Ω–completeness and forcing axioms

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Generalizing Martin's Axiom

This is joint work with Miguel Angel Mota.

Notation: Given a class Γ of partial orders and a cardinal κ, $\mathrm{FA}(\Gamma)_\kappa$ means:

For every $\mathbb{P} \in \Gamma$ and every collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

Definition (Foreman–Magidor–Shelah) MM (Martin's Maximum) is
$\mathsf{FA}(\{\mathcal{P} : \mathcal{P}$ preserves all stationary subsets of $\omega_1\})_{\aleph_1}$.

Theorem: If κ is a supercompact cardinal, then there is a poset $\mathcal{P} \subseteq V_\kappa$ forcing MM. On the other hand, if $\mathcal{P}$ destroys some stationary subset of ω_1, then $\mathsf{FA}(\{\mathcal{P}\})_{\aleph_1}$ is false.

Hence, MM is a provably maximal forcing axiom for collections of $\aleph_1$–many dense sets which is consistent (modulo a supercompact cardinal).

Forcing axioms beyond $H(\omega_2)$

Theorem (Shelah) For every regular cardinal $\kappa > \aleph_1$, FA($\{\mathcal{P} : \mathcal{P}$ preserves all stationary subsets of all regular $\lambda \leq \kappa\})_\kappa$ is false.

In fact, the general picture of forcing axioms at the level of $H(\omega_3)$ or beyond is at present very far from being well understood.

A contribution to forcing axioms beyond $H(\omega_2)$: Generalizing Martin's Axiom

Goal: Generalize Martin's Axiom for a reasonably broad class of $\aleph_2$–c.c. partial orders.

Some limitation is necessary: $\mathrm{Coll}(\omega, \omega_1)$ has size $\aleph_1$, but $\mathrm{FA}(\{\mathrm{Coll}(\omega, \omega_1)\})_{\aleph_1}$ is false.

Definition (Asperó–Mota): A poset $\mathcal{P}$ is *regular* iff

(a) for every $p \in \mathcal{P}$, $p = (\nu, x)$ with $\nu \in \omega_1$, and

(b) for every regular $\lambda \geq |\mathcal{P}|^+$ there is a club $D \subseteq [H(\lambda)]^{\aleph_0}$ such that for all $N_0, \ldots N_m \in D$ ($m < \omega$) and every $(\nu, x) \in \mathcal{P}$, if $\nu < N_i \cap \omega_1$ for all $i \leq m$, then there is $(\nu', x') \leq_{\mathcal{P}} (\nu, x)$ such that (ν', x') is $(N_i, \mathcal{P})$–generic for all i.

(**Note**: We are not asking that $(\nu, x) \in N_i$ for all i.

Let Υ denote the class of all regular posets with the $\aleph_2$–chain condition. Let MA^*_λ denote $\mathrm{FA}(\Upsilon)_\lambda$.

Clearly: ccc $\subseteq \Upsilon \subseteq$ proper

Theorem (Asperó–Mota): Let $\kappa \geq \omega_3$ be a regular cardinal. Suppose CH holds, $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$, and $\Diamond(\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \omega_2\})$ holds. Then there is a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition (in particular, $\mathcal{P}$ preserves all cardinals) such that the following holds in the extension by $\mathcal{P}$:

(1) MA^*_λ for all $\lambda < \kappa$.

(2) $2^{\aleph_0} = \kappa$

Some applications of MA^*_λ

(1) MA^*_λ implies MA_λ.

(2) $\mathrm{MA}^*_{\aleph_2}$ implies that if $\mathbb{P} \in \Upsilon$ and $X \in [\mathbb{P}]^{\aleph_2}$, then there is $Y \in [X]^{\aleph_2}$ such that every $\sigma \in [Y]^{<\omega}$ has a lower bound in $\mathbb{P}$. Hence, the finite support product of any collections of members of Υ has he $\aleph_2$–chain condition. Proof is like the usual proof of productiveness of c.c.c. under $\mathrm{MA}_{\aleph_1}$.

(3) MA^*_λ implies that for every $\mathcal{F}$ consisting of functions $f : \omega_1 \longrightarrow \omega_1$ with $|\mathcal{F}| \leq \lambda$ there is $g : \omega_1 \longrightarrow \omega_1$ such that $\{\nu < \omega_1 \ : \ f(\nu) < g(\nu)\}$ is unbounded in ω_1 for all $f \in \mathcal{F}$. Proof uses the natural poset for adding a club $C \subseteq \omega_1$ by finite conditions.

(4) MA^*_λ implies that for every $\tau < \omega_1$, if $\mathcal{A}$ is a collection of subsets of ω_1 of order type at most τ with $|\mathcal{A}| \leq \lambda$, then there is a club $C \subseteq \omega_1$ such that $C \cap A$ is finite for every $A \in \mathcal{A}$. This conclusion, even for $\tau = \omega$, implies $2^{\aleph_0} > \lambda$.

(5) MA^*_λ implies that for every collection $\mathcal{F}$ of functions with $|\mathcal{F}| \leq \lambda$, if for every $f \in \mathcal{F}$ there is $\alpha < \omega_1$ such that $f : \alpha \longrightarrow \omega$ is a continuous function with respect to the order topology, then there is a club $C \subseteq \omega_1$ such that for all $f \in \mathcal{F}$, $\mathit{range}(f \restriction C) \neq \omega$. Again, this conclusion implies $2^{\aleph_0} > \lambda$.

We don't know how to force the conclusions in (2), and in (3), (4) and (5) for $\lambda \geq \aleph_2$, by any method other than ours. In particular, we don't know how to force them by any 'conventional' forcing iteration.

Proof of (4)

Given $\lambda' \leq \lambda$ and $\vec{A} = (A_i : i < \lambda')$ with $\mathrm{ot}(A_i) \leq \tau$ (for $i < \lambda'$) let $\mathbb{P}_{\vec{A}}$ consist of (f, p) such that

(1) $p \subseteq \omega_1 \times \omega_1$ is a finite function that can be extended to a normal function $F : \omega_1 \longrightarrow \omega_1$,

(2) p is a finite function with $\mathrm{dom}(p) \subseteq \lambda'$, and

(3) for all $i \in \mathrm{dom}(p)$, $p(i) \in [\omega_1]^{<\omega}$ and $p(i) = \mathrm{range}(f) \cap A_i$.

(f_1, p_1) extends (f_0, p_0) if and only if

(i) $f_0 \subseteq f_1$, and

(ii) $p_0 \subseteq p_1$

$\mathbb{P}_{\vec{A}}$ is $\aleph_2$–c.c.: If (f, p_0), $(f, p_1) \in \mathbb{P}_{\vec{A}}$, then $(f, p_0 \cup p_1) \in \mathbb{P}_{\vec{A}}$ (if $i \in \mathrm{dom}(p_0) \cap \mathrm{dom}(p_1)$, then $p_0(i) = A_i \cap \mathit{range}(f) = p_1(i)$).

By coding finite functions $\omega_1 \times \omega_1$ into ordinals $\nu < \omega_1$ it is easy to see that $\mathbb{P}_{\vec{A}}$ admits a regular representation (this depends on the fact that $\tau < \omega_1$ is fixed).

By a simple density argument, $\mathbb{P}_{\vec{A}}$ adds a normal function
$F : \omega_1 \longrightarrow \omega_1$
such that $\mathrm{range}(F) \cap A_i = p(i)$ for some (f, p) in the generic filter.

The proof of (5) is similar.

Consistency proof of MA^*_λ (outline)

Theorem (Asperó–Mota): Let $\kappa \geq \omega_3$ be a regular cardinal. Suppose CH holds, $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$, and $\diamondsuit(\{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) \geq \omega_2\})$ holds. Then there is a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition (in particular, $\mathcal{P}$ preserves all cardinals) such that the following holds in the extension by $\mathcal{P}$:

(1) MA^*_λ for all $\lambda < \kappa$.

(2) $2^{\aleph_0} = \kappa$

The proof involves a finite–support forcing iteration enhanced with certain finite 'symmetric' systems of countable structures as side conditions.

Let $\vec{X} = (X_\alpha \,:\, \alpha \in \mathrm{cf}(\alpha) \geq \omega_2)$ be a $\diamondsuit$–sequence. We build $(\mathcal{P}_\alpha)_{\alpha \leq \kappa}$, together with $(\mathbb{Q}_\alpha \,:\, \alpha < \kappa)$. $\mathcal{P}$ will be $\mathcal{P}_\kappa$. $(\mathcal{P}_\alpha)_{\alpha \leq \kappa}$ is a sequence of posets such that:

(A) Conditions in $\mathcal{P}_0$ are pairs $(\emptyset, \Delta)$, where Δ is a finite set $\{(N_i, 0) : i < m\}$ such that $\{N_i : i < m\}$ is a set of countable elementary substructures with suitable symmetry.

$(\emptyset, \Delta_1) \leq_0 (\emptyset, \Delta_0)$ if and only if $\mathrm{dom}(\Delta_0) \subseteq \mathrm{dom}(\Delta_1)$.

(B) Conditions in $\mathcal{P}_{\alpha+1}$ are pairs $q = (p, \Delta)$ such that:

(B1) p is an $\alpha + 1$–sequence.

(B2) Δ is a finite set of pairs (N, γ) with $\gamma \leq (\alpha + 1) \cap \sup(N \cap \kappa)$, and

(B3) $q|_\alpha := (p \restriction \alpha, \{(N, \min\{\gamma, \alpha\}) : (N, \gamma) \in \Delta\})$ is a condition in $\mathcal{P}_\alpha$.

(B4) $\dot{\mathbb{Q}}_\alpha$ is a $\mathcal{P}_\alpha$–name for a "V–regular" (this is a technical strengthening of "regular") poset with the $\aleph_2$–c.c.

(B5) If X_α codes a $\mathcal{P}_\alpha$–name $\dot{\mathbb{Q}}$ for a V–regular poset with the $\aleph_2$–c.c., then $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}$.

(B6) If $p(\alpha) \neq \emptyset$, then

$$q|\alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \in \mathbb{Q}$$

And, if $(N, \alpha + 1) \in \Delta$ and $N \in \mathcal{M}^{\alpha+1}$, then

$$p|_\alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \text{ is } (N[\dot{G}_\alpha], \mathbb{Q})\text{–generic}$$

($\mathcal{M}^{\alpha+1}$ is a given club of $N \in [H(\kappa)]^{\aleph_0}$ that "see" $\mathcal{P}_\alpha$).

Given $q_0 = (p_0^\frown\langle x_0\rangle, \Delta_0)$ and $q_1 = (p_1^\frown\langle x_1\rangle, \Delta_1)$ in $\mathcal{P}_{\alpha+1}$, $q_1 \leq_{\alpha+1} q_0$ if and only if

(i) $q_1|_\alpha \leq_\alpha q_q|_\alpha$,

(ii) $q_1|_\alpha$ forces in $\mathcal{P}_\alpha$ that x_1 extends x_0 in $\dot{\mathbb{Q}}_\alpha$, and

(iii) $\{N : (N, \alpha + 1) \in \Delta_0\} \subseteq \{N : (N, \alpha + 1) \in \Delta_1\}$

(C) If $\alpha \neq 0$ is a limit ordinal, conditions in $\mathcal{P}_\alpha$ are pairs $q = (p, \Delta)$, where

(C1) p is an α–sequence,

(C2) Δ is a finite set of pairs (N, γ) with $\gamma \leq \beta \cap \sup(N \cap \kappa)$, $q|_\beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and

(C3) $\mathrm{supp}(q) := \{\xi < \alpha \,:\, p(\xi) \neq \emptyset\}$ is finite.

Given $q_0 = (p_0, \Delta_0)$, $q_1 = (p_1, \Delta_1)$ in $\mathcal{P}_\alpha$, $q_1 \leq_\alpha q_0$ if and only if

(i) $q_1|_\beta \leq_\beta q_0|_\beta$ for all $\beta < \alpha$, and

(ii) $\{N \,:\, (N, \beta) \in \Delta_0\} \subseteq \{N \,:\, (N, \beta) \in \Delta_1\}$ (if $\beta < \kappa$).

Easy:

- (∘) $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ for $\alpha < \beta \leq \kappa$. In fact, if $q = (p, \Delta_q) \in \mathcal{P}_\beta$ and $r = (s, \Delta_r) \leq_\alpha q|_\alpha$, then

$$(p \cup (r \restriction [\alpha,\ \beta)), \Delta_q \cup \Delta_r) \in \mathcal{P}_\beta$$

  This is where the markers (the γ's in $(N, \gamma) \in \Delta$) come into play.]
- (∘) For all $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ has the $\aleph_2$–c.c. [The proof is by induction on α and uses CH.]
- (∘) $\mathcal{P}_\kappa$ forces $2^{\aleph_0} \leq \kappa$.

More work:

($\circ$) For all $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is proper:

The proof is by induction on α and uses standard side condition arguments (start from $q \in D$, for the relevant dense set $D \in N$; using correctness of N, find 'nice' $r \in N \cap D$; argue that q and r can be amalgamated into a condition extending both). For limit $\alpha \neq 0$, the case $\mathrm{cf}(\alpha) \neq \omega_1$ is easy. For the case $\mathrm{cf}(\alpha) = \omega_1$, the condition of V–regularity of the 'active poset' at stage β (for all $\beta < \alpha$), together with the symmetry of the system of side conditions, plays a crucial role in the success of the side condition argument.

Finally:

($\circ$) $\mathcal{P}_\kappa$ forces MA^*_λ for all $\lambda < \kappa$:

Let $\dot{\mathbb{Q}}$ be a $\mathcal{P}_\kappa$–name for a regular poset with the $\aleph_2$–c.c. condition. Argue that it suffices to assume $\dot{\mathbb{Q}} \subseteq H(\kappa)$. Find α such that X_α codes $\dot{\mathbb{Q}} \cap N$ for a structure N such that ${}^{\omega_1}N \subseteq N$. Argue that, in $V^{\mathcal{P}_\alpha}$, the poset coded by X_α is V–regular and has the $\aleph_2$–c.c. The rest of the proof is immediate.

Thank you for your attention

Ω–completeness and forcing axioms

ZFC is a very incomplete theory. It would be nice to have a consistent theory extending ZFC and deciding as many properties of the universe as possible.

[Moreover, perhaps we would like this theory to be simply definable / interesting / true, ...].

(Set)–forcing is an extremely powerful method in set theory. In fact it is the most powerful method currently available for proving independence results.

A desirable feature of axioms: Generic absoluteness

A desirable feature of axioms extending ZFC (given that forcing is our main method for proving independence): They should neutralise the effects of (set)–forcing as much as possible.

In other words: If ZFC + A holds, σ is (any) sentence, and G is (any) (set)-forcing over V and $V[G] \models \mathrm{ZFC} + A$, then

$$V \models \sigma$$

if and only if

$$V[G] \models \sigma$$

A useful language (when restricting to Π_2 sentences)

Definition (Woodin) Given $T \cup \{\sigma\} \subseteq Sent_{\mathcal{L}_{ST}}$,

$$T \models_\Omega \sigma$$

if and only if

for every ordinal α and every set-generic G over V, if

$$V_\alpha^{V[G]} \models T$$

then

$$V_\alpha^{V[G]} \models \sigma$$

Definition Suppose $T \subseteq Sent_{\mathcal{L}_{ST}}$. Suppose Γ is a collection of sentences. *T is Ω–complete for Γ* if and only if for every $\sigma \in \Gamma$,

$$T \models_\Omega \sigma$$

or

$$T \models_\Omega \neg\sigma$$

There don't seem to be (m)any examples of Σ_2–sentences whose consistency with the ambient large cardinals can be demonstrated by class–forcing but not by set–forcing (Challenge: Find any).

Hence, we would like our axioms to be as Ω-complete as possible.

Woodin: Suppose there is a proper class of Woodin cardinals. Then, given $T \cup \{\sigma\} \subseteq Sent_{\mathcal{L}_{ST}}$ and given any set-generic G over V,

$$V \models \text{“}T \models_\Omega \sigma\text{”}$$

if and only if

$$V[G] \models \text{“}T \models_\Omega \sigma\text{”}$$

That is: The relation of consequence in Ω–logic is generically invariant in the presence of a proper class of Woodin cardinals.

The resurrection of Σ_2–truths

A key ingredient in the proof of the generic invariance of $\models_\Omega$:

Theorem (Woodin) Suppose δ is a Woodin cardinal, σ is a sentence, $\alpha < \delta$, and

$$V_\alpha \models \sigma$$

Suppose G is generic over V for a forcing of size less than δ. Then in $V[G]$ there is a poset $\mathcal{Q}$ of size δ such that if H is $\mathcal{Q}$–generic over $V[G]$, then

$$V[G][H] \models \text{“ There is } \alpha < \delta \text{ such that } V_\alpha \models \sigma\text{”}$$

Theorem (Woodin) Suppose there is proper class of Woodin cardinals. Then, for every set-generic G over V and every sentence σ,

$$L(\mathbb{R})^{V} \models \sigma$$

if and only if

$$L(\mathbb{R})^{V[G]} \models \sigma$$

In other words, ZFC + “There is a proper class of Woodin cardinals” is Ω–complete for the theory of $L(\mathbb{R})$.

There is no large cardinal axiom A such that ZFC + A is Ω–complete for even the Σ_2–theory of $\langle H(\omega_2), \in\rangle$:

CH can be expressed as $\langle H(\omega_2), \in\rangle \models \sigma$ with σ being Σ_2, and both CH and $\neg$CH can always be forced by set-forcing. On the other hand, if $\mathcal{P}$ is a forcing notion and $|\mathcal{P}| < \kappa$, then forcing with $\mathcal{P}$ preserves the large cardinal properties of κ.

Theorem (Woodin) Suppose there is a proper class of measurable Woodin cardinals. Assume CH. Then, for every set-generic G over V and every Σ_1^2 sentence σ, if

$$V[G] \models \sigma$$

then

$$V \models \sigma$$

In particular, ZFC + CH + “There is a proper class of measurable Woodin cardinals” is Ω–complete for the collection of Σ_1^2 sentences.

The main question (version 1)

Is there *any* recursive consistent theory extending ZFC and which is Ω–complete for the theory of $\langle H(\omega_2), \in \rangle$?

There is an obvious candidate: ZFC + V = L.
In fact, ZFC + V = L is Ω–complete for *all* sentences.

However, V = L is incompatible with even mild large cardinal hypotheses and we have good reasons for liking large cardinals.

The main question (version 2)

Is there any recursive theory extending ZFC, compatible with all consistent large cardinal axioms, and which is Ω–complete for the theory of $\langle H(\omega_2), \in\rangle$?

Well, yes:

ZFC + There is a proper class of indestructible supercompact cardinals

(A supercompact cardinal κ is indestructible if κ remains supercompact after forcing with any $< \kappa$–directed forcing.
The above theory can be forced by class–forcing, starting from a proper class of supercompact cardinals, while preserving all other large cardinals there might be.
On the other hand, by a result of Joel Hamkins, if κ is supercompact and $\mathcal{P}$ is a nontrivial forcing of size less than κ, then κ is not indestructible after forcing with $\mathcal{P}$.)

Thanks to Gunter Fuchs for pointing this out.

There are other theories T compatible with all large cardinals, and such that T does not hold after any nontrivial set-forcing whatsoever:

ZFC + For every set X of ordinals there are unboundedly many ordinals α such that for all $\xi < sup(X)$,

$$2^{\aleph_{\alpha+\xi+1}} = \aleph_{\alpha+\xi+2} \text{ if and only if } \xi \in X$$

Of course, the reason these theories T are Ω–complete (actually for all sentences) is that there is **no** nontrivial set-forcing extension satisfying T.

ZFC + V = L of course has also this feature.

We would like to find a theory T extending ZFC, compatible with all consistent large cardinal axioms, and which is Ω–complete for the theory of $\langle H(\omega_2), \in \rangle$ for some *interesting* reason.

The main question (final version)

Is there any recursive theory T extending ZFC, compatible with all consistent large cardinal axioms, which is Ω–complete for the theory of $\langle H(\omega_2), \in\rangle$, and such that, from some suitable large cardinal assumption, T can always be forced after any set-forcing?

Iterating generic ultrapowers

Let (M, I) be a pair with M a transitive model of ZFC^* and I a normal ideal on ω_1^M. An *iteration of* (M, I) is a sequence

$$\langle M_\alpha, I_\alpha, G_\beta, j_{\beta,\alpha} : \beta < \alpha \leq \rho \rangle$$

such that

(a) $(M_0, I_0) = (M, I)$,

(b) G_β is $\mathcal{P}(\omega_1)^{M_\beta} \setminus I_\beta$ generic over M_β, $M_{\beta+1} = Ult(M_\beta, G_\beta)$, and

$$j_{\beta,\beta+1} : M_\beta \longrightarrow M_{\beta+1}$$

is given by $j_{\beta,\beta+1}(x) = [c_x]_{G_\beta}$,

(c) if $\alpha \leq \rho$ is limit, then M_α, $j_{\beta,\alpha}$ $(\beta < \alpha)$ is the direct limit of the commuting system $M_{\beta'}$, $j_{\beta,\beta'}$ $(\beta < \beta' < \alpha)$, and

(d) all other $j_{\beta,\alpha}$'s are obtained by composition.

Iterable pairs

A pair (M, I) is *iterable* iff for every iteration $\langle M_\alpha, I_\alpha, G_\beta, j_{\beta,\alpha} : \beta < \alpha \leq \rho \rangle$ of (M, I) with $\rho \leq \omega_1$, every M_α is well–founded.

The forcing $\mathbb{P}_{max}$

A condition in $\mathbb{P}_{max}$ is a triple (M, I, a) such that

(1) M is a countable transitive model of $\mathrm{ZFC}^* + \mathrm{MA}_{\omega_1}$,

(2) (M, I) is an iterable pair, and

(3) $a \in \mathcal{P}(\omega_1)^M$ is such that $M \models \omega_1^{L[a]} = \omega_1$.

Given $\mathbb{P}_{max}$ conditions (M_0, I_0, a_0), (M_1, I_1, a_1), $(M_1, I_1, a_1) \leq_{\mathbb{P}_{max}} (M_0, I_0, a_0)$ if and only if, letting $\rho = \omega_1^{M_1}$, there is a (unique) iteration $\langle M_\alpha^0, I_\alpha^0, G_\beta, j_{\beta,\alpha} : \beta < \alpha \leq \rho \rangle \in M_1$ of (M_0, I_0) such that

(i) $j_{0,\rho}(a_0) = a_1$, and

(ii) every set in $\mathcal{P}(\omega_1)^{M_\rho^0} \setminus I_\rho^0$ is I_1–positive.

Theorem (Woodin) (ZFC + There is a proper class of Woodin cardinals)
$\mathbb{P}_{max} \in L(\mathbb{R})$ is a σ–closed forcing definable in $L(\mathbb{R})$ such that if G is $\mathbb{P}_{max}$ generic over $L(\mathbb{R})$:

(1) $L(\mathbb{R})[G]$ satisfies
- ZFC
- The universe is of the form $L(\mathcal{P}(\omega_1))$
- The universe is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$

(2) Suppose A is a set of reals in $L(\mathbb{R})$, $\mathcal{P} \in V$ is a poset, H is $\mathcal{P}$–generic over V, and σ is a Π_2 statement. If

$$\langle H(\omega_2), \in, NS_{\omega_1}, A_H \rangle^{V[H]} \models \sigma$$

then

$$\langle H(\omega_2), \in, NS_{\omega_1}, A \rangle^{L(\mathbb{R})[G]} \models \sigma$$

That is, all Π_2 statements about $\langle H(\omega_2), \in, NS_{\omega_1}, A \rangle$ that can be forced by set-forcing over V hold simultaneously in $L(\mathbb{R})[G]$'s $\langle H(\omega_2), \in, NS_{\omega_1}, A \rangle$.

Definition $(*)$ is the conjunction of the following statements.

(1) $\mathrm{AD}^{L(\mathbb{R})}$

(2) $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.

[(1) is for technical reasons and follows from the existence of enough Woodin cardinals]

If there is a proper class of Woodin cardinals and $(*)$ is Ω–satisfiable (equivalently, if $(*)$ can be forced by set-forcing), then the maximality condition for Π_2 sentences expressed in (2) from above theorem is in fact *equivalent* to $(*)$:

Theorem (Woodin) Suppose there is a proper class of Woodin cardinals. Suppose $(*)$ is Ω–satisfiable. Then the following conditions are equivalent:

(1) Suppose A is a set of reals in $L(\mathbb{R})$, $\mathcal{P} \in V$ is a poset, H is $\mathcal{P}$–generic over V, and σ is a Π_2 statement. If

$$\langle H(\omega_2), \in, NS_{\omega_1}, A_H \rangle^{V[H]} \models \sigma$$

then

$$\langle H(\omega_2), \in, NS_{\omega_1}, A \rangle^{V} \models \sigma$$

(2) $(*)$

ZFC + $(*)$ + "There is a proper class of Woodin cardinals" is Ω–complete for the theory of $\langle H(\omega_2), \in\rangle$:

Let σ be a sentence and let $\mathcal{Q}_0$, $\mathcal{Q}_1$ be posets forcing $(*)$. Let G_0 be $\mathcal{Q}_0$–generic over V and let G_1 be $\mathcal{Q}_1$-generic over V.

Then

$$\langle H(\omega_2), \in\rangle^{V[G_0]} \models \sigma$$

if and only if, in $V[G_0]$,

$$L(\mathbb{R}) \models \text{``} \Vdash_{\mathbb{P}_{max}} \langle H(\omega_2), \in\rangle \models \sigma\text{''}$$

(since $\mathbb{P}_{max}$ is homogeneous) if and only if this is true in $V[G_1]$ (since the theory of $L(\mathbb{R})$ is frozen under set-forcing).

Question (Woodin): Is $(*)$ compatible with all large cardinals?

Assuming some large cardinal axiom. Is $(*)$ Ω–satisfiable? Equivalently: Assuming some large cardinal axiom, can $(*)$ be forced by set-forcing?

Note that an affirmative answer to the second question provides an affirmative answer to the first question.

Note that an affirmative question to the first question provides an affirmative answer to our Main Question.

Definition (Foreman–Magidor–Shelah) Martin's Maximum^{++}, MM^{++}, is the following strengthening of MM:
Suppose $\mathcal{P}$ is a forcing notion preserving all stationary subsets of ω_1. Suppose $\mathcal{D}$ is a collection of dense subsets of $\mathcal{P}$ with $|\mathcal{D}| \leq \aleph_1$, and suppose $\dot{S}_\alpha$ (for $\alpha < \omega_1$) are $\mathcal{P}$–names for stationary subsets of ω_1.

Then there is a filter $G \subseteq \mathcal{P}$ such that

(a) $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$, and

(b) for every $\alpha < \omega_1$,

$$\{\nu < \omega_1 \ : \ \text{there is some } p \in G \text{ such that } \Vdash_{\mathcal{P}} \nu \in \dot{S}_\alpha\}$$

is a stationary subset of ω_1.

The original consistency proof of MM actually yields a model of MM^{++}.

Question (Woodin): Does MM^{++} (together with large cardinals) imply $(*)$?

There is strong evidence that the answer should be yes.

Example:

Theorem (M. Viale) Suppose there is a proper class of Woodin cardinals. Suppose MM^{++} holds. Then $H(\omega_2)^V \prec_{\Sigma_2} H(\omega_2)^{V^{\mathcal{P}}}$ for every $\mathcal{P}$ such that

(1) Forcing with $\mathcal{P}$ preserves stationary subsets of ω_1, and
(2) $\mathcal{P}$ forces BMM.

Conjecture: MM^{++} implies $(*)$

We have a promising scenario for proving this (this is joint work in progress with Ralf Schindler).

If the conjecture is true and if there is a supercompact cardinal, then $(*)$ can be forced by set-forcing (equivalently, $(*)$ is Ω–satisfiable). Hence, assuming there is a supercompact cardinal, $(*)$ is compatible with all consistent large cardinal hypotheses.

This would show that the answer to our Main Question is Yes, as witnessed by $(*)$.

A limitation

Ultimate dream: Assume some reasonable sufficiently strong large cardinal hypothesis (e.g. a proper class of Woodin cardinals). Is it possible to find theories compatible with all large cardinals which, in the presence of large cardinals, are Ω–complete for all (definable) initial segments of the universe, and which moreover are ‘local’ (i.e., can be defined by a Σ_2 sentence, like $(*)$)?

Theorem (Woodin) Suppose there is a proper class of Woodin cardinals and suppose the Ω Conjecture holds. Then there is no consistent Σ_2–definable theory which, in the presence of a proper class of Woodin cardinals, is Ω–complete for the theory of $H(\delta_0^+)$, where δ_0 is the first Woodin cardinal:

$\vdash_\Omega$ is definable, without parameters, in $H(\delta_0^+)$. If $\vdash_\Omega$ and $\models_\Omega$ are equal and V satisfies an Ω–complete Σ_2–definable theory for the theory of $H(\delta_0^+)$, then there is a definition, within $H(\delta_0^+)$ and from no parameters, of truth in $H(\delta_0^+)$.