperature

Control objective

- For r=0 stable behaviour
- For r=0.5 Hopf bifurcation
  - Sustained oscillations
- Use a set of cooling zones
  - stabilise system at r=0.5
    - To behave like system at r=0



r=0



r=0.5

Sampling

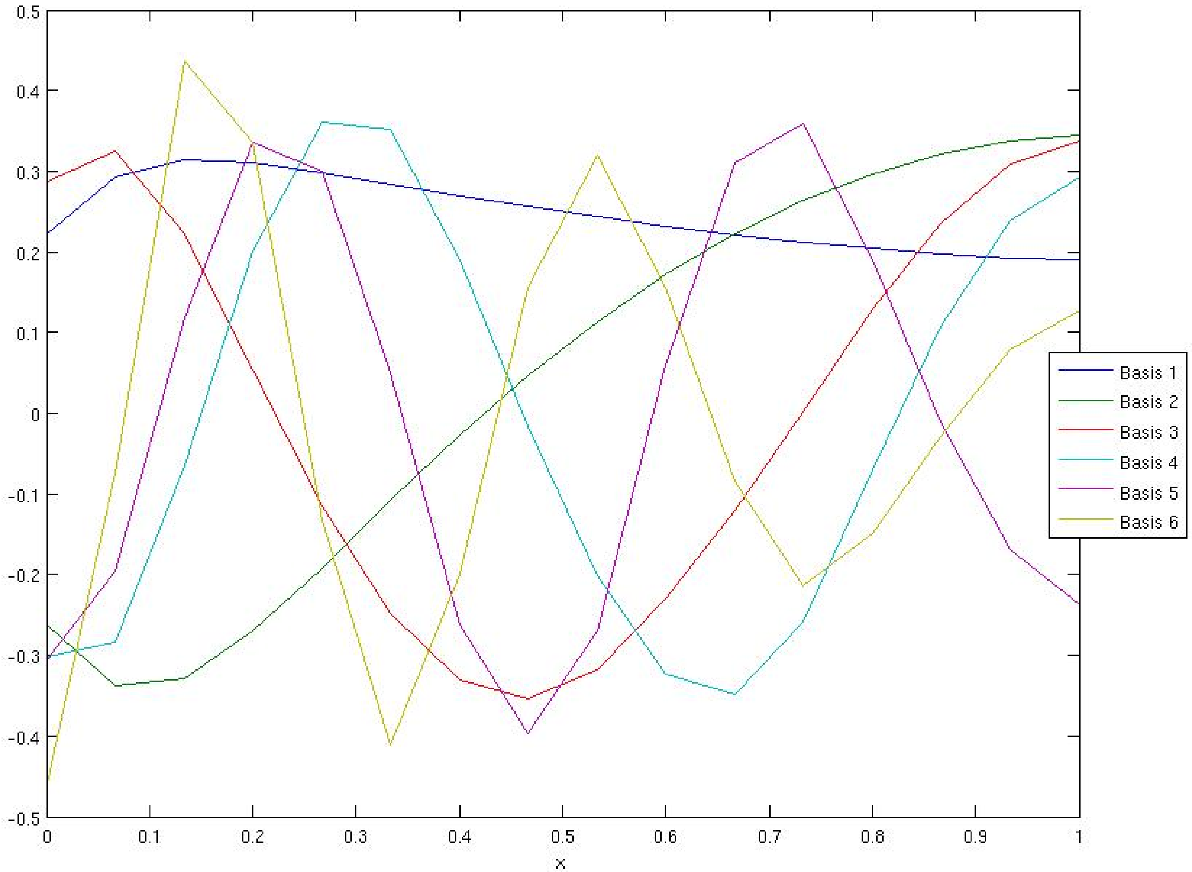
- Heaviside functions
  - for example: 3 actuators have 8 states and 5 actuators have 32 states
- 11 samples for every Heaviside functions
  - Temperature [-0.999,1]
  - Concentration [0,1]

So, 8 x 11=88 samples for 3 actuators,
and 32 x 11=352 samples for 5 actuators

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

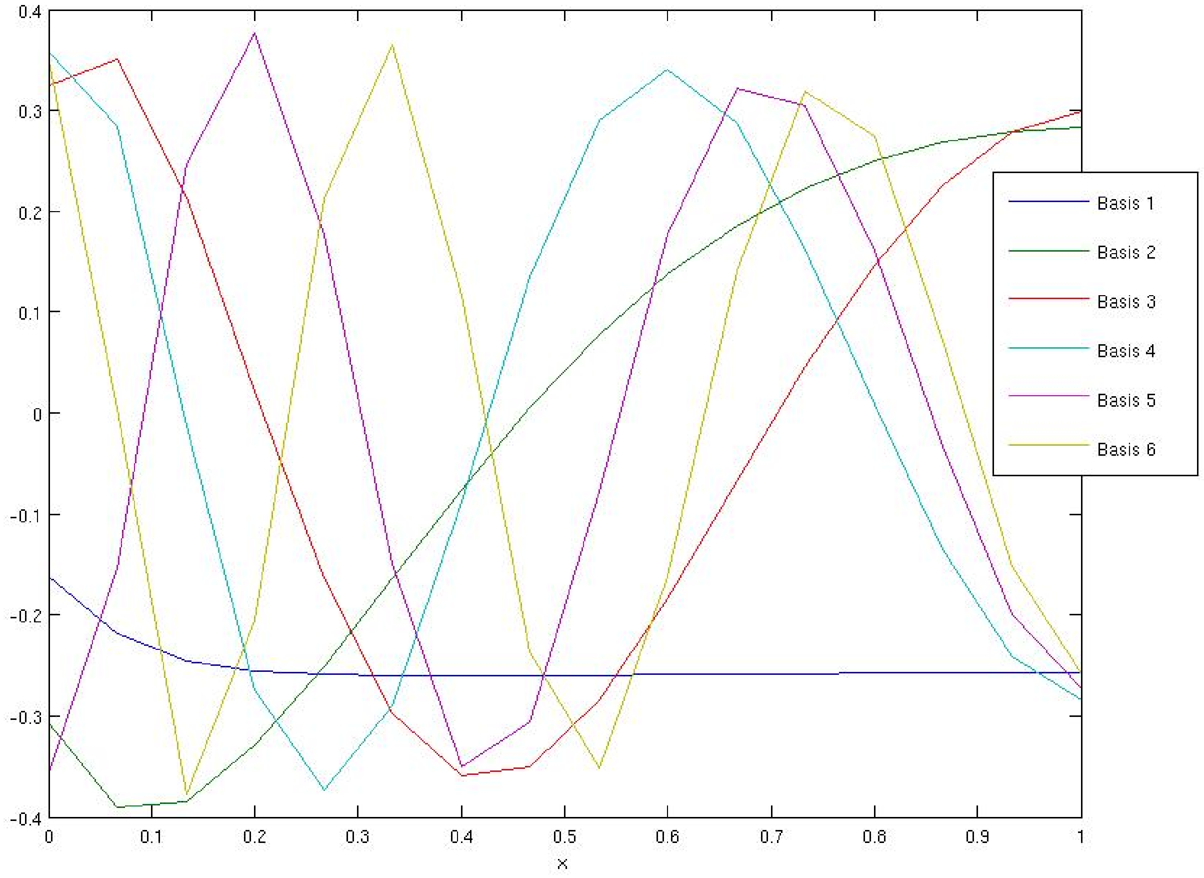
Tubular reactor
- 6 POD basis functions for temperature

- With 5 actuators and Heaviside functions



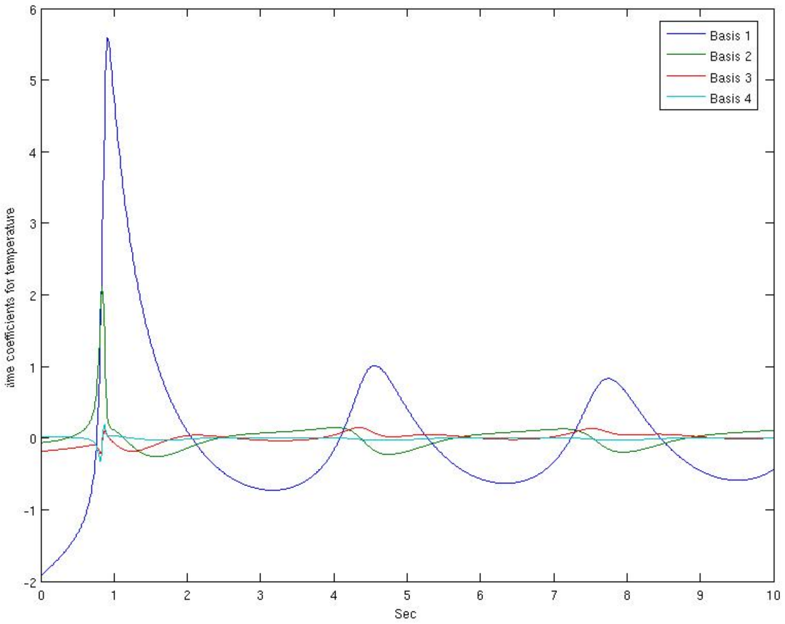
Tubular reactor
- 6 POD basis functions for concentration

- With 5 actuators and Heaviside functions

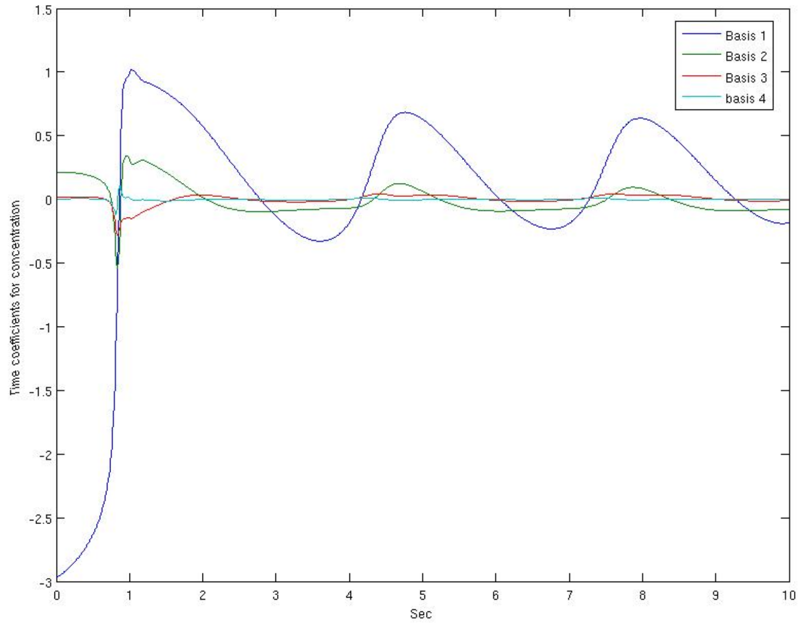


Tubular reactor: Time coefficients

Temperature with r=0.5

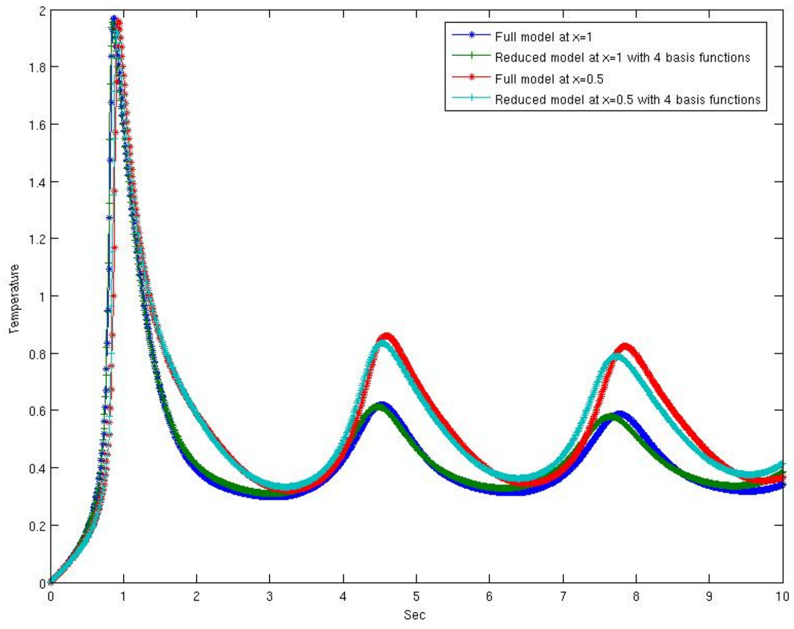


Concentration with r=0.5

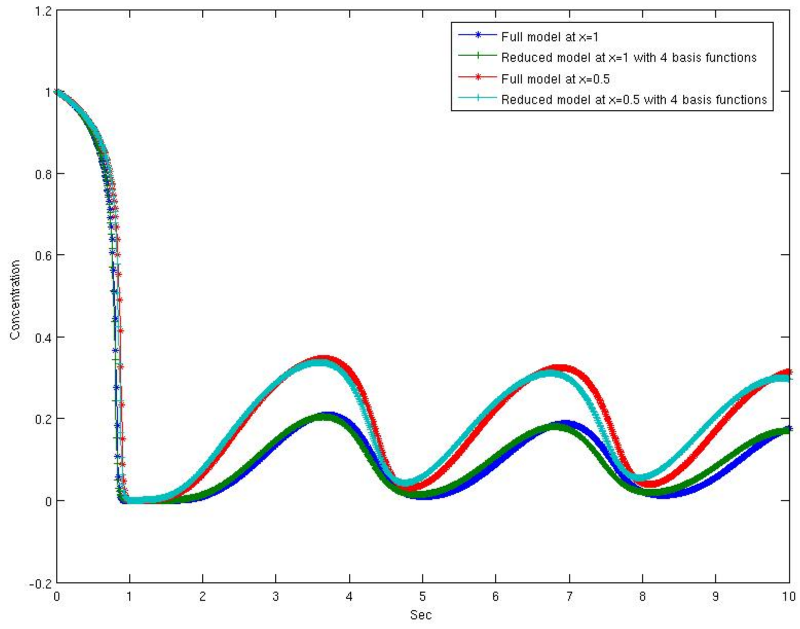


Full vs. reduced model

Temperature with r=0.5



Concentration with r=0.5



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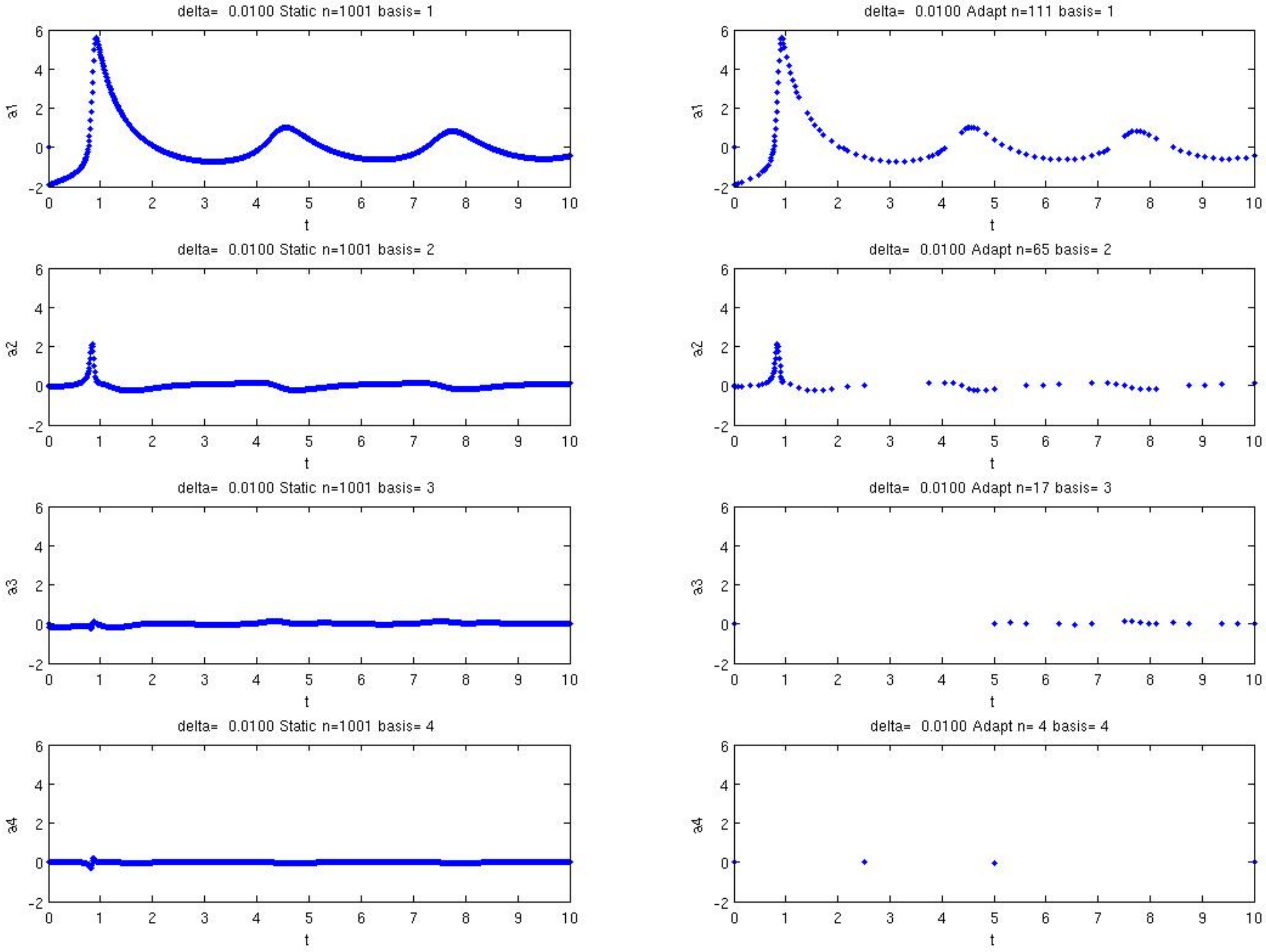
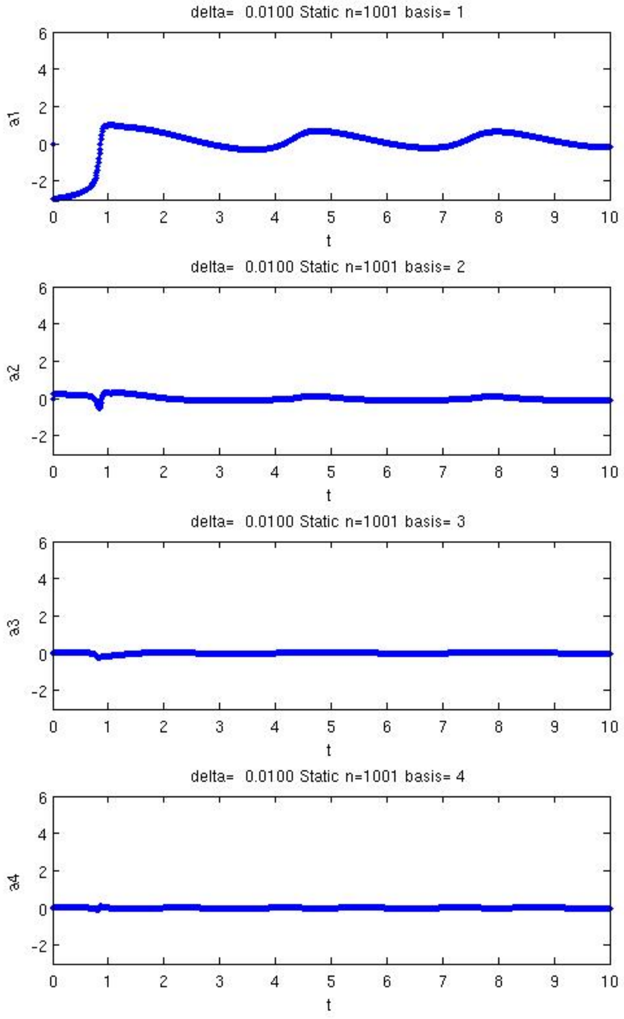
Linearisation of temperature time coefficients

Static | Adaptive

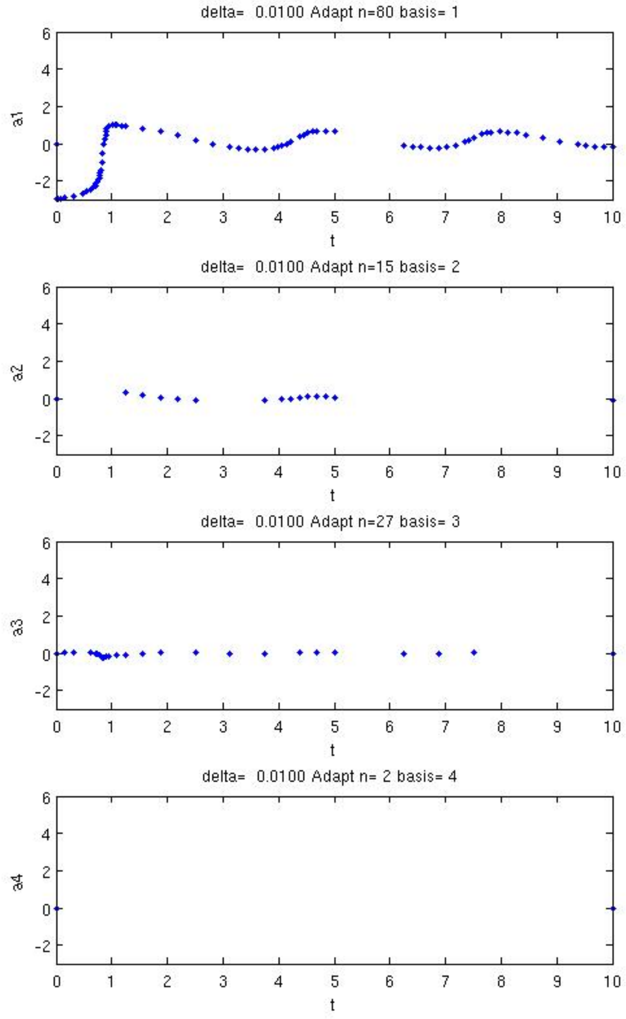


Linearisation of concentration time coefficients

Static



Adaptive



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Control formulation—tubular reactor
with POD method

- Objective function: quadratic due to POD formulation

$$J=\min_{du}\left((\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(x)+\overline{T_{16}})-T_{ref}(t)\right)^{T}Q\left((\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(x)+\overline{T_{16}})-T_{ref}(t)\right)+DU^{T}RDU$$

where, $T_{ref}(t)$ is the reference state with r=0, (equation 24)

DU is control on actuators

- s.t.

$$\frac{\partial(\sum_{k=1}^{m}\alpha_{k\_C}(t)\varpi_{k\_C}(z)+\overline{C(z)})}{\partial t}=-\frac{\partial(\sum_{k=1}^{m}\alpha_{k\_C}(t)\varpi_{k\_C}(z)+\overline{C(z)})}{\partial z}+\frac{1}{Pe_C}\frac{\partial^2(\sum_{k=1}^{m}\alpha_{k\_C}(t)\varpi_{k\_C}(z)+\overline{C(z)})}{\partial z^2}$$

$$-B_C(\sum_{k=1}^{m}\alpha_{k\_C}(t)\varpi_{k\_C}(z)+\overline{C(z)})\exp\left(\frac{\gamma(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}{1+(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}\right)$$

$$\frac{\partial(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}{\partial t}=-\frac{\partial(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}{\partial z}+\frac{1}{Pe_T}\frac{\partial^2(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}{\partial z^2}$$

$$+B_TB_C(\sum_{k=1}^{m}\alpha_{k\_C}(t)\varpi_{k\_C}(z)+\overline{C(z)})\exp\left(\frac{\gamma(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}{1+(\sum_{k=1}^{m}\alpha_{k\_T}(t)\varpi_{k\_T}(z)+\overline{T(z)})}\right)+\beta_T(U-T)$$

(equation 25)

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Control formulation—POD method
with TPWL method

- Piece-wise linear form on State Space Model:

$$\alpha(t+1) = L_1\alpha(t) + B_1U(t)$$

$$\alpha(t+p/t_n) = L_p\alpha(t+(p-1)/t_n) + B_pU(t+(p-1)/t_n)$$

$$y(t) = H\alpha(t) + \overline{T_{16}}$$ (equation 26)

where, $\alpha(t)$ includes time coefficients for concentration and temperature,

$$H = [\underbrace{0,0,\Lambda,0}_{m},\varpi_{1\_T}(z_{16}),\varpi_{2\_T}(z_{16}),\Lambda\ \varpi_{m\_T}(z_{16})]^T$$

Control law for TPWL model

Quadratic Programming applied to obtain the control law:

$$DU = (G_{y1}^T Q G_{y1} + rI)^{-1} G_{y1}^T Q [Y_{ref} - G_1 \alpha(t) - G_{u1} U(t-1)] \qquad \text{(equation 27)}$$

Where, $r = 1$ because only one output;

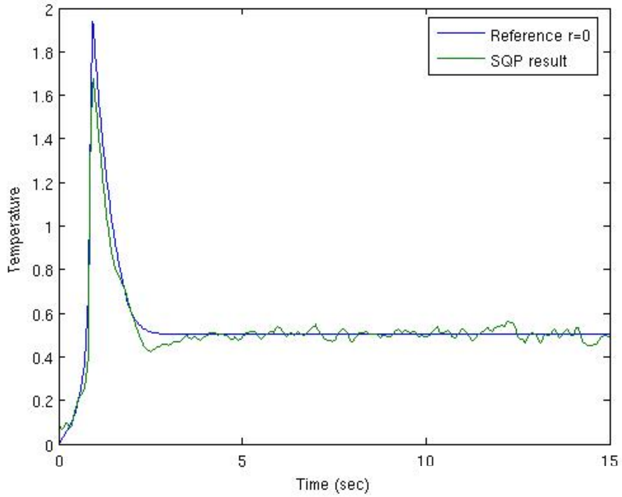
$$G_{y1} = \begin{bmatrix} HB_1 & 0 & \mathrm{K} & 0 \\ HB_2 + HL_2 B_1 & HB_2 & 0 & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{K} & 0 \\ HB_n + HL_n B_{n-1} + \Lambda + HL_n L_{n-1} \Lambda\ L_2 B_1 & HB_n + HL_n B_{n-1} + \Lambda + HL_n L_{n-1} \Lambda\ L_3 B_2 & \mathrm{K} & HB_n \end{bmatrix}$$

$$G_{u1} = \begin{bmatrix} HB_1 \\ HB_2 + HL_2 B_1 \\ \mathrm{M} \\ HB_n + HL_n B_{n-1} + \Lambda + HL_n L_{n-1} \Lambda\ L_2 B_1 \end{bmatrix} \quad \text{and} \quad G_1 = \begin{bmatrix} HL_1 \\ HL_2 L_1 \\ \mathrm{M} \\ HL_n L_{n-1} \Lambda\ L_1 \end{bmatrix}$$

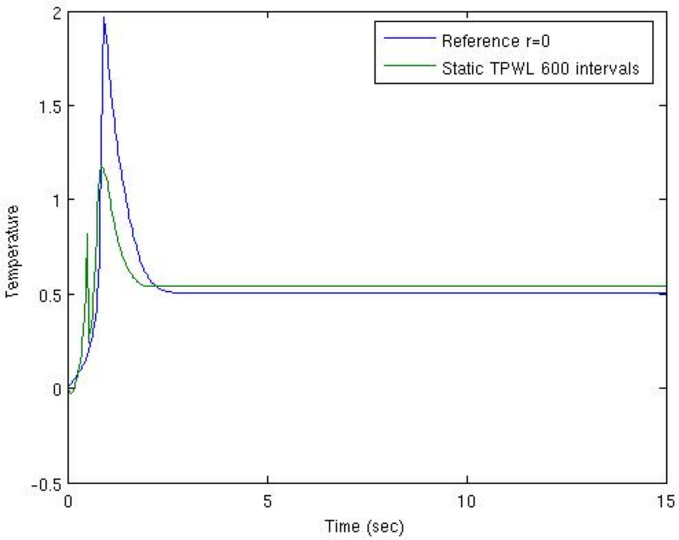
Then the output variables can be calculated using

$$Y = G_1 \alpha(t) + G_{y1} DU(t) + G_{u1} U(t-1) + \overline{T_{16}} \qquad \text{(equation 28)}$$

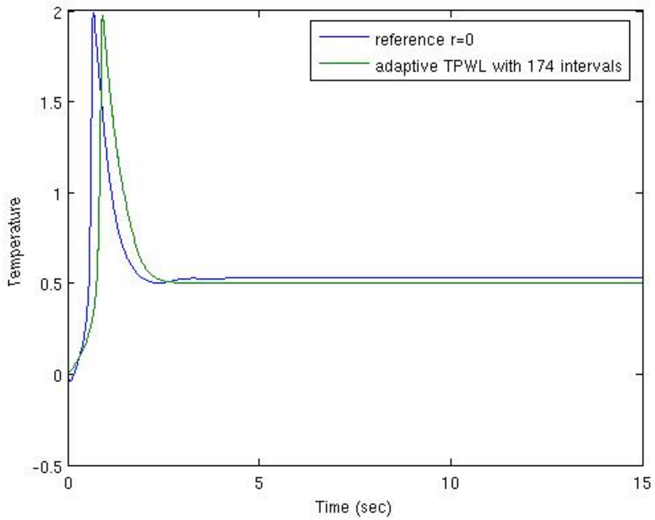
Results
–8 actuators and 15 secs



SQP



Static TPWL



Adaptive TPWL

Conclusions

- A POD-based method has been developed
  - with static or adaptive TPWL
  - for non-linear large scale distributed systems
- Our method catches the dynamics of systems
- Significantly reduces computation time
  - compared to SQP method.
- Applied to both discretised and continuous systems

Future work

- Piece-wise affine reduced model
  - Use PAROS software for parametric control
- Other cases study
  - Microfiltration process
  - Lyophilisation process

Acknowledgements

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- The financial contribution of the EU Progamme ***CAFE*** [KBBE-212754]