Getting Forcing Axioms by Finite Support Iteration

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outline

Search For new axioms

Main requirements for the new axioms:

- The Axiom should be strong enough to decide a large class of statements which are undecidable on the basis of the accepted axioms
- The Axiom Should Produce a coherent elegant theory for some important class of problems.
- The Axiom should have some intuitive or Philosophical appeal.A Slogan
- If possible the axiom should have "testable,verifiable consequences"
- If possible the axiom should be resilient under forcing extensions

Slogans for forcing Axioms

Slogans:

An object that can be imagined to exist and there is no obvious objection to its existence, does exist

The Universe of sets is rich.

Forcing Axioms

Definition

Let P be a partially ordered sets. λ a cardinal, the forcing axiom $FA_\lambda(P)$ is the statement that for every collection $\langle D_\alpha | \alpha < \lambda\rangle$ of dense subsets of P , there is a filter $G \subset P$ such that for all $\alpha < \lambda$, $G \cap D_\alpha \neq \emptyset$.If $\mathcal{P}$ is a class of partially ordered sets then $FA_\lambda(\mathcal{P})$ is the statement that $FA_\lambda(P)$ holds for every $P \in \mathcal{P}$.

For every $\mathcal{P}$ $FA_{\aleph_0}(\mathcal{P})$ is always true. If P is a forcing notion that adds a new real then $FA_{2^{\aleph_0}}(P)$ is false, so the first interesting case is $\lambda = \aleph_1$ when $\aleph_1 < 2^{\aleph_0}$.

Martin's Axiom

MA-Martin's Axiom is the statement that $FA_\lambda(\mathcal{P})$ holds for $\mathcal{P}$ the class of forcings that satisfy the countable chain condition (a.k.a. c.c.c) and for all $\lambda < 2^{\aleph_0}$. (In order to avoid triviality we also assume that *MA* states that *CH* is false. i.e. $2^{\aleph_0} > \aleph_1$.)

Theorem (Martin-Solovay)

MA is consistent with $2^{\aleph_0} = \kappa$ *for every regular* $\kappa > \aleph_1$*.*

MA settles many problems which are other wise independent of Set Theory.(e.g. it settles Souslin Hypothesis in the positive, Whitehead conjecture in the negative , The Moore space conjecture in the negative etc.)

Pattern For Forcing Forcing Axioms

The pattern established by the Martin -Solovay is always the same: Iterate forcing with a sequence of forcing notions which belongs to the class $\mathcal{P}$: $\langle P_\alpha | \alpha \leq \kappa \rangle$ and $\langle Q_\alpha | \alpha < \kappa \rangle$ where P_α is the iteration up to stage α, Q_α is a P_α name of a forcing belonging to the class $\mathcal{P}$,which is the next forcing in the iteration, and κ is a large enough cardinal.
Typically the iteration is determined by the support, namely what one does at limit stage α.
The hope is that for every forcing notion $P \in \mathcal{P}$ and for every list of λ dense subsets of P , $\langle D_\beta | \beta < \lambda \rangle$, there will be a stage $\alpha < \kappa$ such that Q_α is similar enough to the final P so that the generic object we introduce for Q_α will generate the filter $G \subseteq P$ which will intersect every dense set in $\langle D_\beta | \beta < \lambda \rangle$.

Problem (Catching your tail problem)

We want that in the final model every $P \in \mathcal{P}$ and a sequence of dense sets $\langle D_\beta | \beta < \lambda \rangle$ is approximated by some Q_α in the iteration. Two obstacles:

1. *$\mathcal{P}$ is typically a proper class which should be emulated by set many forcing notions.*
2. *If the iteration makes a dramatic change in the universe then the meaning of the class $\mathcal{P}$ in the final model is so different than the the maeing in the intermidiate model , so for some $P \in \mathcal{P}$ of the final model there is no chance of emulating them by Q_α which is in $\mathcal{P}$ of an intermediate stage.e.g. If the iteration collapses ω_1 then in the final model the meaning of being c.c.c. will be completely different than the meaning in any intermediate universe.*

Answer (Martin-Solovay)

1. *A sub order of a c.c.c forcing notion is also a c.c.c. forcing notion so in order to get a model of* MA_λ *it is enough to iterate c.c.c. forcings of cardinality* $\leq \lambda$
2. *If one iterate c.c.c. forcings with finite support then the iteration satisfies c.c.c. . As a result* ω_1 *is not collapsed.*

The c.c.c. condition seems to be essential for dealing with problem no 2 , if one insists on using finite support.

Fact

If we iterate with the finite support infinitely (even only ω *many) non trivial forcings which do not satisfy c.c.c then the iteration collapses* ω_1*.*

The Use of Supercompacts

MA still leaves many central independent problems. e.g. The size of the continuum.Following our slogan we try to get stronger and stronger forcing axioms. The limiting factor is the availability of answers to our two problems.

Answer (1st problem)

Let us assume the existence of a super compact cardinal κ . It implies that every structure (in a countable signature) has a second order elementary structure of cardinality less than κ. (In fact having a reflecting cardinal for second order logic is equivalent to the existence of a supercompact cardinal.)
This strong reflection allows us in many cases, to get a model of $FA_{<\kappa}(\mathcal{P})$ by iterating forcing notions of size $< \kappa$ where the iteration is of length κ.

Proper Forcings

We are looking for larger class of forcing notions that can be iterated without damaging too much the structure of the universe (like collapsing ω_1. One such class is the class of Proper Forcing.

Definition

Let P be a forcing notion . Let κ be large enough such that $P \in H_\kappa$. Let M be an elementary substructure of H_κ such that $P \in M$.

1. $p \in P$ is said to be a master condition for M if p forces that $G \cap M$ is generic over M (where $G \subseteq P$ is the generic filter over P.) It is equivalent to: For every P-name of an ordinal $\tilde{x} \in M$, p forces that $\tilde{x}$ is realized as an ordinal in M.
2. $p \in P$ is said to be η-semi master condition for M if for every P-name of a member of η, $\tilde{x} \in M$, p forces that $\tilde{x}$ is realized as a member of $\eta \cap M$.
3. $p \in P$ is said to be semi master condition for M if it is ω_1 semi master condition for M.

Definition

1. A forcing notion P is said to be proper if for κ large enough such that $P \in H_\kappa$, for every countable $M \prec H_\kappa$ such that $P \in M$ and for every $q \in P \cap M$ there is an extension of q which is a master condition for M
2. A forcing notion P is said to be semi-proper if for κ large enough such that $P \in H_\kappa$, for every countable $M \prec H_\kappa$ such that $P \in M$ and for every $q \in P \cap M$ there is an extension of q which is a semi master condition for M

It is easy to see that semi- proper forcing does not collapse ω_1 (Hence proper forcings do not collapse ω_1.)

Iterating Proper and Semi Proper Forcings

Theorem (Shelah)

1. *The Iteration of proper forcings with countable support is proper.*
2. *There is a variation on countable support iteration (called* ***Revised countable support iteration*** *or RCS) such that if* $\langle P_\alpha | \alpha \leq \kappa \rangle$ *and* $\langle Q_\alpha | \alpha < \kappa \rangle$ *is RCS iteration of semi-proper forcings such that for all* $\alpha < \kappa$

$$\Vdash_{P_{\alpha+1}} |P_{\alpha+1}| = \aleph_1$$

then P_κ *is semi-proper.*

Definition

1. The Proper Forcing Axiom (PFA) is $FA_{<2^{\aleph_0}}(\mathcal{P})$ where $\mathcal{P}$ is the class of proper forcing. (In order to avoid trivialities we also assume $2^{\aleph_0} > \aleph_1$)
2. The Semi Proper Forcing Axiom (SPFA) is $FA_{<2^{\aleph_0}}(\mathcal{P}$ where $\mathcal{P}$ is the class of semi proper forcing. (In order to avoid trivialities we also assume $2^{\aleph_0} > \aleph_1$)

Theorem (Baumgartner, Shelah)

Assuming the consistency of supercompact cardinal one can get a model of SPFA. (Hence of PFA).

Theorem (Todorcevic, Velickovic)

PFA implies $2^{\aleph_1} = 2^{\aleph_0} = \aleph_2$. Hence PFA is equivalent to $FA_{\omega_1}(\mathcal{P})$ for $\mathcal{P}$ the class of proper forcings. Similarly SPFA is equivalent to $FA_{\omega_1}(\mathcal{P})$ for $\mathcal{P}$ the class of semi proper forcings.

Martin's Maximum (MM) is $FA_{<2^{\aleph_0}}(\mathcal{P})$ for $\mathcal{P}$ the class of forcings that do not destroy stationary subsets of ω_1. This class is a maximal class $\mathcal{P}$ of forcings for which the axiom $FA_{<2^{\aleph_0}}(\mathcal{P})$ is consistent with $2^{\aleph_0} > \aleph_1$.

Theorem (Foreman, M. Shelah)

MM is equivalent to SPFA.

PFA and MM settles many independent problems. (For instance: the size of the continuum,the singular cardinals problem, the existence of Aronszjan Tree on ω_2 etc.)

Because of the maximality of MM as an axiom, it has a claim to be the ultimate forcing axiom.
Recall our slogan:
The universe of sets is rich: A set that can be imagined to exist and there is obvious obstacle to its existence, does exit!

MM (and PFA) are essentially statements about sets of size $\aleph_1$. Their impact on sets of larger size is minimal. e.g. they do not decide $2^{\aleph_2}$.

Question

Can we find analogues of strong forcing axioms for larger cardinals. e.g. get a forcing axiom that decides $2^{\aleph_2}$?

Problem

We miss iteration theorems!

Naive Attempts

Take the consistency proof of PFA as a guide. The definition of proper forcing requires the existence of master condition for countable $M \prec H_\kappa$. A natural attempt to generalize it in order to get a forcing axiom for $\aleph_2$ in the same sense that PFA and MM are foricng axioms for $\aleph_1$ is to define

Definition

A forcing notion P is said to be ω_2 proper if for large enough κ such that $P \in H_\kappa$ and for every $M \prec H_\kappa$ such that $|M| < \aleph_2, P \in P$ and for every $q \in M \cap P$ there is an extension of q which is a master condition for M.

What kind of iteration we should use ? Finite support is impossible because for non c.c.c. forcings we surely collapse ω_1.

A terminal Problem for Higher Analogues?

The proof of iteration of proper forcings (in the usual sense) uses at limit stage α diagonalization over all the dense sets in the countable model $M \prec H_\kappa$ using countably many steps. For α of countable cofinality we use directly the countability of M.For α of uncountable cofinality the existence of master condition follows form the countable cofinality case, relying strongly on the fact that we are talking on countable support. For our attempt to generalize the result to ω_2-proper forcings if we still use countable support we run into problems for M of size ω_1 and for support of size ω_1 we run into problems for countable M.Is this problem terminal?

Reviving Finite Support

Can we revise finite support so to avoid its problems?
Let $\langle P_\alpha | \alpha \leq \kappa \rangle$ and $\langle Q_\alpha | \alpha < \kappa \rangle$ be finite support iteration. It means that for limit $\alpha \leq \kappa$ P_α is made up of vectors $\langle q_\gamma | \gamma < \kappa \rangle$ such that for $\gamma < \alpha$ q_γ is a P_γ name such

$$\Vdash_{P_\gamma} q_\gamma \in Q_\gamma$$

and such that

$$\{\alpha < \kappa | q_\alpha \quad \text{is not forced to be the maximal memeber of } Q_\alpha\}$$

is finite.
A way of avoiding collapsing ω_1 is that given a sequence of dense subsets of P_κ $\langle D_n | n < \omega \rangle$ and $p \in P$ we need an extension of p, q such that bellow q each D_n is now countable.

Enter Side Conditions

The idea of finite iteration with side conditions goes back to Baumgartner forcing for introducing clubs in ω_1 with finite conditions and Friedman and Mitchell forcing of clubs in larger cardinals with finite conditions.
Assume that all the forcing notions we talk about are in H_κ. We are also given a class of elementary substructures $\mathcal{T}$ of H_κ.

Definition

A $\mathcal{T}$ side condition is a finite sequence $M_0 \in M_1 \ldots \in M_{n-1}$ of models in $\mathcal{T}$ such that the sequence is closed under intersections. (i.e. for $i, j < n$ the model$M_i \cap M_j$ is also on the sequence.)

Finite Support Iteration with Side Conditions

Intuitive (and not completely accurate) description: The set of conditions is the set of pairs (p, s) where p is a member of the finite support iteration of $\langle P_\alpha | \alpha \leq \kappa \rangle$ and $\langle Q_\alpha | \alpha < \kappa \rangle$. s is a side condition but we require that whenever N is one of the models in s and N happens to be countable and if $\alpha \in N$ then for every $\alpha < \kappa$ for which $p(\alpha)$ is not trivial then

$p \restriction \alpha \Vdash_{P_\alpha} p(\alpha)$ is a master condition for $N[G_\alpha]$ and the forcing Q_α

Theorem (Neeman)

Assume κ is a supercompact cardinal. Then there is finite support iteration with side such that if we force with it , the resulting model satisfies PFA

Why Bother? since we have another way of getting a model PFA prom supercompact.

Answer

There is a possibility of getting higher analogues of forcing axioms for larger cardinals.

In fact Neeman generalized his PFA proof and got such analogue. He defined the class of $\{\aleph_0, \aleph_1\}$ proper forcing,$\mathcal{P}$ and got (from supercompact) the consistency of $FA_{\omega_2}(\mathcal{P})$.

Also Asperó and Mota got interesting class of forcings (containing all c.c.c. forcing but larger than it) for which one can get the consistency of $FA_{\omega_2}(\mathcal{P})$.

But these higher analogues of the forcing axioms for $\aleph_1$ lacks the maximality that we get by assuming *MM*. Can we have higher analogues of MM?

We can not generalize MM verbatim because e.g. if one tries to define $\mathcal{P}$ as the class of forcings that do not kill stationary subsets of ω_1 or of ω_2 then Shelah showed that FA_{ω_2} is inconsistent.

Can finite support iteration of generalization of **semi-proper** forcings lead us to the right higher analogues of MM?

Theorem (Gitik-M.)

Let κ be a supercompact cardinal. There is a version of finite support iteration with side conditions of semi-proper forcing notions that creates a model of MM

The structure of the side conditions is more complicated. Main difference : If N is an countable elementary submodel of H_λ for λ large enough, $Q \in N$ is a proper forcing notion and $q \in Q$ is a master condition for N , then q forces that $N[G] \cap V = N$. This is false if Q is only assumed to be semi proper.

So it is not enough to have one (finite) sequence of models which are the side conditions. We have to tailor the side conditions for each step of the iteration , but without losing the finite structure of the total structure of the side conditions.

Can this lead us to the right higher analogues of MM?
There are some attempts of such higher analogues but it is still very much a work in progress.
The right maximality principle for (say sets of size $\aleph_2$) still eludes us.

Thank you!