Homotopy-initial W-types

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Homotopy type theory

Main fact:

- There is a new class of models for Martin-Löf's type theories, in which types are interpreted as spaces.

Main consequence:

- We have a new geometric intuition to work in type theory, which provides inspiration for new type-theoretic notions, theorems and axioms.

Aim of the talk

Type theory	**Homotopy theory**
$A : \mathsf{type}$	A space
$a : A$	$a \in A$
$x : A \vdash B(x) : \mathsf{type}$	$B \to A$ fibration
$x : A, y : A \vdash \mathsf{Id}_A(x, y)$	$A^{[0,1]} \to A \times A$
$\vdots$	$\vdots$
Inductive types	**?**

Overview

Part I. Strictly initial W-types

- The type theories H and H^{ext}
- W-types
- Characterisation of W-types over H^{ext}

Part II. Homotopy-initial W-types

- Contractibility
- Characterisation of weak W-types over H

Part I

Strictly initial W-types

Type theories

Forms of judgements

$$\Gamma \vdash A : \mathsf{type} \qquad \Gamma \vdash a : A$$

$$\Gamma \vdash A = B : \mathsf{type} \qquad \Gamma \vdash a = b : A$$

Note

- Dependent types, e.g.

$$n : \mathsf{Nat} \vdash \mathsf{List}_n(A) : \mathsf{type}$$

- Definitional vs. propositional equality.

The type theory H

- Standard deduction rules for

$$\mathsf{Id}_A(a, b)\,, \quad (\Sigma x : A)B(x)\,, \quad (\Pi x : A)B(x)\,,$$

with which we can define

$$A \times B\,,\ A \to B\,.$$

- The function extensionality principle, i.e. the type

$$(\Pi x : A)\,\mathsf{Id}_{B(x)}(f(x), g(x)) \to \mathsf{Id}_{(\Pi x:A)B(x)}(f, g)$$

is inhabited.

Note

- The univalence axiom implies function extensionality
- H has models in **Gpd**, **SSet** and **Set**

The type theory H^{ext}

$$\mathsf{H}^{\text{ext}} = \mathsf{H} \quad + \quad \frac{p : \mathsf{Id}_A(a, b)}{a = b : A}$$

Note

- H^{ext} has models in locally cartesian closed categories
- Type-checking becomes undecidable

W-types

Formation rule

$$\frac{x : A \vdash B(x) : \mathsf{type}}{(Wx : A)B(x) : \mathsf{type}}$$

Introduction rule

$$\frac{a : A \qquad t : B(a) \to W}{\mathsf{sup}(a, t) : W}$$

where $W =_{\mathrm{def}} (Wx : A)B(x)$

Elimination rule

$$\frac{\begin{array}{l} w : W \vdash E(w) : \mathsf{type} \\ x : A\,,\ u : B(x) \to W,\ v : (\Pi y : B(x))E(u(y)) \vdash e(x,u,v) : E(\mathsf{sup}(x,u)) \end{array}}{w : W \vdash \mathsf{rec}(w,e) : E(w)}$$

Computation rule

$$\frac{\begin{array}{l} w : W \vdash E(w) : \mathsf{type} \\ x : A\,, u : B(x) \to W\,, v : (\Pi y : B(x))E(u(y)) \vdash e(x,u,v) : E(\mathsf{sup}(x,u)) \end{array}}{x : A, u : B(x) \to W \vdash \mathsf{rec}(\mathsf{sup}(x,u),e) = e(x,u,\ldots) : E(\mathsf{sup}(x,u))}$$

Polynomial functors and their algebras in $\mathsf{H}^{\mathrm{ext}}$

For $x : A \vdash B(x) : \mathsf{type}$ let

$$\begin{aligned} P : \mathbf{Types} \quad &\longrightarrow \quad \mathbf{Types} \\ X \quad &\longmapsto \quad (\Sigma x : A)(B(x) \to X) \end{aligned}$$

Definition.

- P-algebra:

$$\big(X \,,\ s_X : P(X) \to X \,\big)$$

- P-algebra morphism:

$$\begin{array}{ccc} P(X) & \xrightarrow{\;P(f)\;} & P(Y) \\ {\scriptstyle s_X}\big\downarrow & & \big\downarrow{\scriptstyle s_Y} \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

Characterisation over H^{ext}

Theorem (Dybjer, Moerdijk & Palmgren). Over H^{ext} the following are equivalent:

1. Every polynomial functor has an initial algebra
2. The deduction rules for W-types

Note

- Induction vs. recursion
- Strict initiality

Part II

Homotopy-initial W-types

Problem

Within H the deduction rules for W-types imply:

- Existence of

$$\begin{array}{ccc} P(W) & \xrightarrow{\ P(f)\ } & P(X) \\ {\scriptstyle s_W}\big\downarrow & s_f & \big\downarrow{\scriptstyle s_X} \\ W & \xrightarrow[\ f\ ]{} & X \end{array}$$

  where $s_f : \mathsf{Id}\big(f \cdot s_W\,, s_X \cdot P(f)\big)$
- But also propositional uniqueness of f (an η-rule),
- But also propositional uniqueness of the above proof ...
- ...

How can we capture all this?

Contractibility

Definition (Voevodsky)**.** A type X is **contractible** if

$$\mathrm{iscontr}(X) =_{\mathrm{def}} (\Sigma x : X)(\Pi y : X)\mathsf{Id}_X(x, y)$$

is inhabited.

Idea

- Existence and uniqueness

Note

- X contractible $\Leftrightarrow X \simeq \mathsf{1}$
- X contractible $\Rightarrow \mathsf{Id}_X(x, y)$ contractible for all $x, y : X$

P-algebras and weak P-algebra maps in H

Given $x : A \vdash B(x) : \mathsf{type}$, let $P(X) =_{\mathrm{def}} (\Sigma x : A)(B(x) \to X)$

Definition.

- P-algebra:

$$(X, s_X : P(X) \to X)$$

- Weak P-algebra morphism:

$$\begin{array}{ccc} P(X) & \xrightarrow{P(f)} & P(Y) \\ {\scriptstyle s_X}\downarrow & s_f & \downarrow{\scriptstyle s_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

  where

$$s_f : \mathsf{Id}\big(f \cdot s_X \,, s_Y \cdot P(f)\big)$$

Homotopy-initial algebras

Given P-algebras (X, s_X) and (Y, s_Y), we can define the type

$$P\text{-alg}\big[(X, s_X), (Y, s_Y)\big]$$

of weak P-algebra morphisms between them.

Definition. A P-algebra (X, s_X) is **homotopy-initial** if for every P-algebra (Y, s_Y) the type

$$P\text{-alg}\big[(X, s_X), (Y, s_Y)\big]$$

is contractible.

Characterisation over H

Theorem. Over H, the following are equivalent:

1. Every polynomial functor has a homotopy-initial algebra
2. The formation, introduction, elimination and propositional computation rules for W-types.

Propositional computation rule

$$\frac{\vdots}{\ldots \ \vdash \mathsf{comp}(x,u,e) : \mathsf{Id}\big(\mathsf{rec}(\mathsf{sup}(x,u),e),\ e(x,u,\ldots)\big)}$$

Remarks

- These are homotopy-invariant W-types
- Homotopy-initiality implies existence and uniqueness of weak P-algebra maps up to higher and higher identity proofs, since

$$\begin{aligned} P\text{-alg}\big[(X, s_X), (Y, s_Y)\big] &\text{ contractible} \\ &\Rightarrow \mathsf{Id}((f, s_f), (g, s_g)) \text{ contractible} \\ &\Rightarrow \mathsf{Id}((\alpha, s_\alpha), (\beta, s_\beta)) \text{ contractible} \\ &\quad\vdots \end{aligned}$$

- Similar analysis carries over to other inductive types
- In H weak W-types allow us to define weak versions of other inductive types.

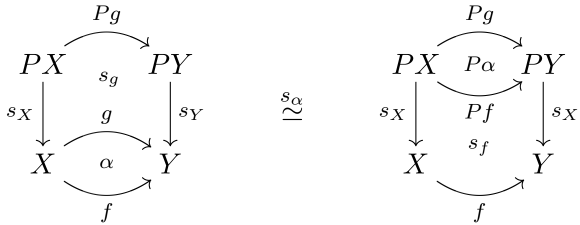
Remarks

Key Lemma. Equivalence between:

- Identity proofs between weak algebra maps (f, s_f), (g, s_g)
- Algebra 2-cells from (f, s_f) to (g, s_g), i.e. pairs (α, s_α) where

$$\alpha : \mathsf{Id}(f, g)$$

and



References

Paper

- S. Awodey, N. Gambino, K. Sojakova
  Inductive Types in Homotopy Type Theory
  LICS 2012

Proofs

- Coq code on Github