
GMC
C STARR

Elliott's program and descriptive set theory I

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a, a, a, a, the, the, the, the.

I shall need this exercise later, someone please solve it

Exercise

If $A = \varinjlim_n A_n$, $B = \varinjlim_n B_n$ and there are morphisms φ_j, ψ_j such that the following diagram commutes

$$
\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & & A \\
\downarrow{\scriptstyle \varphi_1} & \nearrow{\scriptstyle \psi_1} & \downarrow{\scriptstyle \varphi_2} & \nearrow{\scriptstyle \psi_2} & \downarrow{\scriptstyle \varphi_3} & & & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots & & B
\end{array}
$$

then $A \cong B$.

The plan

1. Today:
   1.1 Basic properties of C*-algebras.
   1.2 Classification: UHF and AF algebras.
   1.3 Elliott's program.
2. Friday: Applying logic to 1.2–1.3.
3. Saturday: Convincing you that 1.2–1.3 is logic.

Prologue

A topological space X is *Polish* if it is separable and completely metrizable.
A subset of X is *analytic* if it is a continuous image of a Borel set.
An equivalence relation E on X is analytic if it is an analytic subset of X^2.

Thesis

Almost all classical classification problems deal with analytic equivalence relations on Polish spaces.

Thesis

In almost all cases, the space of invariants has a Polish topology and the computation of invariants is given by a Borel-measurable function.

Hilbert space, inner product

$$v = (v_0, v_1, \ldots, v_n, \ldots) \in \mathbb{C}^{\mathbb{N}}$$

$$(v|u) = \sum_n v_n \bar{u}_n \qquad \text{inner product}$$

$$\|v\| = \sqrt{(v|v)} \qquad \text{norm}$$

$$\ell_2 = \{v \mid \|v\| < \infty\}$$
$$L_2(\mu) = \{f : [0,1] \to \mathbb{C} \mid \int |f|^2 \, d\mu < \infty\}$$

Fact

Two complex Hilbert spaces are isomorphic iff their dimensions are equal. In particular, $\ell_2 \cong L_2(\mu)$.

C*-algebras

H: a complex Hilbert space, ℓ_2
If $a\colon H \to H$ is linear let

$$\|a\| = \sup_{\|\xi\|=1} \|a\xi\|.$$

Define a^* implicitly via

$$(a\eta|\xi) = (\eta|a^*\xi)$$

for all η and ξ in H.
$(\mathcal{B}(H), +, \cdot, ^*, \|\cdot\|)$: the algebra of bounded linear operators on H,

Example

If $\dim(H) = n$ then $\mathcal{B}(H)$ is $M_n(\mathbb{C})$: $n \times n$ complex matrices.

Definition

A *concrete C*-algebra* is a norm-closed subalgebra of $\mathcal{B}(H)$.
An *abstract C*-algebra* is a Banach algebra with involution $(A, +, \cdot, *, \|\cdot\|)$ such that

1. $(a^*)^* = a$
2. $(ab)^* = b^*a^*$
3. $\|a\| = \|a^*\|$
4. $\|ab\| \leq \|a\| \cdot \|b\|$
5. $\|a^*a\| = \|a\|^2$

for all a and b in A.

Every concrete C*-algebra is an abstract C*-algebra.

Theorem (Gelfand–Naimark–Segal, 1942)

Every abstract C-algebra is isomorphic to a concrete C*-algebra.*

Examples of C*-algebras

1. $\mathcal{B}(H)$, $M_n(\mathbb{C})$.
2. If X is a compact Hausdorff space,

$$C(X) = \{f\colon X \to \mathbb{C} | f \text{ is continuous}\}.$$

Gelfand–Naimark duality

A C*-algebra is *unital* if it has a multiplicative unit.

Theorem (Gelfand–Naimark)

Every unital abelian C-algebra is of the form $C(X)$ for some compact Hausdorff space.*

Theorem

Categories of unital abelian C-algebras with *-homomorphisms and compact Hausdorff spaces with continuous maps are equivalent.*

$$X \xrightarrow{\quad \varphi \quad} Y$$

$$C(X) \xleftarrow{\; g \mapsto g \circ \varphi \;} C(Y)$$

Automatic continuity

Proposition

*Every *-homomorphism $\Phi\colon A \to B$ between C*-algebras satisfies*

$$\|\Phi(a)\| \leq \|a\|$$

(C-algebraists call such map a contraction).*

Proof.

First the abelian case:

$$\Phi\colon C(X) \to C(Y)$$

is of the form $\Phi(f) = f \circ \varphi$ for a continuous $\varphi\colon Y \to X$.
For the general case, note that a^*a always generates an abelian algebra, hence $\|\Phi(a^*a)\| \leq \|a^*a\|$.
Then use the C*-equality, $\|a\|^2 = \|a^*a\|$. □

Direct (or inductive) limits

Assume A_λ, for $\lambda \in \Lambda$, is a directed system of C*-algebras with

$$\Phi_{\kappa\lambda}\colon A_\kappa \to A_\lambda$$

for $\kappa < \lambda$ in Λ a commuting system of *-homomorphisms. Since all $\Phi_{\kappa\lambda}$ are contractions, on the 'algebraic' direct limit we have a uniquely defined C*-norm.
Let $\lim_\lambda A_\lambda$ be the closure of the 'algebraic' direct limit.

(Unital) embeddings

A *-homomorphism between unital algebras is *unital* if it sends unit to unit.
Define a unital *-homomorphism

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C})$$

via

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix}$$

CAR algebra (Fermion algebra, M_{2^∞})

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow M_{16}(\mathbb{C}) \hookrightarrow \dots$$

$$M_{2^\infty}(\mathbb{C}) = \varinjlim M_{2^n}(\mathbb{C}).$$

The structure of full matrix algebras

Lemma

*There is a unital *-homomorphism of* $M_m(\mathbb{C})$ *into* $M_n(\mathbb{C})$ *if and only if* m *divides* n*. Moreover, such *-homomorphism is unique up to a unitary conjugacy.*

Partial proof.

If $m|n$ then $a \mapsto \mathrm{diag}(a, a, \ldots, a)$ (a repeated m/n times). $\square$

Uniformly HyperFinite (UHF) algebras are direct limits of full matrix algebras

If $A = \lim_j M_{n(j)}(\mathbb{C})$ is unital, let the *generalized integer* associated to A be the formal infinite product

$$GI(A) = \prod_{p \text{ prime}} p^{k(p)}$$

where $k(p)$ is the supremum of all k such that p^k divides $n(l)$ for some l.

It seems that

$$GI(M_{2^\infty}) = 2^{\aleph_0},$$

but one needs to check that $GI(A)$ is well-defined.

Stability

We need to show that $A = \lim_j M_{n(j)}(\mathbb{C}) = \lim_j M_{k(j)}(\mathbb{C})$ implies

$$(\forall j)(\exists l)(n(j) \text{ divides } k(l)).$$

If C, D are subalgebras of A, write

$$C \subseteq_\varepsilon D$$

if $(\forall c \in C)(\exists d \in D)\|c - d\| \leq \varepsilon\|c\|$.

Lemma

$(\forall k \in \mathbb{N})(\exists \varepsilon > 0)$ *such that for all* A *and subalgebras* C *and* D *of* A *such that* $C \subseteq_\varepsilon D$ *and* $C \cong M_k(\mathbb{C})$*, then there exists an inner automorphism* Φ *of* A *such that* $\Phi[C] \subseteq D$*.*

A C*-algebra A is LM (Locally Matricial) if every finite $F \subseteq A$ is ε-included in some full matrix subalgebra of A, for every $\varepsilon > 0$.

Corollary

If A *is separable then LM implies UHF.*

Classification of UHF algebras

Theorem (Glimm, 1960)

If A and B are unital separable UHF algebras, then $GI(A) = GI(B)$ if and only if $A \cong B$.

Proof.

$\Rightarrow$ Let $A = \lim_j M_{n(j)}(\mathbb{C})$ and $B = \lim_j M_{m(j)}(\mathbb{C})$.
Since $GI(A) = GI(B)$, we may assume

$$n(j) | m(j) \text{ and } m(j) | n(j+1).$$

Then we have φ_j and ψ_j for all $j \in \mathbb{N}$ so that

$$\begin{array}{ccccccccc}
M_{n(1)}(\mathbb{C}) & \longrightarrow & M_{n(2)}(\mathbb{C}) & \longrightarrow & M_{n(3)}(\mathbb{C}) & \longrightarrow & \cdots & & A \\
\varphi_1 \downarrow & \psi_1 \nearrow & \varphi_2 \downarrow & \psi_2 \nearrow & \varphi_3 \downarrow & & & & \\
M_{m(1)}(\mathbb{C}) & \longrightarrow & M_{m(2)}(\mathbb{C}) & \longrightarrow & M_{m(3)}(\mathbb{C}) & \longrightarrow & \cdots & & B
\end{array}$$

commutes, and by the exercise we have $A \cong B$. □

AF algebras

We move on to the classification problem for the next simplest class of unital separable C*-algebras.
A C*-algebra is *AF (Approximately Finite)* if it is a direct limit of finite-dimensional C*-algebras.

Lemma

Every finite-dimensional C-algebra is a direct sum of full matrix algebras.*

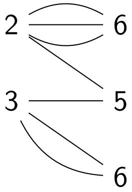
Bratteli diagrams

Consider unital *-homomorphism

$$\Phi\colon M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_6(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_6(\mathbb{C})$$

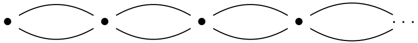
$$(a, b) \mapsto (\mathrm{diag}(a, a, a), \mathrm{diag}(a, b), \mathrm{diag}(b, b)).$$

The *Bratteli diagram* describing this map is the following:

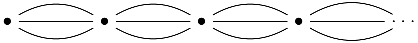


Bratteli diagram determines *-homomorphism Φ uniquely, up to the unitary conjugacy.

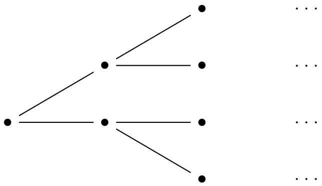
Examples of Bratteli diagrams that describe AF algebras



M_{2^∞}.

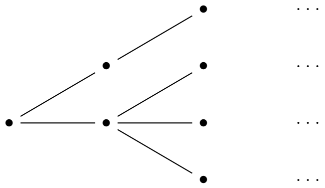


M_{3^∞}.

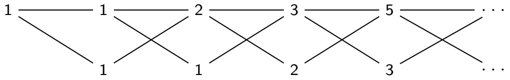


$C(2^{\mathbb{N}})$.

More examples of Bratteli diagrams



$C(\omega + 1)$



Fibonacci algebra: a simple unital AF algebra with a unique trace that is not UHF.

Classification of AF algebras: Stabilization

If $n \in \mathbb{N}$ and A is a C*-algebra, then so is $M_n(A)$: $n \times n$ matrices of elements of A with respect to the matrix operations and the operator norm.
Consider the direct limit of $M_n(A)$, for $n \in \mathbb{N}$, with (non-unital) *-homomorphism

$$\Phi_n \colon M_n(A) \to M_{n+1}(A)$$

defined via

$$\Phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $M_\infty(A) = \lim_n M_n(A)$ is the *stabilization of A*.

Classification of AF algebras: Murray-von Neumann equivalence

Fix a C*-algebra A. Some $p \in A$ is a *projection* if $p^2 = p = p^*$. If p and q are projections in A, then we write $p \sim q$ and say that *p and q are Murray von Neumann equivalent* if there exists $v \in A$ such that

$$vv^* = p \text{ and } v^*v = q.$$

Let $V(A)$ be the set of projections on $M_\infty(A)$ modulo $\sim$, equipped with the operation $\oplus$ defined by

$$[p] \oplus [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

This is an abelian semigroup and its Grothendieck group is $K_0(A)$. $K_0(A)^+$ is the set of elements of $K_0(A)$ that correspond to projections in $M_\infty(A)$.

K-theoretic classification

Example

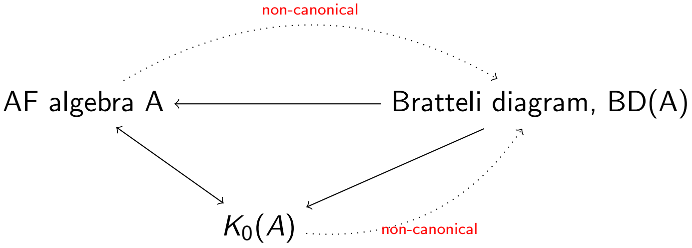
If A is UHF then $K_0(A) = \{k/m : k \in \mathbb{Z}, m \text{ divides } GI(A)\}$.

Theorem (Elliott, 1975)

Separable unital AF algebras are classified by the ordered (countable, abelian) group

$$\mathbf{K}_0(A) = (K_0(A), K_0(A)^+, [1_A]).$$

Categories of AF algebras and their K_0 groups are equivalent.



Elliott's program

Conjecture (Elliott, 1990's)

All nuclear,[1] *separable, simple, unital, infinite-dimensional C*-algebras are classified by the K-theoretic invariant,*

$$\mathrm{Ell}(A): \qquad ((K_0(A), K_0(A)^+, 1), K_1(A), T(A), \rho_A).$$

This conjecture has lead to some spectacular developments.

[1] I shall define nuclear C*-algebras on Saturday. All algebras mentioned today (except $\mathcal{B}(H)$) are nuclear.

Examples

If $\mathbb{K}$ is a class of compact Hausdorff spaces, then $A\mathbb{K}$ *algebras* have building blocks of the form

$$C(X, M_m(\mathbb{C})) = \{f\colon X \to M_m(\mathbb{C}) | X \in \mathbb{K} \text{ and } f \text{ is continuous}\}$$

With $\mathbb{K} = \{[0,1]\}$ we have AI algebras, if $\mathbb{K} = \{\{z \in \mathbb{C} : |z| = 1\}\}$ we have $A\mathbb{T}$ algebras, if $\mathbb{K}$ is the class of all compact metric spaces then we have AH algebras.

Theorem (Elliott–Gong–Li, 2010)

If $\sup\{\dim(X) : X \in \mathbb{K}\} < \infty$ *then simple unital* $A\mathbb{K}$ *algebras are classified by their Elliott invariant.*

It actually suffices to have algebras of *slow dimension growth*.

Nuclear, separable, simple, unital counterexamples

Jiang–Su, 1999

There exists an infinite-dimensional C*-algebra $\mathcal{Z}$ such that $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$ and $\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(A)$ for all A.

Rørdam 2003, Toms, 2004

There are AH algebras A such that $A \not\cong A \otimes \mathcal{Z}$ (yet $\mathrm{Ell}(A) = \mathrm{Ell}(A \otimes \mathcal{Z})$).

The algebra A constructed by Toms cannot be distinguished from $A \otimes \mathcal{Z}$ by any 'reasonable' invariant.

Elliott's conjecture recast

Conjecture (Toms–Winter, 2009)

All nuclear, separable, simple, unital algebras A such that $A \otimes \mathcal{Z} \cong A$ *are classified by Elliott's invariant.*

Theorem (Toms–Winter, 2009)

For 'natural' AH algebras A, $A \otimes \mathcal{Z}$ *has slow dimension growth and is therefore subject to Elliott–Gong–Li classification theorem.*

Toms–Winter conjecture has been confirmed in many cases, largely by the work of Winter.

Next time:

1. What is the descriptive complexity of the isomorphism relation of C*-algebras?