From commutators to Cartan subgroups in the o-minimal setting

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COMMUTATORS IN GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

Introduction

- Let $\mathcal{M} = \langle M, <, \cdots \rangle$ an o-minimal structure.
- Cell decomposition $\rightsquigarrow$ dimension with good properties:

(Finite sets) X definable is finite if and only if $dim(X) = 0$

(Definability) If $f : X \to Y$ is definable, then the set

$$\{y \in Y : dim(f^{-1}(y)) = m\}$$

is definable for each $m \in \mathbb{N}$.

(Additivity) If $f : X \to Y$ is definable and the dimension of the fibers have constant dimension m then

$$dim(X) = dim(Im(f)) + m.$$

(Monotonicity) $dim(A \cup B) = max\{dim(A), dim(B)\}$

Introduction

- Let G be a definable group in an o-minimal structure $\mathcal{M}$.
- Recall that this group can be equipped with a definable manifold structure compatible with the group structure (Pillay'88). For example, if $M = \mathbb{R}$ then we have a real Lie group.
- In particular, we have finite definably connected components and hence we have descending chain condition (DCC) on $\mathcal{M}$-definable subgroups.
- $\mathcal{M}$ eliminates imaginaries for definable subsets of G (Edmundo'01).

Introduction

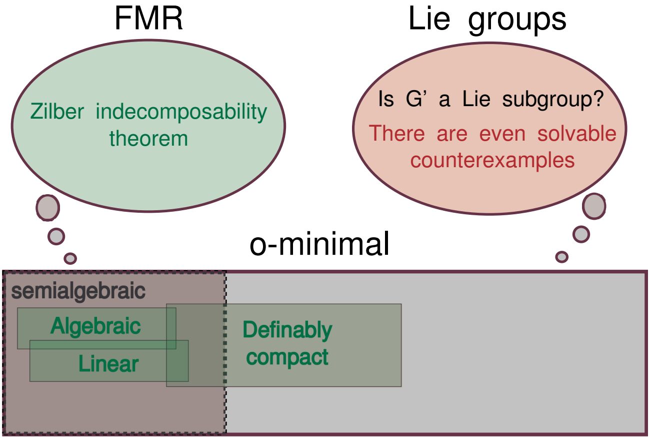
Question

If G is definably connected, is the derived (or commutator) subgroup

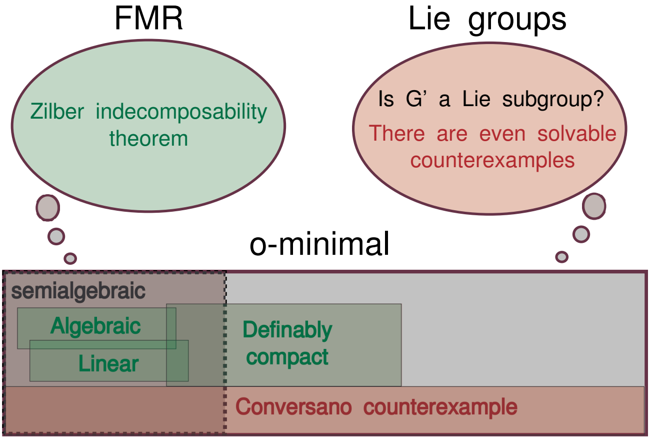
$$G' = [G, G]$$

definable in $\mathcal{M}$ and definably connected?

State of art


FMR
Zilber indecomposability theorem
Lie groups
Is G' a Lie subgroup?
There are even solvable counterexamples
o-minimal
semialgebraic
Algebraic
Linear
Definably compact

State of art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Conversano counterexample

Conversano's counterexample

There exists a connected semialgebraic group G over the real numbers which is a central extension of a simple group

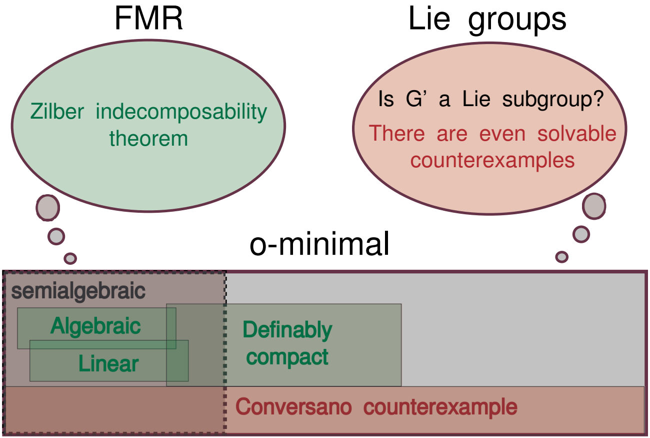
$$1 \to \mathbb{R} \to G \to PSL_2(\mathbb{R}) \to 1$$

with G' equal to the universal cover $\widetilde{PSL_2(\mathbb{R})}$ of $PSL_2(\mathbb{R})$.

(the construction of G uses the fact that $\widetilde{PSL_2(\mathbb{R})}$ can be regarded as a locally definable group.)

So G is a central extension of a (definably) simple group...

State of art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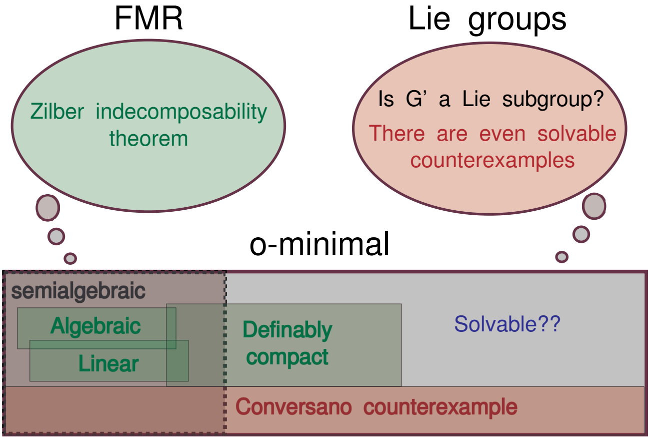
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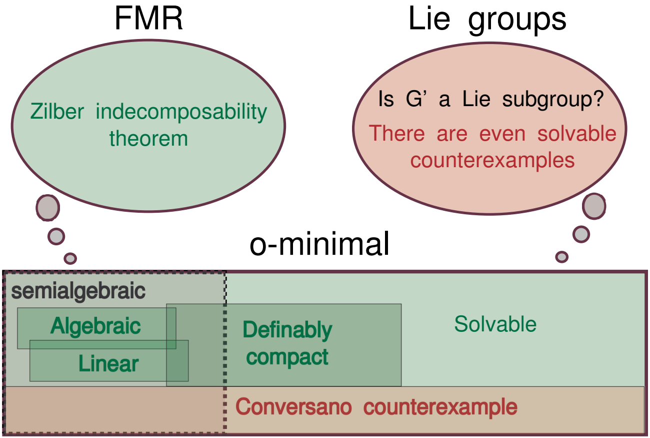
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Result

Conversano's counterexample is the essential obstruction to the definability of the commutator subgroup.

Theorem B.-Jaligot-Otero

Let G be a group definable definably connected in an o-minimal structure $\mathcal{M}$. Suppose that for all definable subgroups $K \trianglelefteq H \leq G$ such that H/K is a (strict) central extension of a definably simple group, we have that the derived subgroup $(H/K)'$ is definable. Then for every definable definably connected subgroups A and B of G which normalize each other we have that $[A, B]$ is definable and definably connected.

Remark

1) In particular, the lower central series $G^{n+1} = [G, G^n]$ and the derived series $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ are definable.
2) Solvable groups satisfy the hypotheses of the theorem.

Solvable case in a general framework

But the solvable case is true even in a more general framework...

Definition

We say that a structure $\mathcal{M}$ has a *dimension* if it eliminates imaginaries and it has a dimension on definable sets with the properties (Finite sets), (Additivity), (Definability) and (Monotonicity).

Theorem

Let $\mathcal{M}$ be a structure with a dimension. Let G be a solvable definable group in $\mathcal{M}$, definably connected and with DCC. Then G' is definable and definably connected.

Proof of the solvable case with DCC and dimension

Lemma

Let G be like in the theorem and nontrivial. Then there exists a proper normal definable definably connected subgroup A of G such that $G' \leq A$.

Proof

Take $A \triangleleft G$ definable, definably connected of maximal dimension.
We show that $H := G/A$ is abelian, so that $G' \leq A$.
Since H is solvable, for some n

$$1 = H^{(n)} < H^{(n-1)} < \cdots < H.$$

Take $m \leq n$ minimal such that $H^{(m)}$ is finite. Then $H_1 := H/H^{(m)}$ is abelian. Indeed, $H_1^{(m-1)}$ is abelian, infinite and normal. Hence $Z(C(H_1^{(m-1)}))$ is definable, abelian, infinite and normal. By maximality $H_1 = Z(C(H_1))$. Since H_1 is abelian, H is abelian.

Proof of the solvable case with DCC and dimension

Lemma

Let G be like in the theorem. Let A be a definable subgroup of G. If $[G, A] < Z(A)$ then $[G, A]$ is definable and definably connected.

Proof

For each $x \in G$ the definable map

$$ad_x : A \mapsto G : a \mapsto [x, a]$$

is homomorphism since $[x, ab] = [x, b][x, a]^b = [x, b][x, a]$. So that each $[x, A]$ is a definable definably connected subgroup of G. Moreover, each $[x_1, A] \cdots [x_k, A]$ is definable definably connected subgroup. Then such a product of maximal dimension must equal $[G, A]$.

Proof of the solvable case with DCC and dimension

Theorem

Let $\mathcal{M}$ be a structure with a dimension. Let G be a solvable definable group in $\mathcal{M}$, definably connected and with DCC. Then G' is definable and definably connected.

Proof

By induction on $dim(G)$. If G is nontrivial, then there is $A \triangleleft G$ definable definably connected subgroup such that $G' \leq A$. By induction A' is definable def-connected. If $A' \neq 1$ then by induction $(G/A')' = G'/A'$ is definable, so that G' is definable. If $A' = 1$ then $[G, A] \leq G' \leq A = Z(A)$ is definable and definably connected. If $[G, A] \neq 1$ then again we are done. If $[G, A] = 1$ then $G' < A < Z(G)$ and G' is definable and definably connected.

Open questions

- The *commutator width* of G is the minimal $m \in \mathbb{N}$ such that

$$G' = [G, G]_m.$$

(if there isn't such m then the commutator width is ∞).

Along the proofs, we show that the commutator is finite.
Moreover, in the solvable case, we have that the commutator width is bounded by the dimension of the group.
In the general case, there are problems to find a bound of the commutator subgroup. But the most important problem is that there is no bound for definably simple groups.
For finite simple groups it is known that the commutator width is 1 (Ore conjecture). For simple Lie groups, it is also a conjecture (Cartan-Doković).

If G is definably simple, is the commutator width of G equal to 1?

Open questions

- If G is definably simply connected, is G' definable?

  This is true for Lie groups. In fact, from this we deduce easily that the derived subgroup is a virtual Lie group (the image of a Lie group by a Lie homomorphism). But the classical proof uses in a crucial way the archimedean property of the real numbers.

CARTAN SUBGROUPS IN GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

Motivation I: Carter subgroups in FMR

Definition

A definable subgroup of a group of finite Morley rank is a Carter subgroup if it is connected, nilpotent, and of finite index in its normalizer.

Example

In $SL_2(\mathbb{C})$ the subgroup of diagonal matrices

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^* \right\}$$

is up to conjugacy the unique Carter subgroup.
Moreover, $D^{SL_2(\mathbb{C})}$ is generic in $SL_2(\mathbb{C})$.

Let G be a group of FMR. We say that a definable subgroup H of G is *generous* if the rank of H^G equals the rank of G.

Motivation I: Carter subgroups in FMR

Theorem — Frécon-Jaligot

Any group G of finite Morley rank contains a Carter subgroup.

Theorem — Jaligot

In a group of FMR, *generous* Carter subgroups are conjugate.

Conjecture

In any group of finite Morley rank,

- Carter subgroups are conjugate,
- Any (at least one) Carter subgroup of a group of finite Morley rank is generous.

Theorem — Frécon/Wagner

In any connected solvable group G of finite Morley rank, Carter subgroups are generous, conjugate, and selfnormalizing.

Motivation II: Definably compact groups in the o-minimal case

Theorem — Berarducci/Edmundo

Let G be a definably connected, definably compact definable group in an o-minimal structure. Then there is a unique maximal definable definably connected abelian subgroup T up to conjugacy. Moreover, T has finite index in $N_G(T)$ and $T^G = G$.

Corollary — Berarducci/Edmundo/Otero

Let G be a definably connected, definably compact definable group in an o-minimal structure. Then G is divisible.

The idea is to replace maximal-tori by Carter subgroups in the non-compact case.

Carter in the o-minimal setting

In the o-minimal setting, Carter subgroups...

- exist?
- are conjugate?
- are generous?

In the o-minimal setting, the notion of generic is ambiguous.

Definition

Let G be a definable group in an o-minimal structure. We say that a definable subset X of G is...

- *weakly generic* if $dim(X) = dim(G)$,
- *large* if $dim(G \setminus X) < dim(G)$,
- is *weakly generous* if X^G is weakly generic in G,
- is *largely generous* if X^G is large in G.

Carter in the o-minimal setting

In the o-minimal setting, Carter subgroups...

- exist?
- are conjugate?
- are largely generous?

In the o-minimal setting, the notion of generic is ambiguous.

Definition

Let G be a definable group in an o-minimal structure. We say that a definable subset X of G is...

- *weakly generic* if $dim(X) = dim(G)$,
- *large* if $dim(G \setminus X) < dim(G)$,
- is *weakly generous* if X^G is weakly generic in G,
- is *largely generous* if X^G is large in G.

Example: $SL_2(\mathbb{R})$

In $SL_2(\mathbb{R})$ there are two Carter subgroups up to conjugacy

$$C_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$

$$C_2 = SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}$$

Moreover,

$$C_1^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : tr(A) > 2\} \cup \{Id\}$$

$$C_2^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : |tr(A)| < 2\} \cup \{Id, -Id\}$$

No conjugacy (but there is finitely many conjugacy classes).
No largely generous (but weakly generous).
The union of all Carter is not large.

From Carter to Cartan

In the o-minimal setting we have to replace Carter by Cartan.

Definition

A subgroup Q of G is *Cartan* if and only if

- is nilpotent and maximal with this property,
- if X is a normal subgroup of finite index of Q then X has finite index in $N_G(X)$.

In the o-minimal setting Cartan and Carter have a good relation...

Proposition

Let G be a definable group in a o-minimal structure. If Q is a Cartan subgroup of G then it is definable. Moreover, a definable group Q is Cartan if and only if Q^0 is Carter and Q is a maximal nilpotent subgroup (among the nilpotent subgroups of $N_G(Q^0)$).

Example: $SL_2(\mathbb{R})$

In $SL_2(\mathbb{R})$ there are two Cartan subgroups up to conjugacy

$$Q_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^* \right\}$$

$$Q_2 = SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}$$

Moreover,

$$Q_1^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : |tr(A)| > 2\} \cup \{Id\}$$

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Finite number of conjugacy classes of Cartan subgroups.
Each Cartan subgroup is weakly generous.
The union of all Cartan subgroups is large.

Cartan subgroups in the solvable case

Theorem B.-Jaligot-Otero

Let G be a definable definably connected solvable group in an o-minimal structure. Then

- Cartan subgroups exists, are definably connected (then also Carter) and selfnormalizing.
- All Cartan subgroups are conjugate.
- Cartan subgroups are largely generous in G.

Cartan subgroups in the semisimple case

Theorem B.-Jaligot-Otero

Let G be a definable definably connected semisimple group in an o-minimal structure. Then

- Cartan subgroups exists,
- there exists only a finite number of conjugacy classes of Cartan subgroups,
- each Cartan subgroup is weakly generous,
- the union of all Cartan subgroups is large in G,
- if Q_1, Q_2 are Cartan subgroups and $Q_1^0 = Q_2^0$ then $Q_1 = Q_2$.

Cartan subgroups in the general case

Theorem B.-Jaligot-Otero

Let G be a definable definably connected group in an o-minimal structure. Then

- Cartan subgroups exists,
- there exists only a finite number of conjugacy classes of Cartan subgroups,
- each Cartan subgroup is weakly generous,
- if Q_1, Q_2 are Cartan subgroups and $Q_1^0 = Q_2^0$ then $Q_1 = Q_2$.

Open problem

Is the union of all Cartan subgroups large in G?

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