Human effective computability and Absolute Undecidability

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Structure of the talk

1. Human effective computability
2. Epistemic Church's Thesis
3. Absolute undecidability
4. Is Epistemic Church's Thesis true?

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Machine effective computability and human effective computability

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Kreisel (1972) draws a distinction between:

- machine effective computability
- human effective computability

Thesis

machine effective computability = algorithmic computability

human effective computability = ?

Kreisel on human effective computability

" [in human effective computability], 'effective' means humanly performable and not only mechanical"

"[human] effectively definable functions as the analogue of provable theorems"

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Kreisel on human effective computability

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Definition

A function f is human effective computable iff, **recognisably**, for every number m given in canonical notation, a canonically given number n exists such that the statement $f(m) = n$ is humanly **provable**.

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A priori knowability

⇒ How should the epistemic notion involved be understood?

It must be an **iterable** notion

- not as informal mathematical provability
- but as *a priori knowability*

Church's Thesis for human effective computability

Let $\phi(x, y)$ be a total functional predicate.

Thesis (HCT)

If $\phi(x, y)$ is human effectively computable, then there is a Turing machine e such that for all $m \in \mathbb{N} : \phi(m, e(m))$.

$\Rightarrow$ Is HCT true?

Idealisation (I)

Any [. . .] theory [of human effective computability] would seem to need an idealisation far removed from our ordinary experience (of human performances in mathematics). Consequently, we have not one, but two difficulties. If **experience presents itself in such a way that the proper idealisation is difficult to find** *then, for the same reason, the idealisation may be difficult to apply even if it is found. In particular, there will now be a genuine problem of formulating principles of evidence or adequacy conditions for the validity of idealisations. Besides when idealisations are difficult to find there will, in general, be competing theories and hence the problem of discovering (observational) consequences which can be used to decide between different theories. (Kreisel 1972)*

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Idealisation (II)

1. It is reasonable to take the subject of our notion of a priori knowability to be the human community as a whole.
2. The subject does not have any fixed finite limitations of memory space or life span.
3. It is reasonable to take a priori knowability to have a discretely ordered temporal structure. (This may or may not be a *branching* temporal structure.)
4. At every given point in time, it is reasonable to take what is a priori known to be closed under logical consequence.
5. At every given point in time, the extension of what is a priori known is recursively axiomatisable.

Epistemic Arithmetic

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The language of Epistemic Arithmetic ($\mathcal{L}_{EA}$) consists of the language of PA plus an epistemic operator $\Box$ (a priori knowability).

$$EA = PA + S4$$

Human effective computability and calculability

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Definition (Shapiro, 1985)

A total functional expression $\phi(x, y)$ is *calculable* iff $\Box \forall x \exists y \Box \phi(x, y)$.

Thesis

For any total functional expression $\phi(x, y)$*:*
$\phi(x, y)$ *is human effective computable iff* $\phi(x, y)$ *is calculable*

Epistemic Church's Thesis

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Thesis (ECT, Shapiro (1985))

$\Box \forall x \exists y \Box \phi(x, y) \rightarrow$ *"ϕ is Turing-computable"*

Thesis

ECT is a good formalisation of HCT

Gödel's Disjunction

Thesis (Gödel, 1951)

Either the Human Mathematical Mind is not a Turing machine, or there are absolutely undecidable statements.

Question

Can we be more specific?

This is hard. . .

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Question

Can we be more specific?

This is hard...

We will argue for an analogue of Gödel's disjunction.

Absolute undecidability in $\mathcal{L}_{EA}$

"ϕ is absolutely undecidable" can be expressed in $\mathcal{L}_{EA}$ as

$$\neg\Box\phi \wedge \neg\Box\neg\phi.$$

Definition (McKinsey, $S4.1$)

$$\neg\Box(\neg\Box\phi \wedge \neg\Box\neg\phi)$$

Absolutely undecidable arithmetical statements

Thesis

Provable unprovability of an arithmetical proposition supervenes on a provable negative arithmetical fact.

Axiom (A)

$\Box\neg\Box\phi \rightarrow \Box\neg\phi$ *for* ϕ *any sentence of the language of PA.*

Proposition

If A *is true, then there are no provably absolutely undecidable arithmetical sentences.*

So A entails $S4.1$ restricted to arithmetical sentences.

Other undecidables

- Fitch's argument
  - We are interested only in noncontingent statements here
- knower sentences
  - We are interested only in grounded statements here
- set theoretic undecidables
  - We are interested only in sentences of $\mathcal{L}_{EA}$ here

A new disjunctive thesis

Theorem
If ECT is true, then there are Π_3 absolutely undecidable sentences expressible in the language of EA.

- That *ECT* entails the existence of absolutely undecidables is easy to see (erasing $\Box$s).
- To establish the lower bound we have to do a little more work...

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`Equivalently...:`

Either CT for human effective computability fails, or there are absolute undecidables of low complexity.

ECT and Absolute Undecidability II

Proof by contraposition. Suppose that there are no absolutely undecidable Π_3 sentences in $\mathcal{L}_{EA}$, i.e.:

$$\Box\Psi \leftrightarrow \Psi \text{ for all } \Pi_3 \text{ sentences } \Psi \in \mathcal{L}_{EA}.$$

Choose a Turing uncomputable total functional Π_1 relation $\phi(x, y) \in \mathcal{L}_{PA}$. From elementary recursion theory we know that such $\phi(x, y)$ exist. Then $\forall x \exists y \phi(x, y)$. But then we also have $\forall x \exists y \Box\phi(x, y)$. The reason is that $\Pi_1 \subseteq \Pi_3$, so for every m, n, $\phi(m, n)$ (being a Π_1 statement) entails $\Box\phi(m, n)$. But now $\forall x \exists y \Box\phi(x, y)$ is a Π_3 statement of $\mathcal{L}_{EA}$. So from our assumption again, it follows that $\Box\forall x \exists y \Box\phi(x, y)$. So for the chosen $\phi(x, y)$, the antecedent of *ECT* is true, whereas its consequent is false. So, for the chosen $\phi(x, y)$, *ECT* is false.

Consequences

Corollary

If ECT holds, then its converse fails.

- the converse of Church's Thesis is "trivial".

Corollary

If $\Box(ECT)$, then $S4.1$ does not hold.

- Recall that $S4.1$ restricted to $\mathcal{L}_{PA}$ perhaps does hold...

Corollary

If ECT holds, then the antecedent of ECT is intensional.

- ECT is not an adequate formalisation of CT.

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Intensionality

Proof.

Let ψ be true but absolutely unprovable, and let g be the constant 0 function. Define:

$$\begin{cases} \forall x \; f(x) := g(x) \text{ if } \psi \\ \forall x \; f(x) := g(x) + 1 \text{ if } \neg\psi \end{cases}$$

Then we have that f is co-extensional with g but f is not provably coextensive with g. The function g is calculable; the function f is not. □

A dubious thesis

Is *ECT* true?

Yes if . . .

A dubious thesis

Is *ECT* true?

Yes if ...

Thesis

The only way in which a statement of the form $\forall x \exists y \Box \phi(x, y)$ can be priori known is by giving an algorithm for computing ϕ.

`Problem:` Can it be that for some functional expression $\phi(x, y)$, it is a priori knowable *in a non-constructive way* that $\forall x \exists y \Box \phi(x, y)$?

A dubious thesis

Is *ECT* true?

Yes if ...

Thesis

The only way in which a statement of the form $\forall x \exists y \Box \phi(x, y)$ *can be priori known is by giving an algorithm for computing* ϕ*.*

`Problem:` Can it be that for some functional expression $\phi(x, y)$, it is a priori knowable *in a non-constructive way* that $\forall x \exists y \Box \phi(x, y)$?

We will 'test' ECT in a class of models...

Branching time models

The language of Modal-Epistemic Arithmetic $\mathcal{L}_{MEA}$: Split the a priori knowability operator in an a priori knowledge operator P and a possibility operator $\Diamond$

$$\text{“}\phi \text{ is a priori knowable”} \approx \Diamond P\phi$$

Definition

A model for $\mathcal{L}_{MEA}$ is an ordered triple $\langle W, R, f \rangle$, with

- W a set of epistemic states of the idealised epistemic agent
- R a partial ordering relation
- $f : W \mapsto \mathcal{P}(\mathcal{L}_{MEA})$ assigns a collection of a priori known sentences of $\mathcal{L}_{MEA}$ to epistemic states

Acceptable models

Condition
For every $w \in W$, $f(w)$ is a Σ_1-definable set.

Condition
if a sentence $P\phi$ is true in a state w, then ϕ also has to be true in state w.

Condition
For every w, $f(w)$ is closed under logic.

Condition
$wRw' \Rightarrow f(w) \subseteq f(w')$

Non-clairvoyance

Definition

If a model $\mathfrak{M}$ is given, then the truncation of $\mathfrak{M}$ at w ($\mathfrak{M} \mid w$) is the structure that results from removing every world accessible from w and different from w from the model $\mathfrak{M}$.

Condition (non-clairvoyance)

If $\mathfrak{M}$ is an acceptable model, then $\mathfrak{M} \mid w$ must also be an acceptable model.

ECT and clairvoyance

Lemma

If $\mathfrak{M}$ is a non-clairvoyant model, then

$$\mathfrak{M} \models P\Diamond P\phi \rightarrow P\phi$$

Proposition

ECT holds in acceptable models meeting the non-clairvoyance condition.

Proof

Proof.

Consider any given one such models $\mathfrak{M}$ in which $\forall x \exists y \Diamond P\phi(x, y) \in f(w)$ for any given w. Now consider the truncated model $\mathfrak{M} \mid w$. In this model, w "sees" no epistemic states other than itself. By the soundness condition, for every m there is an n such that $P\phi(m, n)$ is true at w in $\mathfrak{M} \mid w$. So the same holds for $\mathfrak{M}$. But $f(w)$ is Σ_1. So there is a Turing machine e such that for all m, $\phi(m, e(m))$. So ECT is true in $\mathfrak{M}$. □

Some open technical questions

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Question (?)

Is $EA + ECT$ arithmetically conservative over PA?

Question

Is $EA + ECT$ conservative over HA under Gödel's translation?

References

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