



Greg Hjorth, 1963–2011.

Elliott's program and descriptive set theory II

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A detailed presentation of the material from talk #1 and most of talk #2 is available in my lecture notes from the Singapore Summer School, available upon request.

The plan

1. Yesterday:
 - 1.1 Basic properties of C^* -algebras.
 - 1.2 Classification: UHF and AF algebras.
 - 1.3 Elliott's program.
2. Today: Applying logic to 1.2–1.3.
 - 2.1 Set theory: Abstract classification.
 - 2.2 C^* -algebras (review).
 - 2.3 More set theory.
3. Saturday: Convincing you that 1.2–1.3 is logic.

Basic definitions

A topological space X is *Polish* if it is separable and completely metrizable.

A subset of X is *analytic* if it is a continuous image of a Borel set. An equivalence relation E on X is analytic if it is an analytic subset of X^2 .

Thesis

Almost all classical classification problems deal with analytic equivalence relations on Polish spaces.

“Proof” .

Objects are separable and the equivalence of A and B is witnessed by another separable object F in a Borel fashion.

The set of such triples (A, B, H) is Borel, and its projection is analytic. □

The only classical-ish example for which this does not seem to work is homeomorphism relation of Polish spaces.

Smoothness

Definition (Mackey)

An equivalence relation E on X is *smooth* if there is a Borel-measurable $f: X \rightarrow \mathbb{R}$ such that

$$x E y \Leftrightarrow f(x) = f(y).$$

Example

Similarity of $n \times n$ complex Hermitian matrices is smooth. Associate to M the list of its eigenvalues (in the increasing order, with multiplicities).

A criterion for non-smoothness

Proposition

If $G \curvearrowright X$ is a Polish group action on a Polish space such that all orbits are dense and meager (i.e., of first category) then the orbit equivalence relation E_G^X is not smooth.

Proof.

If $f: X \rightarrow \mathbb{R}$ is Borel, then we can find a dense G_δ subset Y of X such that the restriction of f to Y is continuous. The set $\{x \in X : g.x \in Y\}$ is comeager for all $g \in G$. Therefore we can find $x \in X$ such that $\{g \in G : g.x \in Y\}$ is comeager in G . Therefore $[x] \cap Y$ is dense. Then f is constant on $[x]$ and (by continuity) on Y . □

Example (Vitali equivalence relation)

On \mathbb{R} let $x \sim y$ iff $x - y \in \mathbb{Q}$. All orbits are countable and dense, hence \sim is not smooth.

Borel reducibility

Definition (H. Friedman, Kechris)

Assume E, F are equivalence relations on Polish spaces X, Y , respectively. Then E is *Borel reducible* to F , or $E \leq_B F$, if there is a Borel-measurable map $f: X \rightarrow Y$ such that

$$x E y \Leftrightarrow f(x) F f(y).$$

Interpretations:

1. Borel cardinality of X/E is \leq than the Borel cardinality of Y/F .
2. Classification problem for E is simpler than the classification problem for F .
3. F -Equivalence classes are complete invariants for E -equivalence classes.

Glimm-Effros Dichotomy

Definition

On $2^{\mathbb{N}}$, let $x E_0 y$ if

$$(\exists k)(\forall n \geq k)x(n) = y(n).$$

Theorem (Harrington–Kechris–Louveau, 1990)

If E is a Borel equivalence relation on a Polish space then either E is smooth or $E_0 \leq_B E$.

This is false for analytic equivalence relations, since there is one with exactly \aleph_1 equivalence classes.

Combinatorics of the proof comes from Glimm's theorem to the effect that every non-type I C^* -algebra has M_{2^∞} as a subquotient.

Whatever I say three times is true

Thesis

Almost all classical classification problems deal with analytic equivalence relations on Polish spaces.

Thesis

In almost all cases, the space of invariants has a Polish topology and the computation of invariants is given by a Borel-measurable function.

Example 1: Polish space of countable groups

Every countable group G is isomorphic to one of the form (\mathbb{N}, \cdot_G) , and the latter is coded by

$$\{(a, b, c) \in \mathbb{N}^3 : ab = c\}$$

Therefore the space \mathbb{G} of countable discrete groups is a Borel subspace of the compact metric space $\mathcal{P}(\mathbb{N}^3)$.

The isomorphism \cong^G is an analytic equivalence relation, because (by S_∞ we denote the Polish group of all permutations of \mathbb{N})

$$\{(G, H, f) \in \mathbb{G}^2 \times S_\infty, f: G \rightarrow H \text{ is an isomorphism}\}$$

is Borel.

Example 1a: Polish space of countable models

A construction analogous to \mathbb{G} gives a Borel space of all countable models in a fixed countable language.

Models of a fixed first-order theory form a Borel set.

The isomorphism relation is an S_∞ -orbit equivalence relation.

Classification by countable structures

An equivalence relation (X, E) is *classified by countable structures* if there is a countable language L and a Borel map f from X into the space of countable L -models such that

$$x E y \text{ iff } f(x) \cong f(y).$$

Proposition

An analytic equivalence relation E is classified by countable structures iff it is \leq_B to an S_∞ -orbit equivalence relation.

But what if we are classifying structures that are merely separable instead of countable?

Hyperspaces

Assume (K, d) is a compact metric space. The space $F(K)$ of all compact subsets of K equipped with the Hausdorff metric

$$d(F, G) = \inf\{\varepsilon : F \subseteq_\varepsilon G \text{ and } G \subseteq_\varepsilon F\}$$

(with $F \subseteq_\varepsilon G$ iff $(\forall a \in F)(\exists b \in G)d(a, b) \leq \varepsilon$) is also compact. Since every compact metric space is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$, $F([0, 1]^{\mathbb{N}})$ is the compact metric space of all compact metric spaces.

This does not work for non-compact Polish spaces X , since the Hausdorff metric is not separable on $F(X)$.

Effros Borel space

For a Polish space X let $F(X)$ be the space of closed subsets of X . Consider σ -algebra Σ on $F(X)$ generated by sets

$$\{A \in F(X) \mid A \cap U \neq \emptyset\}$$

where U ranges over open subsets of X .

Theorem (Effros)

$(F(X), \Sigma)$ is a standard Borel space (i.e., Σ is the σ -algebra of Borel sets for some Polish topology on X).

Example

Every separable Banach space is isometric to a closed subspace of $C([0, 1])$. Therefore

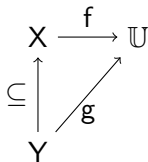
$$\{X \in F(C([0, 1])) : X \text{ is a closed subspace}\}$$

is 'the standard Borel space of all separable Banach spaces.'

Urysohn space, \mathbb{U}

This is a separable complete metric space which is universal for separable metric spaces and satisfies the following extension property:

for all finite metric $X \subseteq Y$, every isometry $f: X \rightarrow \mathbb{U}$ extends to an isometry $g: Y \rightarrow \mathbb{U}$.



Theorem (Clemens, Gao–Kechris, 2000)

Translation action of the isometry group $\text{Iso}(\mathbb{U})$ on $F(\mathbb{U})$ is the maximal orbit equivalence relation of a Polish group action.

C*-algebras review

1. Concrete C*-algebra is a norm-closed algebra of operators on a complex Hilbert space.
2. (Gelfand–Naimark–Segal, GNS) Abstract C*-algebra is a Banach algebra with involution $*$ that satisfies $\|a\|^2 = \|aa^*\|$ for all a .
3. (Gelfand–Naimark) Compact metric spaces are complete isomorphism invariants for separable unital abelian C*-algebras.
4. UHF algebras are direct limits of full matrix algebras, $M_n(\mathbb{C})$.
5. (Glimm, 1960) Generalized integer is a complete isomorphism invariant for unital UHF algebras.
6. AF algebras are direct limits of finite-dimensional C*-algebras.
7. (Elliott, 1974) Pre-ordered abelian group \mathbf{K}_0 is a complete isomorphism invariant for AF algebras.
8. In (3), (5), and (7) we even have equivalence of categories.

A nonseparable digression

Theorem (F.–Katsura, 2011)

For any uncountable cardinal κ there are 2^κ nonisomorphic unital UHF algebras of character density κ with the same K_0 .

Also, there are unital LM algebras of character density \aleph_2 that are not UHF.

However, this is a completely different story. All other C*-algebras in my talks will be separable, unital, simple and nuclear.

(This is not going to be on the exam.)

Standard Borel space of C^* -algebras, I

Lemma

Every separable C^ -algebra is isomorphic to a subalgebra of $\mathcal{B}(H)$, for **the** separable, infinite-dimensional complex Hilbert space H .*

A silly proof.

GNS followed by Löwenheim–Skolem. □

However, $\mathcal{B}(H)$ is not norm-separable, hence the space of its norm-closed subalgebras is not standard Borel.

Theorem (Junge–Pisier, 1995)

There is no universal separable C^ -algebra.*

Standard Borel space of C^* -algebras, II

Definition (Kechris, 1995)

Endow $\mathcal{B}(H)$ with the Borel structure of the strong operator topology. Then $\Gamma = \mathcal{B}(H)^{\mathbb{N}}$ is a standard Borel space. Every $\gamma \in \Gamma$ 'codes' the C^* -algebra $C^*(\gamma)$ generated by it.

Proposition

The following subsets of Γ are Borel.

1. (easy) $\{\gamma \mid C^*(\gamma) \text{ is unital}\}$,
2. (Effros) $\{\gamma \mid C^*(\gamma) \text{ is nuclear}\}$,
3. (F.-Toms-Törnquist) $\{\gamma \mid C^*(\gamma) \text{ is simple}\}$.

Review: Elliott's program

Conjecture (Elliott, 1990's)

All nuclear,¹ separable, simple, unital, infinite-dimensional C-algebras are classified by the K-theoretic invariant,*

$$\text{Ell}(A) : \quad ((K_0(A), K_0(A)^+, 1), K_1(A), T(A), \rho_A).$$

The conjecture is false, but it has led to some spectacular mathematics and many instances of its revised version have been confirmed.

¹I shall define nuclear C*-algebras tomorrow. All algebras mentioned today (except $\mathcal{B}(H)$) are nuclear.

Everything is Borel

Theorem (F.–Toms–Törnquist, 2011)

*There is a standard Borel spaces **EII** of Elliott invariants, and the computation of EII is given by a Borel map.*

$$\text{Separable } C^*\text{-algebras} \leftarrow \Gamma \xrightarrow{\Phi} \mathbf{EII} \rightarrow \text{Elliott invariants}$$

Corollary

*The isomorphism relation of unital UHF algebras is smooth.
The isomorphism relation of AF algebras is classifiable by countable structures.*

Proof.

Combine the above with Glimm's and Elliott's theorems. □

Hjorth's turbulence

An action $G \curvearrowright X$ is *turbulent* if

1. all orbits are dense
2. all orbits are meager
3. $\forall x \in X, \forall U \in \mathcal{O}(e), \forall V \in \mathcal{O}(x)$ the graph on U defined by

$$\{z, y\} \in E \Leftrightarrow (\exists g \in U)g.z = y$$

is such that the closure of the connected component of x intersects every orbit.

Example

The action of $c_0 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \lim_n |x_n| = 0\}$ on $\mathbb{R}^{\mathbb{N}}$ by translation.

Theorem (Hjorth, 1997)

If $G \curvearrowright X$ is turbulent then the orbit equivalence relation E_G^X is not classified by countable structures.

Compact metrizable spaces I

Proposition (Folklore?)

Homeomorphism relation of closed subsets of $[0, 1]$ is classifiable by countable structures.

Proof.

If $K \subseteq [0, 1]$ is compact then it has only two types of connected components: singleton and interval. Use the 'tagged' version of Cantor–Bendixson analysis. □

Corollary (trust me)

The isomorphism relation of unital abelian C^ -algebras **generated by a single self-adjoint element** is classifiable by countable structures.*

Compact metrizable spaces II

Proposition (F.–Toms–Törnquist, after Hjorth)

Homeomorphism relation of closed subsets of $[0, 1]^2$ is not classifiable by countable structures.

Corollary

*The isomorphism relation of **singly-generated** unital abelian C^* -algebras is not classifiable by countable structures.*

Question

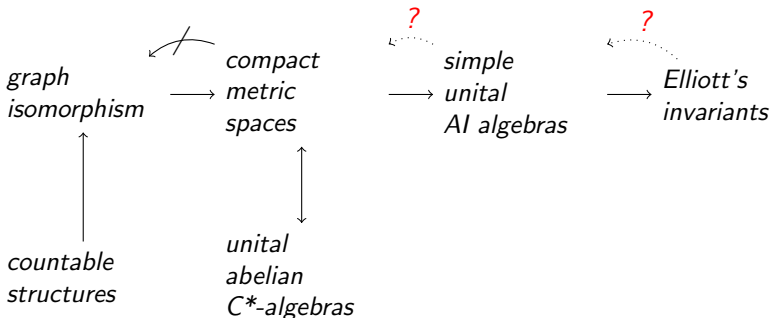
Does the complexity of the isomorphism relation for unital abelian separable C^ -algebras increase if the number of generators increases?*

AI algebras are not classifiable by countable structures ... although they are classifiable by Elliott's invariant

AI algebras are direct limits of $C([0, 1], M_n(\mathbb{C}))$ for $n \in \mathbb{N}$.
Using a Borel version of a result of Thomsen and Elliott's classification result for AI algebras, we obtained the following:

Theorem (F.–Toms–Törnquist, 2011)

We have the following Borel-reductions



The dark side

On $[0, 1]^{\mathbb{N}}$ define

$$x E_1 y \text{ if and only if } (\forall^\infty n) x(n) = y(n)$$

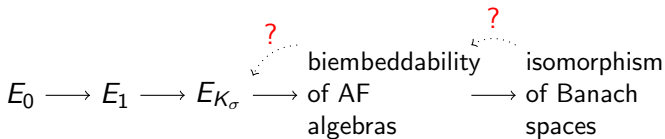
Theorem (Kechris–Louveau, 1997)

If $E_1 \leq_B E$ then E is not Borel-reducible to any orbit equivalence relation of a Polish group action.

Theorem (Ferenczi–Louveau–Rosendal, 2009)

Isomorphism of separable Banach spaces is the \leq_B -maximal analytic equivalence relation.

Together with another result of F.-Toms-Törnquist, this gives



Theorem (folklore)

Isomorphism of von Neumann factors with separable predual is below an orbit equivalence relation.

Therefore classifying von Neumann factors is easier than classifying Banach spaces (up to the isomorphism).²

²Of course this statement has to be taken with a grain of salt. In this context classifying finite simple groups is strictly easier than comparing real numbers.

Theorem (F.–Toms–Törnquist, 2011)

The isomorphism relations in the following categories are below an orbit equivalence relation.

1. *Separable, simple, nuclear, unital C^* -algebras.*
2. *Elliott invariants.*

Proof of (1) uses a Borel version of a very difficult result of Kirchberg and does not appear to be amenable to generalizations.

Polish groupoids

Partially following A. Ramsay, we say that a structure $(\mathcal{O}, \mathcal{A})$ (objects and arrows) is a *Polish groupoid* if

1. It is a groupoid,
2. Both \mathcal{O} and \mathcal{A} carry a Polish topology,
3. Operations $s: \mathcal{A} \rightarrow \mathcal{O}$ and $r: \mathcal{A} \rightarrow \mathcal{O}$ ('source' and 'range') are continuous,
4. Composition is continuous on the set $\{(f, g) \in \mathcal{A}^2 \mid f \circ g \in \mathcal{A}\}$.

On \mathcal{O} define $x E_{(\mathcal{O}, \mathcal{A})} y$ iff

$$(\exists f \in \mathcal{A})(s(f) = x \text{ and } r(f) = y).$$

Proposition (Coskey–Elliott–F.–Lupini, 2012)

$E_1 \not\leq_B E_{(\mathcal{O}, \mathcal{A})}$ for any Polish groupoid.

However...

Question

If A is a separable C^ -algebra, does the groupoid whose objects are subalgebras of A and arrows are $*$ -isomorphisms between them carry a Polish groupoid structure?*

Question

Is every $E_{(\mathcal{A}, \mathcal{O})}$ Borel-reducible to an orbit equivalence relation of a Polish group action?

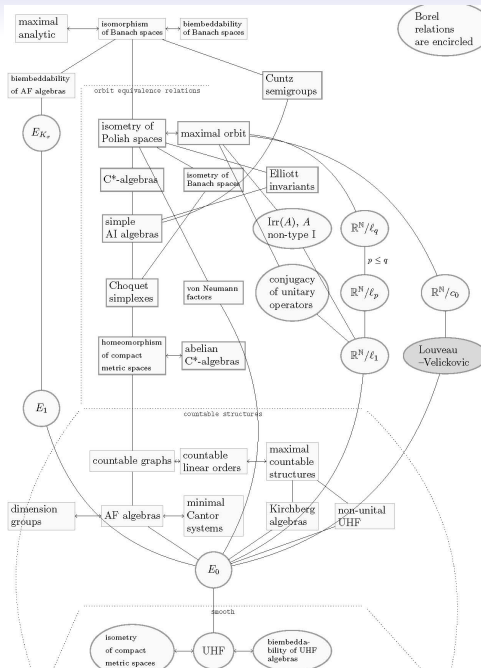
At least I know the answer to the following question:

Question

Is the isomorphism of separable (simple) C^ -algebras below an orbit equivalence relation?*

We will find this out tomorrow.

Borel reductions diagram



Borel reductions diagram, sideways

