# Two principles of dynamic constructivism

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# preamble - declaration of intents

the “classical foundation” can be justified only on the base of a double magic:

- one magically has access to a supernatural world of platonic ideas
- platonic ideas magically correspond to reality

magic (absolute truth,...) is not good in a global and complex world

we should replace: rigid, static, explanation from above (magic, absolute truth), singular, childish, objective,...

with **flexible, dynamic, explanation from below (evolution), plural, adult, intersubjective,...**

general aim: show that it is useful to build a dynamic view of mathematics, including management of information, and that we can actually do it

aim today: two common sense suggestions

# What is mathematics? a dynamic view

every culture has its own mathematics

the mathematical method is useful to man for survival (a continuation of natural evolution)

it is **simpler** and **more effective** to manipulate symbols than things:

mathematical manipulation

$$\longrightarrow$$

abstraction $\uparrow$ $\downarrow$ application

$$\longrightarrow$$

reality

mathematics = study of abstract structures for counting (algebra), measuring (analysis), organizing space (geometry), deducing (logic), etc.

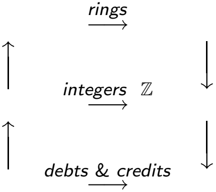
**why is mathematics so effective? by definition**

# Pluralism in mathematics

foundation = choice of what kind of information is relevant (cannot be forgotten)

many ways to abstract = many kinds of mathematics, that is **pluralism**

several different levels of abstraction




levels of abstraction for the mind are like **gears** for a car

the reason for abstraction, what makes it meaningful, is the subsequent application = going back to a more concrete level

# Enriques' criterion

*"We must, above all, avoid the errors of the past. Therefore we should take notice that* **language**, *which we use to express our thoughts,* **is**, *in the last analysis,* **a system of symbolic representation of things**. *Since language furnishes a process of schematizing, rising by degrees to the expression of more general facts, it allows us to reason about abstract ideas, very far from the immediate reality which appeals to our senses. But the use of this* **powerful instrument**, *which comes to the aid of our mental weakness, is not without its* **dangers**. *Taking flight towards the lofty realms of thought,* **we run the risk of forgetting the meaning of words, which become void of sense as soon as they cease to represent things**.
*Having reached this point, nothing is easier than to use symbols formally, while the development of thought tending toward generality, no longer finds any check in the concrete world, to which it remains foreign. If then you would not lose yourself in a dream devoid of sense, you should not forget* **the supreme condition of positivity**, *by means of which the critical* **judgement must affirm or deny**, *in the last analysis,* **facts** *either particular or general."*

**F. Enriques,** *Problemi della scienza*, **1906, Engl. transl. 1914**

# Principle 1: a theoretical form of Enriques' criterion

application, or going back to concrete, means that our mathematics should admit:

- computational interpretation
- Kleene's realizability interpretation
- computer implementation (in a "proof-assistant")

**However, this is not sufficient...**

# Abstraction vs. idealization

abstraction = forget some information

idealization = add fictitious entities or properties, to organize better our knowledge

Examples:

- every equation, $x^2 = -1$ in particular, has a solution
- every two lines intersect in a point
- even: every number has a successor

**abstraction is justified by application; idealization is justified by ???**

# The problem of foundations in 1800s

problem of foundations in 1800s: what is the meaning of abstract mathematics

Cantor-Dedekind 1872: real numbers as actual infinite entities

Cantor: set theory, actual infinite in mathematics

paradoxes

# Thesis - Antithesis

Hilbert: recall we are mathematicians, and we got out of similar problems by the method of ideal elements

what are ideal elements in this case? actual infinite sets (they exist nowhere)

**Thesis** : consistency of a formal system is sufficient to justify any idealization (Hilbert's Program)

**Antithesis** : no ideal elements are allowed, everything must have a computational interpretation (Markov, Bishop, Martin-Löf)

**Times are ripe to look for a synthesis! Let us take a suggestion from nature**

# Continuous vision

visual inputs are discrete ("only" 2-3 million receptors in our retina) but our vision is continuous

continuous vision is an ideal organization of space to include all inputs, changing with time and with perspective

ideal entities are part of every day life, even if they do not exist in reality

**if nature has found it useful, why should we abstain from it?**

# Principle 2: conservativity

continuous vision would be of no help if it should produce hallucinations (visions which are not backed by reality)

idealization is justified by the fact it does not interfere with previous knowledge

technically: ideal notions must be conservative over real ones

$\mathbb{C}$ does not modify $\mathbb{R}$; Argand-Gauss plane = conservativity proof

adding points and lines at infinity does not modify our knowledge of all other points and lines

**conservativity is such an obvious condition that it is mostly given for granted**

# Synthesis

we indeed **can** have ideal mathematics **together** with real mathematics

principle 1: real mathematics must satisfy Enriques' criterion = $T$ must have a realizability interpretation

principle 2: ideal mathematics must be **conservative** over real mathematics

(analogy with: every product must be recyclable, energy must be renewable, etc.)

we need a foundational theory $T$ which is consistent with all this (that is, which allows us to believe that we can satisfy the two principles)

# the minimalist foundation

"set for the computer" $=$ inductively generated set,

we need a distinction between set and collection (e.g. $\mathcal{P}X$)

"operation for the computer" $=$ computable operation

$\mathtt{CT}$ (*internal Church Thesis*): every operation from $\mathbb{N}$ to $\mathbb{N}$ is recursive.

proofs-as-programs: every proof in $T$ of a proposition/specification must become a program fulfilling the specification

$\mathtt{AC}$ (*Axiom of Choice*): every total relation $R$ from $X$ to $Y$ must contain the graph of an operation from $X$ to $Y$:
$(\forall x \in X)(\exists y \in Y)R(x,y) \rightarrow (\exists p : OP(X,Y))(\forall x \in X)R(x,p(x))$.

**our criterion: consistency with** $\mathtt{CT} + \mathtt{AC}$

# the minimalist foundation

We **cannot** have:

$\mathtt{LEM}$ (*Law of Excluded Middle*) $\varphi \vee \neg\varphi$ true for every proposition $\varphi$

$$T + \mathtt{CT} + \mathtt{AC} + \mathtt{LEM} \vdash \bot$$

$\mathtt{PSA}$ (*Power Set Axiom*) if $X$ is a set, also $\mathcal{P}X$ is a set,

$$T + \mathtt{AC} + \mathtt{PSA} \vdash \mathtt{LEM}$$

So $T$ must be intuitionistic and predicative

# two levels of abstraction

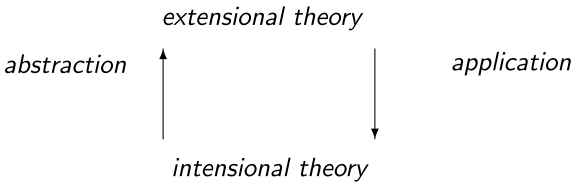
To **actually do** mathematics, we need extensionality:

> $\texttt{ExtOp}$: if two operations $p, p'$ from $X$ to $Y$ satisfy $(\forall x \in X)(p(x) = p'(x))$, then $p = p'$.

But one can prove:

$$T + \texttt{CT} + \texttt{AC} + \texttt{ExtOp} \vdash \bot$$

Impossible? No, we are adding an assumption which is not there: “flatness” of $T$.



# the minimalist foundation

*set theories: CZF, topos th., ZFC,...*
*minimal extensional theory*

*abstraction* ↑ ↓ *application*
$=$ *forget some information* $=$ *implementation*
$=$ *pass to quotients* $=$ *restore information*

*minimal intensional theory*
*type theories: ML, Coq,...*

In practice, the intensional theory is a variant of Martin-Löf type theory, in which *Prop* $\neq$ *Set*
Abstraction is simply closure under quotients. That is, we *forget* the information given by proof-terms, or elements of a set, and we identify some or all of them.

# Minimalist foundation

minimalist in postulates = maximalist in conceptual distinctions (and in epistemological complexity; against reductionism)

no LEM $\Rightarrow$ can have a positive notion of existence

no PSA $\Rightarrow$ can have a notion of generated set

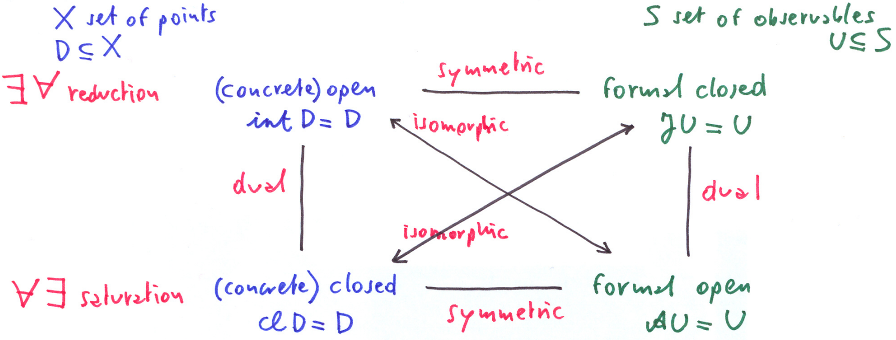
no AC! $\Rightarrow$ can keep functions distinct from operations

sets are real, effective (finite number of rules to generate all elements); e.g. $\mathbb{N}$, $\mathbb{Q}$

collections are ideal (no induction); e.g. $\mathcal{P}X$, $\mathbb{R}$

realizability interpretation is proved to be possible

# New mathematics: symmetry and duality in topology



# Real and ideal in mathematics

real (effective, computable) / ideal (infinitary, uncomputable)

constructive topology: predicative $\rightarrow$ pointfree approach

we need a notion of pointfree topology + a notion of ideal point over it

opens are real (and form a set) / points are ideal (and form a collection)

special case: choice sequences = ideal points over a positive topology over $\mathbb{N}^{\mathbb{N}}$

choice sequences $\neq$ lawlike sequence because of no $\mathtt{AC}$!

Bar Induction: $\forall\alpha(\alpha \Vdash k \rightarrow \alpha \between U) \rightarrow k \lhd U$

Conjecture: the theory of choice sequences with Bar Induction is conservative over pointfree topology

# Future

**Beyond Turing machines:**

what is the notion of computability corresponding to a formal system with two levels of abstraction?

**Mathematizing existential statements:**

the dark side of the moon:

overlap $\between$, positivity relation $\ltimes$, a notion of closed in pointfree terms
overlap algebras, putting topology in positive and algebraic terms

**many other potentialities...**