Infinite graphs

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LC'12

Graph: (V, X), where $X \subseteq [V]^2$,
V: vertices, X: edges

(W, Y) is a *subgraph* of (V, X) if $W \subseteq V$, $Y \subseteq X$.

(W, Y) is an *induced subgraph* of (V, X) if
$W \subseteq V$, $Y = X \cap [W]^2$

Chromatic number: least number of colors, there is a good coloring of vertices
$f : V \to \mu$, if $\{x, y\} \in X$, then $f(x) \neq f(y)$
Notation: $\mathrm{Chr}(X)$

Theorem. (Galvin-K): AC is equivalent to the statement that every graph has chromatic number.

Theorem. (Erdős–de Bruijn) n is a natural number and each finite subgraph of the graph X can be good colored with n colors, then X can be good colored with n colors.

Theorem. (Blanche Descartes) If $n = 2, 3, \ldots$ then there is a finite graph with no C_3 which is n-chromatic.

Theorem. (Erdős–Rado) If κ is an infinite cardinal then there is a triangle-free graph (V, X) with $\mathrm{Chr}(X) > \kappa$ and $|V| = 2^\kappa$.
Improved to $|V| = \kappa^+$.

Theorem. (Erdős) If n, k are natural numbers, then there is a finite graph (V, X) which does not contain $C_3, C_4, \ldots, C_k$ and $\mathrm{Chr}(X) > n$.

Theorem. (Erdős–Hajnal) If the graph X omits C_4 (or any circuit of even length), then $\mathrm{Chr}(X) \leq \aleph_0$.

Theorem. (Erdős–Hajnal) If κ is a cardinal, n is a natural number, then there is a graph X which does not contain $C_3, C_5, \ldots, C_{2n+1}$ and $\mathrm{Chr}(X) > \kappa$.

Definition. (Erdős-Hajnal) If (V, X) is a graph, its *coloring number*, $\mathrm{Col}(X)$, is the least cardinal μ such that there is a well order $<$ of V, such that each vertex is joined into $< \mu$ smaller vertices.

The vertex set V can be good colored with μ colors with a transfinite recursion by $<$ and so

$$\mathrm{Chr}(X) \leq \mathrm{Col}(X)$$

Theorem. (Erdős-Hajnal) If $\mathrm{Col}(X) > \aleph_0$, then X contains a C_4 (4-circuit), in fact every C_{2k}, in fact $K_{n,\aleph_1}$ for each $n < \omega$.

Obligatory graph: isomorphic to a subgraph of X if $\mathrm{Col}(X) > \aleph_0$. What are the obligatory graphs?

Theorem. (K) There is a countable graph Γ and a graph Δ of cardinality $\aleph_1$ such that Γ is the largest countable obligatory graph and Δ is the largest obligatory graph.

Theorem. (Shelah) If λ is singular, X is a graph of cardinality λ, all whose smaller subgraphs have coloring number at most μ, then $\mathrm{Col}(X) \le \mu$.

Theorem. If κ is regular, X is a graph on κ, all whose smaller subgraphs are of coloring number at most μ, then $\mathrm{Col}(X) > \mu$ iff

$$S = \{\alpha < \kappa : \exists\beta \ge \alpha, |N(\beta) \cap \alpha| \ge \mu\}$$

is stationary. Here $N(\beta)$ denotes the set of neighbors of β.

Theorem. A graph X has $\mathrm{Col}(X) > \mu$ iff it contains either
(1) a bipartite graph on sets A, B with $|A| = \lambda^+$, $|B| = \lambda$, with all vertices in A joined into μ vertices of B
or else
(2) a graph (isomorphic to a graph) on some regular cardinal κ such that stationary many points α are joined into a cofinal subset of α of order type μ.

Theorem. (Erdős–Hajnal) If $\mathrm{Chr}(X) > \aleph_0$, then every finite bipartite graph appears in X and each finite nonbipartite graph may be omitted.

What are the obligatory families of graphs?

Theorem. (Erdős–Hajnal–Shelah, Thomassen) If $\mathrm{Chr}(X) > \aleph_0$, then X contains all of $C_{2n+1}, C_{2n+3}, \ldots,$ for some n.

Corollary. If $\mathrm{Chr}(X) > \aleph_0$, $\mathrm{Chr}(Y) > \aleph_0$, there is a 3-chromatic graph embeddable into both (a long odd circuit).

Conjecture. (Erdős) If $\mathrm{Chr}(X) > \aleph_0$, $\mathrm{Chr}(Y) > \aleph_0$ there is a 4-chromatic graph embeddable into both.

If $\mathrm{Chr}(X) > \aleph_0$ then let f_X be the following function. $f_X(n)$ is the number of vertices in the smallest n-chromatic subgraph of X. $f_X(n)$ exists by Erdős–de Bruijn and clearly $f_X(n) \geq n$. Therefore $f_X(n) \to \infty$.

Question. (Erdős–Hajnal) Can f_X increase arbitrarily fast?

Theorem. (Shelah) It is consistent that for every function $f : \mathbb{N} \to \mathbb{N}$ there is a graph X with $\mathrm{Chr}(X) = \aleph_1$ and $f_X(n) \geq f(n)$ $(n \geq 3)$.

The Taylor conjecture (Erdős–Hajnal–Shelah, Taylor) If X is a graph with $\mathrm{Chr}(X) > \aleph_0$, then for each cardinal λ there is a graph Y whose finite subgraphs are the same as those of X and $\mathrm{Chr}(Y) > \lambda$.

Theorem. (K) Consistently there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ and if Y is a graph all whose finite subgraphs occur in X then $\mathrm{Chr}(Y) \leq \aleph_2$.

Theorem. (K) It is consistent, that if $\mathrm{Chr}(X) \geq \aleph_2$, then there are arbitrarily large chromatic graphs with the same finite subgraph as X.

The Erdős-de Bruijn phenomenon does not hold for the coloring number (Erdős-Hajnal), however

Theorem. (K) If n is a natural number and $\mathrm{Col}(X) = n + 1$, then X has a subgraph Y with $\mathrm{Col}(Y) = n$.

What about the chromatic number?

If $\mathrm{Chr}(X) \geq n$, then there is a subgraph Y with $\mathrm{Chr}(Y) = n$.

If $\mathrm{Chr}(X) \geq \aleph_0$, then there is a subgraph Y with $\mathrm{Chr}(Y) = \aleph_0$.

Galvin asked if the chromatic number has the Darboux property, i.e., if $\mathrm{Chr}(X) = \lambda$ and $\kappa < \lambda$, then there is a subgraph $Y \subseteq X$ with $\mathrm{Chr}(Y) = \kappa$? Wlog $\aleph_0 < \kappa$.

Theorem. (Galvin) If $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2}$, then there is a graph X with $\mathrm{Chr}(X) > \aleph_1$, which does not have an *induced* subgraph Y with $\mathrm{Chr}(Y) = \aleph_1$.

Theorem. (K) It is consistent that there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_2$ with no subgraph Y with $\mathrm{Chr}(Y) = \aleph_1$.

If X is a graph, define

$$I(X) =$$

$$\{\mathrm{Chr}(Y) : Y \text{ is an ind. subgr. of } X\} - \{0, 1, \ldots, \aleph_0\}.$$

Then $I(X)$ is closed under taking limits and if $\lambda \in I(X)$ is singular, then $\lambda \in I(X)'$.

Further, if A is a nonempty set consisting of uncountable cardinals having these properties, then there is a ccc forcing which gives a model with a graph X such that $I(X) = A$.

If X is a graph, let

$$S(X) = \{\mathrm{Chr}(Y) : Y \text{ is a subgr. of } X\} - \{0, 1, \ldots, \aleph_0\}.$$

If $\lambda \in S(X)$ is a singular cardinal, then $\lambda \in S(X)'$ and if $\lambda \in S(X)'$ is singular, then $\lambda \in S(X)$.

It may not be closed at regular cardinals:

Theorem. (K) If it is consistent that there is a measurable cardinal, then it is consistent that there is a graph X such that $S(X)$ is not closed at a regular cardinal.

If $\mathrm{Chr}(X) > \aleph_0$, then there is a connected subgraph Y with $\mathrm{Chr}(Y) > \aleph_0$. (One of X's connected component.)
A graph is *n-connected* if it is connected and stays connected after the removal of $< n$ vertices.

Theorem. (K) If n is finite, X is an uncountably chromatic graph then there is an uncountably chromatic n-connected subgraph $Y \subseteq X$ such that all vertices of Y have uncountable degree.

Theorem. (K) It is consistent that each graph (V, X) with $|X| = \mathrm{Chr}(X) = \aleph_1$ contains an $\aleph_0$-connected subgraph Y with $\mathrm{Chr}(Y) = \aleph_1$.

Theorem. (K) It is consistent that there is an $\aleph_1$-chromatic graph of cardinality $\aleph_1$ which does not contain an $\aleph_0$-connected subgraph of cardinality $\aleph_1$.

Problem. (Erdős-Hajnal) Is it true that every graph with uncountable chromatic number contains an infinitely connected subgraph?

If (V, X) is a graph, then its *list-chromatic number* $\mathrm{List}(X)$ is the least cardinal μ such that the following holds. If $F(v)$ is a set with $|F(v)| = \mu$ $(v \in V)$ then there is a good coloring f with $f(v) \in F(v)$ $(v \in V)$.

Lemma. If X is a graph then

$$\mathrm{Chr}(X) \leq \mathrm{List}(X) \leq \mathrm{Col}(X).$$

Theorem. (K) It is consistent that if X is a graph of cardnality $\aleph_1$ then
$\mathrm{List}(X) = \aleph_1 \iff \mathrm{Chr}(X) = \aleph_1$.

Theorem. (K) It is consistent that if X is a graph of cardinality $\aleph_1$ then

$$\mathrm{List}(X) = \aleph_1 \iff \mathrm{Col}(X) = \aleph_1.$$

Theorem. (K) It is consistent that if $\mathrm{Col}(X)$ is infinite (X is of arbitrary size) then

$$\mathrm{List}(X) = \mathrm{Col}(X).$$

Theorem. (K) It is consistent that GCH holds and there exists a graph X with $|X| = \mathrm{Col}(X) = \aleph_1$ and $\mathrm{List}(X) = \aleph_0$.

Theorem. (K) (GCH) $\mathrm{Col}(X) \leq \mathrm{List}(X)^+$.