On the Strongly Bounded Turing Degrees of Simple Sets

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Overview

- Simple sets and Post's Program
- The strongly bounded Turing reducibilities: identity bounded Turing reducibility and computable Lipschitz reducibility
- Simple sets and the strongly bounded Turing reducibilities
- Strongly bounded Turing degrees of simple sets

Simple sets and Post's Program

Post's Problem and Post's Program (1944)

- Post raised the question of whether there are computably enumerable (c.e.) sets which are neither (Turing-) complete nor computable ("Post's Problem")

  In terms of degrees this can be rephrased by

  "Is there a c.e. Turing degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$?"

- Post proposed to solve this problem by a structural approach (Post's Program):

  "Find a property $\mathcal{P}$ of c.e. sets such that a set A with this property is neither (Turing-) complete nor computable"

  In particular he conjectured that a sufficiently strong "thinness" or "simplicity" property $\mathcal{P}$ may have this property.

Simple sets (Post 1944)

DEFINITION. A c.e. set A is simple, if

- the complement $\overline{A}$ of A is infinite but
- $\overline{A}$ does not contain any infinite c.e. set as a subset.

- Note that a simple set A is not computable since its complement is not c.e.
- Also note that a coinfinite c.e. set A is simple iff A does not contain any infinite computable set as a subset (since any infinite c.e. set contains an infinite computable set).

Simple sets are m-incomplete but can be tt-complete (Post 1944)

Post showed that if we replace Turing-reducibility by stronger reducibilities then his program works:

- Simple sets are not complete under many-one- (m-) reducibility (one oracle query; positive evaluation).
- But this is not true anymore if we replace m-reducibility by truth-table- (tt-) reducibility (computable bound on the number of (nonadaptive) oracle queries; evaluation of the queries by a priori specified truth tables).

  Post also showed, however, that if we match the additional power of a tt-reduction in comparison with an m-reduction (a growing finite number of queries in place of a single query) by a corresponding stronger simplicity concept then this property guarantees not only m-incompleteness but also tt-incompleteness.

Hyper-simple sets are tt-incomplete (Post 1944)

DEFINITION. A c.e. set A is hyper (h-) simple, if

- the complement $\overline{A}$ of A is infinite but
- for any computable function f such that, for $x \neq y$, $D_{f(x)}$ and $D_{f(y)}$ are disjoint there is a number x such that $D_{f(x)} \subseteq A$.

- Here D_n is the n-th finite set given by its canonical index.
- Note that a simple set A has the above properties if we require that $|D_{f(x)}| = 1$.

THEOREM (Post 1944; Friedberg and Rogers 1959). Let A be h-simple. Then A is not tt-complete; in fact, not wtt-complete. (Here A is weak-truth-table or bounded Turing reducible to B if A is Turing reducible to B and there is a computable bound on the oracle queries.)

THEOREM (Dekker 1954). There is a Turing-complete h-simple set. In fact, any c.e. T-degree $\neq \mathbf{0}$ contains an h-simple set.

Hyper-hyper-simple sets (Post 1944) and the failure of Post's Program

Post suggested to match the difference between a (weak) truth-table reduction and a Turing reduction by a further strengthening of h-simplicity:

DEFINITION. A c.e. set A is hyper-hyper (hh-) simple, if

- the complement $\overline{A}$ of A is infinite but
- for any computable function f such that, for $x \neq y$, $W_{f(x)}$ and $W_{f(y)}$ are disjoint finite sets there is a number x such that $W_{f(x)} \subseteq A$.
- Here W_n is the n-th c.e. set given by its c.e. index.

But this property - and even the strongest possible simplicity property in the sense of Post, namely maximality - does not guarantee Turing-incompleteness, which may be interpreted as the failure of Post's program in its original, narrow sense.

THEOREM (Yates 1965). There is a maximal - hence hh-simple - Turing complete set.

Solution of Post's Problem (Friedberg and Muchnik 1956)

In the early 1950s Post worked on some alternative constructive approach (Kleene and Post 1954) which eventually led to the solution of Post's Problem based on the invention of the priority method which became the major tool for the analysis of the (Turing) degrees of the c.e. sets:

THEOREM (Friedberg and Muchnik 1956). There are Turing-incomparable c.e. sets.

This development is often interpreted as follows:

- The structural approach is very powerful for the strong reducibilities but of limited use for the weak reducibilities (in particular for Turing reducibility) where brute force arguments are more promising.

Still Post's Program is of interest also for the weaker reducibilities, and - taken in a broader sense - it also led to a number of solutions of Post's Problem.

Solutions of Post's Problem in the sense of Post's Program

Some generalizations and variations of the "thinness" properties of Post provide solutions of Post's Problem along the lines of Post's Program by giving properties of c.e. sets guaranteeing Turing-incompleteness:

- (Degtev 1973, Marchenkov 1976) η-hh-simple semirecursive sets (based on equivalence relations η)
- (Harrington and Soare) "Tardy" sets (inspired by Maass's promptly simple sets combining simplicity with a dynamical property)

"Generalized" Post's Program

Post's Program has the goal to prove the existence of an incomplete c.e. degree $> \mathbf{0}$ by giving a property $\mathcal{P}$ such that

- $\mathbf{0}$ and $\mathbf{0}'$ do not have property $\mathcal{P}$ (i.e., do not contain sets with this property)
- there is a c.e. degree **a** with property $\mathcal{P}$

The same goal can be achieved by giving a property $\mathcal{P}$ such that

- $\mathbf{0}$ and some c.e. degree $> \mathbf{0}$ do not have property $\mathcal{P}$
- there is a c.e. degree **a** with property $\mathcal{P}$

(Generalized Post's Program)

"Generalized" Post's Program vs. Post's Program

	PP				GPP			
	m	tt	wtt	T	m	tt	wtt	T
simple	+	−	−	−	+	+	−	−
h-simple	+	+	+	−	+	+	+	−
hh-simple	+	+	+	−	+	+	+	+

Jockusch (1981): There is a nonsimple c.e. tt-degree $> \{0\}$.

Martin (1963): The hh-simple Turing degrees are the high c.e. Turing degrees.

So, in particular, hh-simplicity solves Post's Problem in the sense of "Generalized" Post's Program.

Simple sets and strongly bounded Turing reducbility

In the following we will explore the relations between simplicity and the strongly bounded Turing reducibilities (identity bounded Turing reducibility and computable Lipschitz reducibility).

These reducibilities are obtained by putting very strong bounds on the size of the oracle queries but at the same time do not restrict the evaluation process.

So they are incompatible with the truth table type reducibilities m and tt (but stronger than wtt).

The strongly bounded Turing reducibilities:

identity bounded Turing reducibility and computable Lipschitz reducibility

The (strongly) bounded Turing reducibilities

In contrast to the truth-table-type reducibilities of Post, the bounded Turing reducibilities are obtained by imposing bounds $b(x)$ on the oracle queries in a Turing reduction $A(x) = \Phi^B(x)$ but not limiting the evaluation process:

- $b(x)$ computable: weak truth-table (wtt) (or bounded Turing (bT))
- $b(x) = id(x) = x$: identity bounded Turing (ibT)
- $b(x) = id(x) + c = x + c$: computable Lipschitz (cl) (or strong weak truth-table (sw))

In the following we refer to ibT and cl as the strongly bounded Turing (sbT) reducibilities.

Origins and Properties

- cl-Reducibility was introduced by Downey, Hirschfeldt and LaForte (2001) in the context of computable randomness. The special case of ibT-reducibility was introduced by Soare (2004) in the context of some applications of computability theory to differential geometry (Nabutovski and Weinberger).
- cl-Reducibility preserves Kolmogorov complexity: For a set A which is cl-reducible to a set B, the Kolmogorov complexity of $A \upharpoonright n$ is bounded by the Kolmogorov complexity of $B \upharpoonright n$ up to an additive constant. Moreover, Downey, Hirschfeldt and LaForte have shown that, on the computably enumerable (c.e.) sets, cl-reducibility coincides with Solovay reducibility which may be viewed as a relative measure of the speed by which a real number can be effectively approximated by rational numbers.
- For $r = \mathrm{ibT}, \mathrm{cl}$, r-reducibility is a preordering, closed under finite variants, but not computably invariant.

Examples of ibT-reductions on the c.e. sets

Some typical, frequently used examples of ibT-reductions on the c.e. sets are the following.

- Splitting

$$A = B \dot{\cup} C \;\Rightarrow\; B \leq_{\mathrm{ibT}} A \;\text{ and }\; C \leq_{\mathrm{ibT}} A$$

  In fact, $deg_r(A) = deg_r(B) \vee deg_r(C)$ for $r = \mathrm{ibT}, \mathrm{cl}$.

- Permitting

$$x \in A_{at\ s} \;\Rightarrow\; \exists\, y \leq x\, (y \in B_{at\ s})$$

  In fact, any ibT-reduction $A \leq_{\mathrm{ibT}} B$ among c.e. sets may be represented by a permitting reduction (by replacing A and B by appropriate ibT-equivalent subsets).

Note that neither of the above reductions is a truth-table reduction.

Strongly Bounded Turing Degrees of C.E. Sets

- The study of the partial orderings $(\mathbf{R}_{\mathrm{ibT}}, \leq)$ and $(\mathbf{R}_{\mathrm{cl}}, \leq)$ of the strongly bounded Turing degrees of the computably enumerable sets is a fairly new subject.

- The investigations use the methods known from the Turing degrees (in particular priority arguments) but also some techniques specific for the sbT-degrees (e.g. shift and density arguments).

  Moreover some transfer techniques have been developed which allow to transfer certain results on the wtt-degrees to the sbT-degrees (though these degree structures look quite different).

The partial orderings $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ of the c.e. sbT-degrees

- $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ is a partial ordering with least element $\mathbf{0} = \{A : A \text{ comput.}\}$ but neither an upper semi-lattice nor a lower semi-lattice (though every degree is join-reducible and meet-reducible). Moreover, $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ is neither dense (though downward dense) nor distributive.
- $(\mathrm{R}_{\mathrm{sbT}}, \leq)$ does not have maximal (hence no greatest) elements (Barmpalias 05). So there are no sbT-complete sets (hence Post's Program in the narrow sense does not make sense for these reducibilities!).
  - For ibT the proof is very simple: For any noncomputable set A, $A <_{\mathrm{ibT}} A - 1 := \{x : x + 1 \in A\}$
  - For cl a similar idea works using unbounded shifts: Given a noncomputable c.e. set A and an infinite computable subset B of A, $A <_{\mathrm{cl}} (A \setminus B)_B$ where $(A \setminus B)_B$ is the compression of $A \setminus B$ using the space in B.

Moreover, there are *maximal pairs*, i.e., c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ s.t. there is no c.e. degree $\mathbf{c} \geq \mathbf{a}, \mathbf{b}$ (Barmpalias 05; Fan and Lu 05).

Simple sets and the strongly bounded Turing reducibilities

Simple sets and the strongly bounded Turing reducibilities

- By Dekker's Theorem, any c.e. Turing degree $> \mathbf{0}$ is simple, i.e., contains a simple set.
- This has been strengthened as follows: any c.e. wtt-degree $> \mathbf{0}$ is simple.

  In fact, any linearly bounded ($c \cdot x$) Turing degree of a noncomputable c.e. set contains a simple set.
- Can this be further improved by showing that any nonzero c.e. cl-degree or even ibT-degree contains a simple set?

  Before we answer this question, we first note that the answers for cl and ibT have to be the same.

Coincidence Theorem

Theorem (Coincidence Theorem)

For any c.e. set A the following are equivalent.

(i) $deg_{\mathrm{ibT}}(A)$ *is simple.*

(ii) $deg_{\mathrm{cl}}(A)$ *is simple.*

So in order to show that there is a noncomputable c.e. set A which is not cl-equivalent to any simple set it suffices to construct a noncomputable c.e. set A which is not ibT-equivalent to any simple set.

Nonsimple sbT-degrees

Theorem

There is a noncomputable c.e. set A such that $deg_{\mathrm{ibT}}(A)$ is not simple.

Corollary

There is a noncomputable c.e. set A such that $deg_{\mathrm{cl}}(A)$ is not simple.

A c.e. set A with the required properties is constructed by a finite-injury argument.

A nonsimple c.e. ibT-degree: Requirements

- Noncomputability requirements:

$$\mathfrak{R}_{2e} : \; A \neq \varphi_e$$

- Nonsimplicity requirements:

$$\mathfrak{R}_{2e+1} : \; [A = \hat{\Phi}_{e_1}^{W_{e_0}} \,\&\, W_{e_0} = \hat{\Phi}_{e_2}^{A}] \;\Rightarrow\; W_{e_0} \text{ is not simple}$$

where $(\hat{\Phi}_n)_{n \geq 0}$ is an effective enumeration of the ibT-functionals.

A nonsimple c.e. ibT-degree: Noncomputability requ's

- Strategy for meeting a noncomputability requirement (standard)

$$\mathfrak{R}_{2e} : \; A \neq \varphi_e$$

  - Wait for a stage $s > 2e$ such that $\varphi_{e,s}(x) = 0$ for some $x \in \omega^{[2e]}$.

    ACTION. Put x into A at stage $s+1$ thereby ensuring $A(x) \neq \varphi_e(x)$.

(Here the infinite computable set $\omega^{[n]}$ is reserved for requirement $\mathfrak{R}_n$. Numbers from this set may be put into A only by $\mathfrak{R}_n$.)

A nonsimple c.e. ibT-degree: Nonsimplicity requ's

- Strategy for meeting a nonsimplicity requirement

$$\mathfrak{R}_{2e+1} : \ [A = \hat{\Phi}_{e_1}^{W_{e_0}} \ \& \ W_{e_0} = \hat{\Phi}_{e_2}^{A}] \ \Rightarrow \ W_{e_0} \text{ is not simple}$$

  - Wait for a stage $s > 2e+1$ such that, for some $x \in \omega^{[2e+1]}$,
    - ★ $x \in W_{e_0,s}$ and
    - ★ $A_s(x) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(x) = 0 \ \& \ W_{e_0,s} \upharpoonright x = \hat{\Phi}_{e_2,s}^{A_s} \upharpoonright x$.

    ACTION. Put x into A at stage $s+1$ and restrain all numbers $< x$ from A.

    Assuming $A = \hat{\Phi}_{e_1}^{W_{e_0}} \ \& \ W_{e_0} = \hat{\Phi}_{e_2}^{A}$,
    - ★ enumerating x into A forces $W_{e_0} \upharpoonright x+1$ to change after stage s. In fact, by $x \in W_{e_0,s}$, a number $< x$ has to enter W_{e_0} after stage s!
    - ★ restraining the numbers $< x$ from A, however, ensures that $\hat{\Phi}_{e_2}^{A} \upharpoonright x = \hat{\Phi}_{e_2,s}^{A_s} \upharpoonright x$.
    - ★ Hence $W_{e_0} \upharpoonright x \neq \hat{\Phi}_{e_2}^{A} \upharpoonright x$ and thus the hypothesis of $\mathfrak{R}_{2e+1}$ fails!

A nonsimple c.e. ibT-degree: Nonsimplicity requ's

- Strategy for meeting a nonsimplicity requirement

$$\mathfrak{R}_{2e+1} : \; [A = \hat{\Phi}_{e_1}^{W_{e_0}} \;\&\; W_{e_0} = \hat{\Phi}_{e_2}^{A}] \;\Rightarrow\; W_{e_0} \text{ is not simple}$$

  - What happens if we cannot act?

    Then there is no stage $s > 2e + 1$ and no $x \in \omega^{[2e+1]}$ such that

    - ⋆ $x \in W_{e_0,s}$ and
    - ⋆ $A_s(x) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(x) = 0 \;\&\; W_{e_0,s} \upharpoonright x = \hat{\Phi}_{e_2,s}^{A_s} \upharpoonright x$

    So - if the hypothesis of requirement $\mathfrak{R}_{2e+1}$ holds - then

$$\omega^{[2e+1]} \cap W_{e_0} = \emptyset.$$

    So W_{e_0} is not simple!

Strongly bounded Turing degrees of simple sets

Strongly bounded Turing degrees of simple sets

It is natural to ask how the simple and nonsimple degrees sbT-degrees are distributed among all c.e. sbT-degrees.

In case of the many-one degrees the class $\mathbf{S}$ of the simple degrees has the following properties:

- $\mathbf{S} \cup \{\mathbf{0}\}$ is an ideal in $(\mathbf{R}_m, \leq)$.
- There is a nonsimple degree which is minimal, hence incomparable with all simple ones (Odifreddi).

In case of the sbT-degrees, however, the structure of the class $\mathbf{S}$ of the simple degrees is much less nice. In the following we list some properties of $\mathbf{S}$.

Strongly bounded Turing degrees of simple sets

- For any c.e. sbT-degree $\mathbf{a} > \mathbf{0}$, there is a simple sbT-degree above $\mathbf{a}$, below $\mathbf{a}$, and incomparable with $\mathbf{a}$.
- For any c.e. sbT-degree $\mathbf{a} > \mathbf{0}$, there is a nonsimple sbT-degree $\neq \mathbf{0}$ below $\mathbf{a}$ and incomparable with $\mathbf{a}$ (but, in general, not above $\mathbf{a}$).
- For any wtt-complete set A, $deg_{\mathrm{sbT}}(A)$ is simple.
- For any c.e. set A which is not wtt-complete there is a nonsimple sbT-degree above $deg_{\mathrm{sbT}}(A)$.
- Neither the simple nor the nonsimple degrees are dense in $(\mathbf{R}_{\mathrm{sbT}}, \leq)$.
- Any c.e. sbT-degree is the join of two nonsimple c.e. sbT-degrees whereas the class of the nonzero c.e. sbT-degrees is not generated by the simple sbT-degrees under join.

  CONJECTURE. Any c.e. sbT-degree is the meet of two simple c.e. sbT-degrees (whereas, by the above, the class of the nonzero c.e. sbT-degrees is not generated by the nonsimple sbT-degrees under meet).

SbT-degrees and simple sets: Conclusion

- Structure of sets matters for the analysis of the strongly bounded Turing degrees of c.e. sets.
- In particular, the simple sets define a substructure of the c.e. sbT-degrees. In contrast to the case of the simple m-degrees, however, this structure is less nice.
- It seems that other structural properties are of more interest for the study of the c.e. sbT-degrees. In particular, density properties (scattered or sparse sets) which are of no or only small interest for other degree structures have been successfully used in the sbT-degrees.

  For instance, for a set $A \subseteq \{2n : n \geq 0\}$, $deg_{\mathrm{ibT}}(A+1)$ is the greatest degree which does not cup to $deg_{\mathrm{ibT}}(A)$ (from which one can deduce that the theory of the c.e. ibT-degrees is not $\aleph_0$-categorical).

THANK YOU !