



Happy Bastille Day!

model theory

Elliott's program and descriptive set theory
~~descriptive set theory~~
theory III

Ilijas Farah

(joint work with Bradd Hart and David Sherman
and with

George Elliott, Vern Paulsen, Christian Rosendal, Andrew Toms and Asger Törnquist)

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As logicians, we do our subject a disservice by convincing others that logic is first order, and then convincing them that almost none of the concepts of modern mathematics can really be captured in first order logic. (Jon Barwise)

The plan

1. Thursday:
 - 1.1 Basic properties of C^* -algebras.
 - 1.2 Classification: UHF and AF algebras.
 - 1.3 Elliott's program.
2. Yesterday: Applying logic to 1.2–1.3.
 - 2.1 Set theory.
 - 2.2 C^* -algebras (review).
 - 2.3 More set theory.
3. Today: Convincing you that 1.2–1.3 is logic.
 - 3.1 Review.
 - 3.2 Logic of metric structures.
 - 3.3 A proof from the book.
 - 3.4 It is all logic.

Review I: C^* -algebras

C^* -algebras are norm-closed subalgebras of $\mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space H .

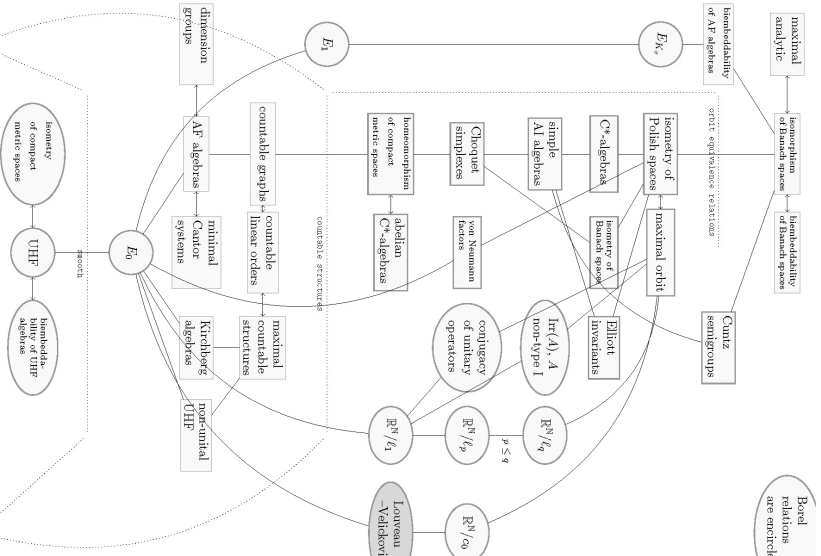
Separable unital algebras that are direct limits of finite-dimensional C^* -algebras (UHF and AF algebras) were classified by Glimm and Elliott.

Elliott's program: Classify separable, unital, simple, nuclear C^* -algebra by K-theoretic invariants.

Are there set-theoretic obstructions to this?

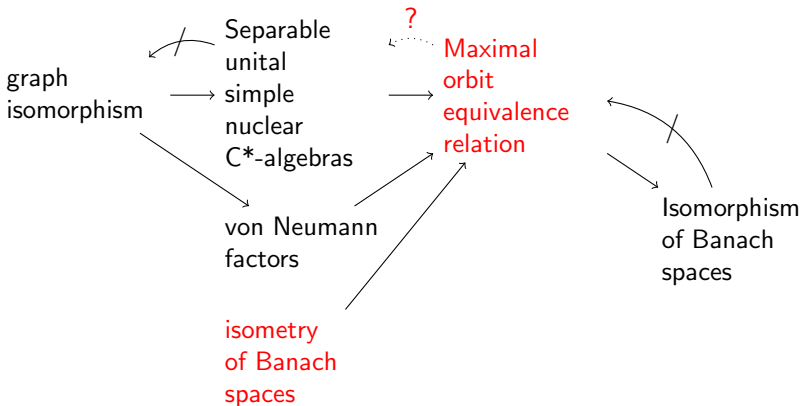
Review II: Borel reductions

$E \leq_B F$ iff there exists a Borel function f such that
 $x E y$ iff $f(x) F f(y)$.



Review II: Set theory

By results of F.–Toms–Törnquist, Ferenczi–Louveau–Rosendal and Melleray:



Question

Is the isomorphism of separable C^* -algebras \leq_B an orbit equivalence relation of a Polish group action?

Review: Urysohn space, \mathbb{U}

It is a separable complete metric space which is universal for separable metric spaces and such that for all finite metric $X \subseteq Y$, every isometry $f: X \rightarrow \mathbb{U}$ extends to an isometry $g: Y \rightarrow \mathbb{U}$.

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{U} \\ \uparrow \subseteq & \nearrow g & \\ Y & & \end{array}$$

Theorem (Clemens–Gao–Kechris, 2000)

The orbit equivalence relation of $\text{Iso}(\mathbb{U}) \curvearrowright F(\mathbb{U})$ is the \leq_B -maximal among orbit equivalence relations of Polish group actions.

Logic of metric structures

Developed by C.W. Henson, I. Ben Ya'acov, A. Berenstein, and A. Usvyatsov.

I shall describe only the 'logic of C^* -algebras' as modified¹ by F.–Hart–Sherman.

¹several times

Logic of C^* -algebras: Syntax

Language: $\{+, \cdot, *\}$.

Terms (s, t, \dots) :

noncommutative $*$ -polynomials.

Atomic formulas (φ, ψ, \dots) :

$\|t\|$ for a term t .

Formulas (φ, ψ, \dots) :

The smallest set \mathbb{F} that satisfies

1. all atomic formulas are in \mathbb{F} ,
2. if $g: \mathbb{R}^n \mapsto \mathbb{R}$ is uniformly continuous and $\varphi_1, \dots, \varphi_n$ are in \mathbb{F} then

$$g(\varphi_1, \dots, \varphi_n)$$

is in \mathbb{F} ,

3. $\sup_{\|x_i\| \leq 1} \varphi$ and $\inf_{\|x_i\| \leq 1} \varphi$ are in \mathbb{F} whenever φ is in \mathbb{F} .

Logic of C^* -algebras: Semantics

If A is a normed metric structure with operations $+$, \cdot , $*$ that are **uniformly continuous** on bounded sets and $\varphi(x)$ is a formula then

$$\varphi(x)^A$$

is interpreted in the natural way.

Its interpretation is a function into \mathbb{R} that is **uniformly continuous** on bounded sets.

Example

Fix C^* -algebra A .

1. If $\varphi_P(x)$ is $\|x^2 - x\| + \|x - x^*\|$ then the zero-set of φ_P

$$\{a \in A \mid \varphi_P(a)^A = 0\}$$

is the set of projections in A .

2. If

$\psi_{MvN}(x, y) = \varphi_P(x) + \varphi_P(y) + \inf_{\|z\| \leq 1} (\|x - zz^*\| + \|y - z^*z\|)$,
then the zero set of ψ_{MvN} is $\{(p, q) : p \sim q\}$.

Theory of a C^* -algebra A , $\text{Th}(A)$

A theory of a C^* -algebra A is the set

$$\{\varphi : \varphi^A = 0\}.$$

Alternatively, one could define the theory of A as the map from the set of all sentences into \mathbb{R}^+ :

$$\varphi \mapsto \varphi^A.$$

With any natural Borel space of models and Borel space of formulas, one has the following

Lemma

The map $A \mapsto \text{Th}(A)$ is Borel.

A short intermission

Your theorem is not as good as you think when you prove it and it is not as bad as you think five days later.

(Gert K. Pedersen)

He [G.K. Pedersen] was obsessed with being witty.

(Anonymous)



Corollary (Elliott–F.–Paulsen–Rosenthal–Toms–Törnquist)

The isomorphism relation of separable C^ -algebras is \leq_B an orbit equivalence relation of a Polish group action.*

A perfect analogy

countable
models

\longleftrightarrow Actions of S_∞

complete
separable
metric
models

\longleftrightarrow Actions of $\text{Iso}(\mathbb{U})$

Problem

Develop a method for distinguishing orbit equivalence relations of turbulent actions of different Polish groups.

The definition of nuclear C^* -algebras, finally

There are several equivalent ways to define nuclear algebras. I will use one that is most convenient for my purposes. It will take some time to define it.

Positivity

An element a of a C^* -algebra is *positive* if $a = b^*b$ for some b .

A linear map $\Phi: A \rightarrow B$ is *positive* if it sends positive elements to positive elements.

It is *completely positive* if

$$M_n(A) \ni (a_{ij})_{i,j \leq n} \mapsto (\Phi(a_{ij}))_{i,j \leq n} \in M_n(B)$$

is positive for all n .

Example

1. Every $*$ -homomorphism is completely positive.
2. The transpose map on $M_2(\mathbb{C})$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

is positive but not completely positive.

Positivity II

Proposition

If $\Phi: A \rightarrow B$ is a $*$ -homomorphism and $p \in B$ is a projection, then

$$a \mapsto p\Phi(a)p$$

is completely positive.

ucp:= unital completely positive

ucp maps $\varphi: A \rightarrow \mathbb{C}$ (aka *states*) play a key role in the GNS theorem.

Completely Positive Approximation Property (CPAP)

Definition

A unital C^* -algebra A is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps φ_j and ψ_j for $j \in \mathbb{N}$

$$A \begin{array}{c} \searrow \\ \varphi_j \\ \rightarrow \\ M_{n(j)}(\mathbb{C}) \end{array} \begin{array}{c} \nearrow \\ \psi_j \\ \rightarrow \\ A \end{array}$$

such that $\psi_j \circ \varphi_j$ converges to id_A pointwise.

Lemma

1. Each $M_n(\mathbb{C})$ is nuclear.
2. Direct limits of nuclear algebras are nuclear.
3. $UHF \Rightarrow AF \Rightarrow$ nuclear.
4. abelian \Rightarrow nuclear.
5. A nuclear, X cpct Hausdroff $\Rightarrow C(X, A)$ nuclear.

Please bear with me - I'll put nuclear algebras on hold for a couple of slides.

Ultrapowers

According to David Sherman, functional analysts discovered ultrapowers before us.

F. B. Wright, A reduction for algebras of finite type, *Ann. of Math.* (2) 60 (1954), 560–570.

K. Łos, Quelques remarques, théorèmes et problèmes sur les classes d'efinissables d'algèbres, *Mathematical interpretation of formal systems*, pp. 98–113. North-Holland Publishing Co., Amsterdam, (1955).

Ultrapowers II

If A is a C^* -algebra and \mathcal{U} is an ultrafilter on \mathbb{N} then let

$$L^\infty(A) = \{(a_n) \in A^{\mathbb{N}} \mid \sup_n \|a_n\| < \infty\}$$

and

$$c_0(\mathcal{U}) = \{(a_n) \in L^\infty(A) : \lim_{n \rightarrow \mathcal{U}} \|a_n\| = 0\}.$$

The *ultrapower* of A is

$$\prod_{\mathcal{U}} A = L^\infty(A)/c_0(\mathcal{U}),$$

usually denoted $A^{\mathcal{U}}$ by operator algebraists.

Here is a sample of what I originally planned to talk about

In the following identify B with its diagonal copy in $\prod_{\mathcal{U}} B$.

Exercise

Assume A is a subalgebra of a separable algebra B , \mathcal{U} is an ultrafilter on \mathbb{N} , and the ultrapower $\prod_{\mathcal{U}} B$ has automorphisms Φ_n for $n \in \mathbb{N}$ such that (identifying A and B with their diagonal copies in the ultrapower)

1. Φ_n fixes all elements of A ,
2. $\lim_{n \rightarrow \infty} \text{dist}(\Phi_n(b), \prod_{\mathcal{U}} A) = 0$ for all $b \in B$.

Then $A \cong B$.

This is used e.g., to characterize C^* -algebras A such that $A \otimes \mathcal{Z} \cong A$ (\mathcal{Z} is the notorious Jiang–Su algebra).

Some unnerving facts

Theorem (Junge–Pisier, 1995)

There is a finite set $F \subseteq \mathcal{B}(H)$ such that any C^ -algebra A such that $F \subseteq A \subseteq \mathcal{B}(H)$ is not nuclear.*

Nuclear algebras form a 'nonstationary set!'

Lemma

An ultrapower of a UHF algebra is not nuclear.

Nuclear algebras are not axiomatizable!

(And the same applies to UHF, AF, AI, AT, AH, ...).

UHF algebras revisited

Lemma

A separable C^ -algebra is UHF if and only if it is LM (locally matricial), i.e., if*

Every finite $F \subseteq A$ is ε -included in some full matrix subalgebra of A , for every $\varepsilon > 0$.

Proposition

For every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a type $\mathbf{t}_\varepsilon(x_0, \dots, x_{n-1})$ in the theory of C^ -algebras over \emptyset such that in every C^* -algebra A , type \mathbf{t}_ε is realized by a_0, a_1, \dots, a_{n-1} iff no full matrix subalgebra ε -includes $\{a_0, \dots, a_{n-1}\}$.*

Glimm revisited

Corollary

There is a sequence of types $\mathfrak{t}_{1/k}$ for $k \in \mathbb{N}$ such that a C^ -algebra A is UHF iff it omits all of those types.*

Proofs of both the above and the following use a bit of what I called 'stability' in my first talk.

Theorem (Glimm, 1960)

Separable unital C^ -algebras that omit all $\mathfrak{t}_{1/k}$ are isomorphic iff they are elementarily equivalent.*

(Not surprisingly, this fails in the nonseparable case by F.–Katsura.)

Revisiting AF... but not Elliott

Proposition

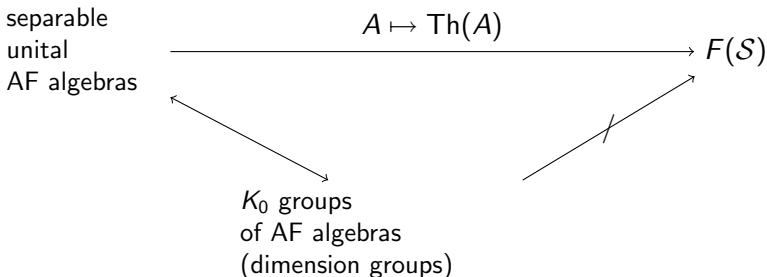
There is a sequence of types $\mathfrak{s}_{1/k}$ for $k \in \mathbb{N}$ such that a C^ -algebra A is AF iff it omits all of those types.*

Proposition

There are separable, unital AF algebras that are elementarily equivalent but nonisomorphic.

Proof.

Let \mathcal{S} be the set of all sentences in the language of C^* -algebras.



K-theory is good

Problem

Is there a model-theoretic interpretation of Elliott's theorem?

All known obstructions to \aleph_1 -saturation of the Calkin algebra and other corona algebras are of K-theoretic nature. (F.-B. Hart–N. C. Phillips).

Question

Is K-theory the only obstruction to \aleph_1 -saturation of the Calkin algebra?

What about the nuclearity?

Definition

A unital C^* -algebra A is *nuclear* if there are $n(j) \in \mathbb{N}$ and ucp maps φ_j and ψ_j for $j \in \mathbb{N}$

$$\begin{array}{ccc} A & & A \\ & \searrow \varphi_j & \nearrow \psi_j \\ & M_{n(j)}(\mathbb{C}) & \end{array}$$

such that $\varphi_j \circ \psi_j$ converges to id_A pointwise.

Conjecture

There is a sequence of types such that the nuclear algebras are exactly the C^ -algebras omitting those types.*

Thesis

We have only scratched the surface.

