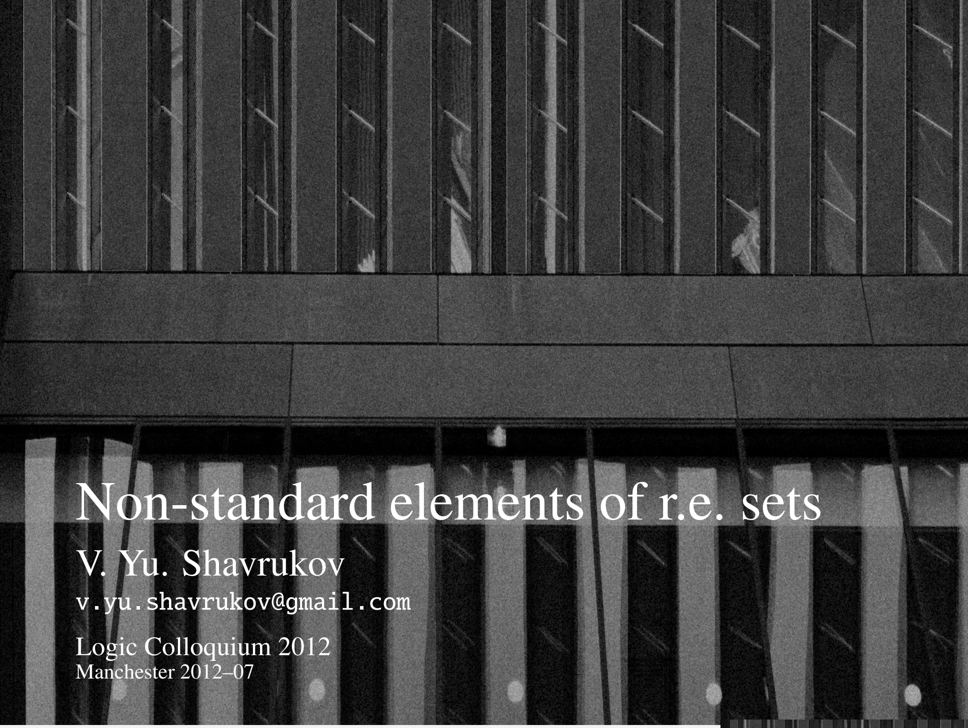
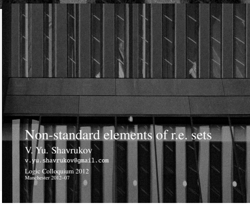
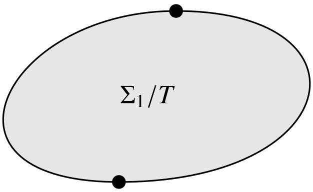
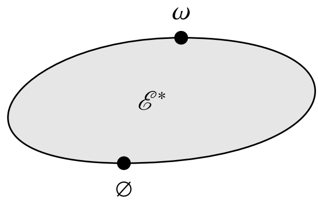




- Good morning.
- I want to thank the Programme and Organizing Committees for my presence here today. I am thrilled to be able to borrow your ears.

$\mathscr{E}$, $\mathscr{E}^*$, and Σ_1/T





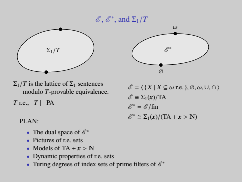
Σ_1/T is the lattice of Σ_1 sentences modulo T-provable equivalence.

T r.e., $T \vdash$ PA

$$\mathscr{E} = \langle \{ X \mid X \subseteq \omega \text{ r.e.} \}, \varnothing, \omega, \cup, \cap \rangle$$
$$\mathscr{E} \cong \Sigma_1(\boldsymbol{x})/\mathrm{TA}$$
$$\mathscr{E}^* = \mathscr{E}/\mathrm{fin}$$
$$\mathscr{E}^* \cong \Sigma_1(\boldsymbol{x})/(\mathrm{TA} + \boldsymbol{x} > \mathbb{N})$$

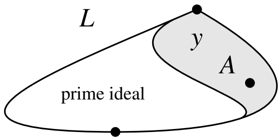
PLAN:

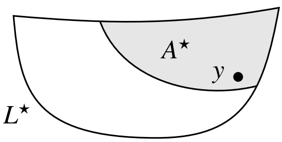
- The dual space of $\mathscr{E}^*$
- Pictures of r.e. sets
- Models of TA $+ \boldsymbol{x} > \mathbb{N}$
- Dynamic properties of r.e. sets
- Turing degrees of index sets of prime filters of $\mathscr{E}^*$



- Let us introduce the main characters of today's story. We shall be primarily concerned with the lattice $\mathscr{E}$ of r.e. sets
- or, rather, its quotient modulo finite differences, $\mathscr{E}^*$. This is a well studied object. We shall explore a perhaps novel approach to it which is inspired by certain aspects of the study of its sister lattice,
- the lattice Σ_1/T of Σ_1 sentences modulo provability in an r.e. theory T
- containing Peano Arithmetic.
- Its kinship to $\mathscr{E}$ is underscored by the fact that an r.e. set is really like a Σ_1 formula with a free variable, so $\mathscr{E}$ is isomorphic to the lattice of Σ_1 formulas with parameter $\boldsymbol{x}$ modulo True Arithmetic. We consider 1st order True Arithmetic in the usual 0-1-plus-times language.
- When you quotient the lattice by finite differences you are in fact strengthening the theory with the assertion that the parameter $\boldsymbol{x}$ is non-standard. So what happened to standard numbers? Well, they did not make it through the quotient: the non-standard numbers are precisely those that are insensitive to finite differences between r.e. sets they belong to. This connects us to the title of our talk.
- So here is the plan:
- The particular elements of the study of Σ_1/T we plan to apply to $\mathscr{E}^*$ concern the Priestley dual of the lattice, so we begin with reviewing Priestley duality and first properties of the dual space of $\mathscr{E}^*$. Along the way we point out similarities and differences with the dual space of Σ_1/T, also because it is considered good practice in this type of talk to mention a few things the speaker actually knows something about.
- Next we are going to see what some familiar classes of r.e. sets look like in the dual space, for one of the things the present talk is intended to accomplish is to show *the* correct way to draw pictures of r.e. sets.
- Then we turn to models of true arithmetic with a distinguished non-standard number as representations of points in the dual space and see how they can be helpful.
- We also take a look at the connection with dynamic properties of r.e. sets and illustrate our approach with applications to small subsets and promptly simple sets.
- Finally, time permitting, we look at the relation between the ordering in the dual space and Turing degrees of prime filters in $\mathscr{E}^*$ — this is a theme directly motivated by similar developments in the lattice of Σ_1 sentences.

Prime filters and Priestley duality



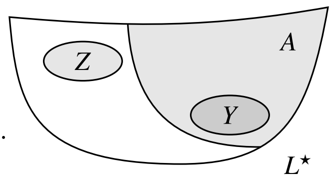


Let L be a bounded distributive lattice.
A filter $y \subseteq L$ is a *prime filter* if $L - y$ is a (prime) ideal.

EXAMPLES. $\mathrm{Th}_{\Sigma_1} M = \{\sigma \in \Sigma_1 \mid M \models \sigma\}$
is a prime filter in Σ_1/T when $M \models T$.
$\mathrm{Th}_{\Sigma_1}(M, \boldsymbol{x}) = \{\text{r.e. } X \mid M \models \boldsymbol{x} \in X\}$
is a prime filter in $\mathscr{E}^*$ when $(M, \boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$.

$L^\star = \langle \{\text{prime filters of } L\}, \subseteq, \mathscr{P} \rangle$, the *dual space of* L.
For $A \in L$, $A^\star = \{y \in L^\star \mid A \in y\}$, a $\uparrow$-closed $\mathscr{P}$-clopen.

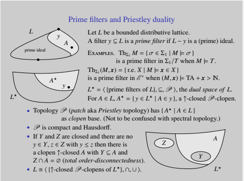
- Topology $\mathscr{P}$ (*patch* aka *Priestley* topology) has $\{A^\star \mid A \in L\}$ as *clopen* base. (Not to be confused with spectral topology.)
- $\mathscr{P}$ is compact and Hausdorff.
- If Y and Z are closed and there are no $y \in Y$, $z \in Z$ with $y \leq z$ then there is a clopen $\uparrow$-closed A with $Y \subseteq A$ and $Z \cap A = \varnothing$ (*total order-disconnectedness*).
- $L \cong \langle \{\uparrow\text{-closed } \mathscr{P}\text{-clopens of } L^\star\}, \cap, \cup \rangle$.



Non-standard elements of r.e. sets

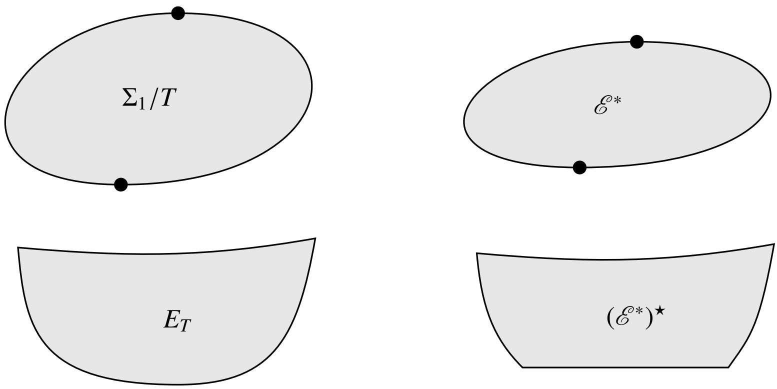
Priestley duality

Prime filters and Priestley duality



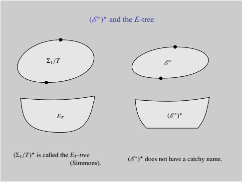
- The upcoming frame is a refresher course in Priestley duality for bounded (i.e. with 0 and 1) distributive lattices. So let L be such a lattice.
- Recall that a filter y of the lattice is *prime*
- if its complement is an ideal (which will then also have to be prime). I.o.w., a prime filter slashes the lattice into two halves, the upper half closed under meet, and the lower half closed under join.
- An example of prime filter in Σ_1/T is the collection of all Σ_1 sentences holding in some model of T. Moreover, each prime filter has this form for an appropriate model.
- Similarly, a prime filter in $\mathscr{E}^*$ can be given as a collection of all r.e. sets X that have as an element some distinguished non-standard number $\boldsymbol{x}$ in a model of True Arithmetic.
- The *dual space* $L^\star$ of the lattice is the collection of all of its prime filters ordered by inclusion together with a topology $\mathscr{P}$ about which a few words in a minute. So a prime filter in the lattice corresponds to a point in the dual space.
- If A is an element of the lattice then it is represented in the dual space by its picture $A^\star$, the collection of those prime filters to which the element A belongs
- although we will oftentimes be forgetting the *. Note that $A^\star$ is upwards closed, for once an element belongs to some prime filter, it will have to belong to all larger ones. $A^\star$ is also a closed and open set in the topology $\mathscr{P}$ because
- $\mathscr{P}$ is defined by taking all sets of the form $A^\star$ as a clopen basis, which means that one takes all elements of this form together with their complements as an open basis. This is known as *Priestley* or *patch* topology.
- One should not confuse it with spectral topology where you take the collection of all $A^\star$ as an *open* basis.
- The Priestley topology is compact and Hausdorff.
- If Y and Z are closed subsets in the dual space such that no element of Y is smaller than any element of Z
- then there is an upwards closed clopen set A that includes Y but is disjoint from Z. I.o.w., an upward closed clopen set can always separate two closed sets unless it is impossible for trivial reasons.
- We shall refer to this property as *total order-disconnectedness,* although its original formulation is ostensibly weaker than ours. Typical closed sets in the dual space are clopen sets, single points, as well as upward or downward closures of closed sets.
- Finally, the original lattice L can be recovered from its dual space as the lattice of all upwards closed sets that are closed and open in topology $\mathscr{P}$. So in particular the set A in the last picture is actually an element of the lattice L. General clopen sets correspond to Boolean combinations of elements of the lattice. Among these, those that are upwards closed belong to the lattice itself.
- The moral is that the dual space carries full information about the lattice and not a bit more. This is as much duality theory as we are going to need today.

$(\mathscr{E}^*)^\star$ and the E-tree



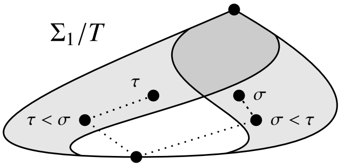
$(\Sigma_1/T)^\star$ is called the E_T-*tree* (Simmons).

$(\mathscr{E}^*)^\star$ does not have a catchy name.



- So to the lattice of Σ_1 sentences there corresponds
- its dual space,
- just as the lattice of r.e. sets (modulo finite differences)
- has a dual space all of its own. Observe that the two asterisks in th term $(\mathscr{E}^*)^\star$ are very different: the inner one stands for a farewell to finite differences while the outer star denotes the passage to the dual space.
- The dual space of Σ_1 sentences is called the E-tree perhaps with a subscript to identify the theory. The name was supplied by Harold Simmons and it has been studied on and off since about mid-70s.
- The dual space of r.e. sets does not have a catchy name — I am in fact taking suggestions. A sufficient reason appears to lie in the fact that this object has to my best knowledge never been considered. I find this rather surprising because on the one hand the lattice $\mathscr{E}^*$ has seen a lot of investigation for more than half a century and, on the other hand, consideration of the dual space is a fairly standard step in the study of any distributive lattice. By way of explanation one could perhaps speculate that in the study of distributive lattices Priestley duality plays the role of abstract nonsense whereas contemporary recursion theorists are very much no-nonsense individuals.

Forests and Reduction

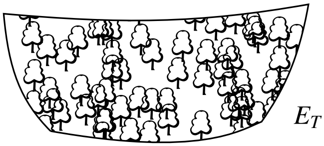


Consider two incomparable prime filters in Σ_1/T.

$$\tau < \sigma \quad \equiv \quad \exists x\,(x : \tau \wedge \neg(x : \sigma))$$

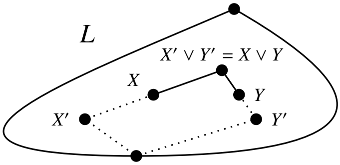
$$T \vdash (\tau < \sigma \wedge \sigma < \tau) \leftrightarrow \bot$$

Hence no proper prime filter can contain both.



Hence E_T is forest-like.

So is $(\mathscr{E}^*)^\star$ because of Reduction Property.

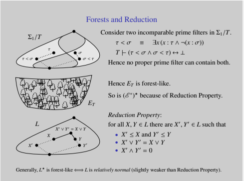


Reduction Property:

for all $X, Y \in L$ there are $X', Y' \in L$ such that

- $X' \leq X$ and $Y' \leq Y$
- $X' \vee Y' = X \vee Y$
- $X' \wedge Y' = 0$

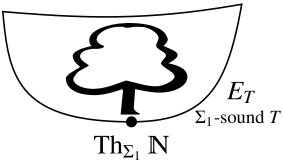
Generally, $L^\star$ is forest-like $\Longleftrightarrow$ L is *relatively normal* (slightly weaker than Reduction Property).



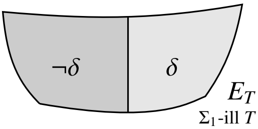
- The dual space of Σ_1/T is called the E-tree. We now explain why its ordering is indeed forest-like.
- Consider two incomparable points in the dual space. They are prime filters in Σ_1/T that probably look somewhat like this. We shall show that there is no proper prime filter including both of them.
- Let τ and σ be two Σ_1 sentences witnessing the incomparability of the two prime ideals.
- Now consider the sentence "τ is less than σ" saying that there is a stage x by which τ is witnessed but σ is not. If you think of the left prime filter as the collection of sentences holding in a fixed model then clearly, since τ holds but σ does not, the sentence $\tau < \sigma$ must also hold in the model and hence be an element of the left prime filter. Further, $\tau < \sigma$ provably implies τ.
- By symmetric considerations, $\sigma < \tau$ lies in the right prime filter.
- The two witness comparison sentences are clearly provably inconsistent with one another, i.o.w., they meet at 0.
- Hence no proper prime filter will contain both sentences and hence no proper prime filter can extend both the left and the right prime filter.
- This proves that no two incomparable points can be topped by a single point in the E-tree. Equivalently, the E-tree is a forest.
- The success of our argument is due to what is known as *Reduction Property*:
- For any two elements of L
- there are two smaller elements
- that join exactly where the original two elements joined
- but now meet at 0.
- R.e. sets also enjoy Reduction Property (this is shown similarly by a version of stage comparison — for X' one considers the set of those elements that get into one set sooner than into the other), so $(\mathscr{E}^*)^\star$ is also forest-like.
- In general, the forest-likeness of the dual space is equivalent to *relative normality* of the lattice (which is a first order property similar to Reduction, modelled on a characteristic property of lattices of open sets of hereditarily normal topological spaces.
- Relative normality is ever so slightly weaker than Reduction — although they are equivalent for finite lattices.)

Minimal points

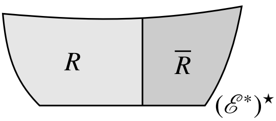
In $L^\star$, below every point there is a minimal one
(for the intersection of a chain of prime filters is a prime filter).



If T is Σ_1*-sound* ($T \vdash \sigma \;\Rightarrow\; \mathbb{N} \models \sigma$) then
$\mathrm{Th}_{\Sigma_1}\,\mathbb{N}$ is the smallest prime filter on Σ_1/T.
Hence the E_T-tree is a rooted tree.



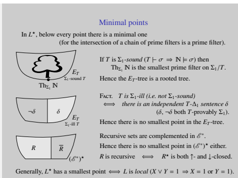
FACT. *T is Σ_1-ill (i.e. not Σ_1-sound)*
$\iff$ *there is an independent T-Δ_1 sentence δ*
(δ, $\neg\delta$ both T-provably Σ_1).
Hence there is no smallest point in the E_T-tree.



Recursive sets are complemented in $\mathscr{E}^*$.
Hence there is no smallest point in $(\mathscr{E}^*)^\star$ either.
R is recursive $\iff$ $R^\star$ is both $\uparrow$- and $\downarrow$-closed.

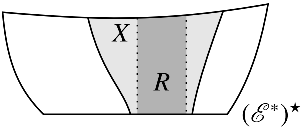
Generally, $L^\star$ has a smallest point $\iff$ L is *local* ($X \vee Y = 1 \;\Rightarrow X = 1$ or $Y = 1$).

Non-standard elements of r.e. sets
Priestley duality
Minimal points



- We are going to look at minimal points in dual spaces. In general, there is a minimal point below every given point because the intersection of a chain of prime filters is again a prime filter. (The same goes for maximal points.)
- Consider the E-tree of a Σ_1-sound theory (which means that all Σ_1 sentences proved by that theory are true in the standard model).
- Then the Σ_1 theory of the natural numbers is the smallest prime filter of the Σ_1 lattice. It is then the smallest point in the E-tree.
- In particular, the E-tree is in the Σ_1-sound case a rooted tree.
- A theory is Σ_1-ill, that is, it proves some false Σ_1 sentence
- if and only if there are independent Δ_1 sentences in that theory.
- A sentence is Δ_1 in a given theory if both that sentence and its negation are provably (in that theory) equivalent to Σ_1 sentences. In the dual space they are both upwards and downwards closed.
- Both parts have to possess minimal points, so there are at least two minimal points in the E-tree of a Σ_1-ill theory. In fact, there are continuum many of those.
- Complements are also present in the lattice of r.e. sets. They are exactly the recursive sets.
- Thus there is no smallest point in $(\mathscr{E}^*)^\star$ either. This makes it look much more like the E-tree of a Σ_1-ill then of a Σ_1-sound theory.
- For future reference, we observe that an r.e. set R is recursive if and only if its picture in the dual space is closed both upwards, as any r.e. set is, and also its complement is closed upwards, or, equivalently, the picture of R is closed downwards.
- In general bounded distributive lattices, the dual space has a least point if and only if the lattice is *local,*
- that is, no two elements smaller than the unit can join at the unit.

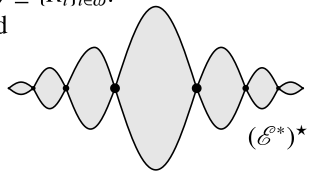
Depth



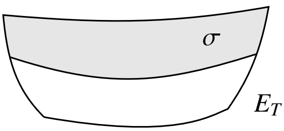
Every ∞ r.e. set X has an ∞ recursive subset R.
Hence $X^\star \cap \min(\mathscr{E}^*)^\star \neq \varnothing$.

FACT. $\min(\mathscr{E}^*)^\star \cap \max(\mathscr{E}^*)^\star \neq \varnothing$.

CONSTRUCTION. Fix a sequence $(R_i)_{i\in\omega}$ of ∞ recursive sets with $R_{i+1} \subseteq R_i$, and $R_i \subseteq W_i$ or $R_i \cap W_i = \varnothing$. Let $y \supseteq \{R_i\}_{i\in\omega}$. Then $y \in \max(\mathscr{E}^*)^\star$ because y cannot be extended to any set $W_i \notin y$ as $R_i \in y$ and $R_i \cap W_i = \varnothing$.
$y \in \min(\mathscr{E}^*)^\star$ because $w \leq y \Rightarrow w \supseteq \{R_i\}_{i\in\omega} \Rightarrow w \in \max(\mathscr{E}^*)^\star \Rightarrow w = y$. ∎



FACT (after Shore). *Every $\phi \in \operatorname{Aut}\mathscr{E}^*$ is uniquely determined by its action on* $\min(\mathscr{E}^*)^\star$.

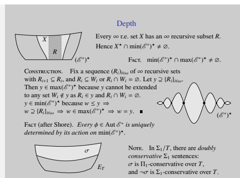


NOTE. In Σ_1/T, there are *doubly conservative* Σ_1 sentences: σ is Π_1-conservative over T, and $\neg\sigma$ is Σ_1-conservative over T.

Non-standard elements of r.e. sets

Priestley duality

Depth

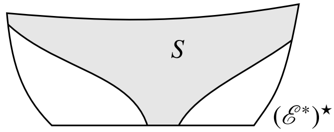


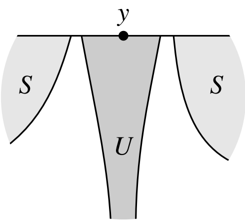
- Let us draw an infinite r.e. set in the dual space.
- It could perhaps look like this.
- Let us now recall that every infinite r.e. set has an infinite recursive subset. That infinite recursive subset will have to reach all the way down to the bottom of the dual space because pictures of recursive sets are downwards closed.
- So the correct picture probably looks like this.
- Thus every infinite r.e. set has to contain some minimal points of $(\mathscr{E}^*)^\star$. This already suggests that $(\mathscr{E}^*)^\star$ may be kind of flat and shallow.
- This sentiment is further supported by the presence in $(\mathscr{E}^*)^\star$ of points that are both minimal and maximal.
- Here is how they can be constructed. Construct inductively a decreasing sequence of infinite recursive sets R_i
- that eventually decides every r.e. set in the following sense: either the ith element of the sequence is a subset of or disjoint from the ith r.e. set. Given a next r.e. set, our current recursive set either has an infinite intersection with it, in which case we select an infinite recursive subset of that intersection, or the intersection is finite in which case we just throw away the finitely many tainted elements. Note that the sequence does not have to be effective or anything.
- Let y be a prime filter containing all of the recursive sets from the sequence we constructed.
- We claim that y is a maximal point of $(\mathscr{E}^*)^\star$, for y cannot be extended to contain any r.e. set that it does not already contain because of disjointness of that r.e. set from an appropriate element of our sequence of recursive sets.
- To show that y is minimal assume w is smaller than y.
- Then w will have to be covered by all the recursive sets from our sequence just as y is because recursive sets are downward closed in the dual space.
- By the previous argument then, w is maximal.
- Hence $w = y$ and y is maximal as well. In fact, one can show that there are minimax points in any infinite r.e. set: just run the same construction starting from that r.e. set.
- So this could be a more accurate picture of $(\mathscr{E}^*)^\star$.
- Perhaps one can offer another piece of evidence that the minimal points of $(\mathscr{E}^*)^\star$ have a tight grip on all of the dual space: Any automorphism of $\mathscr{E}^*$ (which has then to be an automorphism of the dual space as well) is uniquely determined by its action on the set of minimal points — this follows from some old results of Shore.
- Comparing the situation to the E-tree, no matter whether Σ_1-ill or -sound, here the situation appears to be seriously different: There are Σ_1 sentences that are called *doubly conservative.*
- This means that σ is Π_1-conservative over T, and that the negation of σ is Σ_1-conservative. In the dual space this means that the picture of σ covers all maximal and none of the minimal points.
- In fact we can have a dense chain of sentences like this layered within the E-tree. To me this suggests a measure of depth that sets the E-tree apart from $(\mathscr{E}^*)^\star$.

Simple sets

Definition (Post). An r.e. S is *simple*
$\iff$ S intersects every infinite r.e. set.

Fact. *S is simple* $\iff$ $S^\star \supseteq \max(\mathscr{E}^*)^\star$.

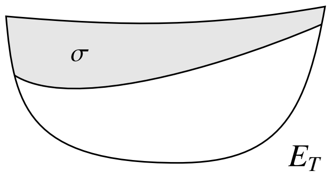


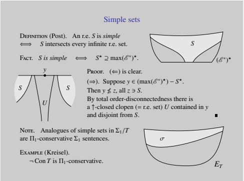


Proof. ($\Leftarrow$) is clear.
($\Rightarrow$). Suppose $y \in (\max(\mathscr{E}^*)^\star) - S^\star$.
Then $y \nleq z$, all $z \ni S$.
By total order-disconnectedness there is a $\uparrow$-closed clopen (= r.e. set) U contained in y and disjoint from S. ∎

Note. Analogues of simple sets in Σ_1/T are Π_1-conservative Σ_1 sentences.

Example (Kreisel).
$\neg$ Con T is Π_1-conservative.





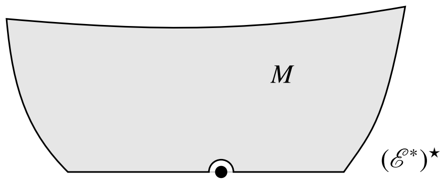
- We shall now have a look at some familiar classes of r.e. sets through the new spectacles.
- Recall the definition of a simple set …
- We are going to show that simple sets are recognizable in the dual space as those containing all maximal points.
- First of all, it is clear that once the set covers all maximal points, there is no room for a disjoint infinite r.e. set, for the latter will have to contain some maximal point which would mean that the intersection in $\mathscr{E}^*$ is non-zero.
- Now suppose a set failed to cover a maximal point y.
- Then y does not lie below any point of $S^\star$.
- By total order-diconnectedness there must be an upwards closed clopen set containing y but disjoint from $S^\star$. That's our infinite r.e. set U disjoint from S.
- In Σ_1/T, a Σ_1 sentence is Π_1-conservative if it implies in T no Π_1 consequences that are not already provable in T. Equivalently, it refutes no T-irrefutable Σ_1 sentences. This is a direct translation of simplicity into Σ_1/T. It looks pretty much like a simple set except that it need not generally cover any minimal points of the dual space.
- An example of Π_1-conservative Σ_1 sentence is the usual statement that the theory T is inconsistent.

Maximal sets

DEFINITION. An r.e. M is *maximal*

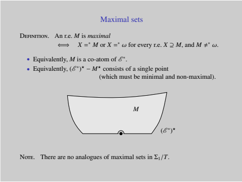
$$\iff \quad X =^* M \text{ or } X =^* \omega \text{ for every r.e. } X \supseteq M, \text{ and } M \neq^* \omega.$$

- Equivalently, M is a co-atom of $\mathscr{E}^*$.
- Equivalently, $(\mathscr{E}^*)^\star - M^\star$ consists of a single point (which must be minimal and non-maximal).



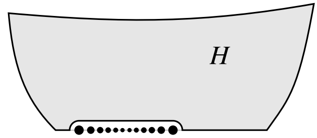
NOTE. There are no analogues of maximal sets in Σ_1/T.

Non-standard elements of r.e. sets
Pictures
Maximal sets



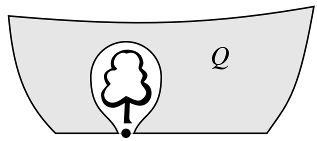
- An r.e. set is *maximal* if each of its r.e. supersets is either cofinite or almost equal to M.
- Equivalently, M is a co-atom of the lattice $\mathscr{E}^*$.
- Equivalently, $M^\star$ covers all but a single point in the dual space.
- This point will have to be minimal and no, it cannot be maximal because every maximal r.e. set is simple, and simple sets cover all maximal points.
- In Σ_1/T, there are no elements like that.

Hyperhypersimple and r-maximal sets



DEFINITION. An r.e. M is *hyperhypersimple*
$\iff$ the interval $[H, \omega]$ of $\mathscr{E}^*$
is a non-trivial Boolean algebra.

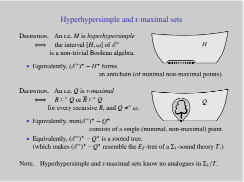
- Equivalently, $(\mathscr{E}^*)^\star - H^\star$ forms
an antichain (of minimal non-maximal points).



DEFINITION. An r.e. Q is *r-maximal*
$\iff$ $R \subseteq^* Q$ or $\overline{R} \subseteq^* Q$
for every recursive R, and $Q \neq^* \omega$.

- Equivalently, $\min(\mathscr{E}^*)^\star - Q^\star$
consists of a single (minimal, non-maximal) point.
- Equivalently, $(\mathscr{E}^*)^\star - Q^\star$ is a rooted tree.
(which makes $(\mathscr{E}^*)^\star - Q^\star$ resemble the E_T-tree of a Σ_1-sound theory T.)

NOTE. Hyperhypersimple and r-maximal sets know no analogues in Σ_1/T.

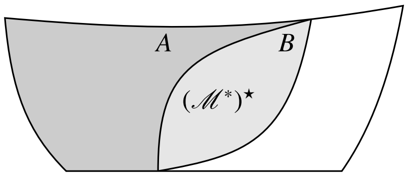


- The class of maximal sets can be generalized in two different directions. The first is hyperhypersimple sets. An r.e. set is *hyperhypersimple* if the lattice of its r.e. supersets forms a non-trivial Boolean algebra.
- Equivalently, the complement of $H^\star$ is an antichain — that's because a Priestley space is that of a Boolean algebra if and only if the ordering is trivial.
- The points in the antichain will have to be minimal (otherwise it would not be an antichain) and none of them can be maximal.
- An infinite r.e. set Q is *r-maximal* if no recursive set splits Q into two infinite, co-infinite halves.
- This happens if and only if there is just one minimal point in the complement of $Q^\star$, for any two distinct minimal points can be separated by some recursive set — this is a consequence of the Reduction Property.
- Equivalently, the complement of $Q^\star$ must be a rooted tree
- which makes that complement resemble the E-tree of a Σ_1-sound theory.
- In lattices of Σ_1 sentences, there are no analogues of hyperhypersimple nor of r-maximal sets.

Major subsets

Definition. Let $A \subseteq B$, both r.e. A is a *major* subset of B ($A \subset_{\mathrm{m}} B$)
$\iff$ A and B have the same recursive subsets, and $A \neq^* B$.

- Equivalently, $A^\star \cap \min(\mathscr{E}^*)^\star = B^\star \cap \min(\mathscr{E}^*)^\star$.
- This implies $A^\star \cap \max(\mathscr{E}^*)^\star = B^\star \cap \max(\mathscr{E}^*)^\star$ (because A not simple in B $\implies$ $\exists$inf. rec. $R \subseteq B - A$).



Theorem (Maass & Stob). *All intervals* $[A, B]$ *of* $\mathscr{E}^*$ *with* $A \subset_{\mathrm{m}} B$ *are isomorphic.*

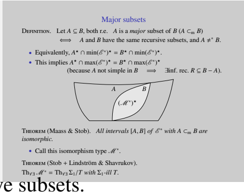
- Call this isomorphism type $\mathscr{M}^*$.

Theorem (Stob + Lindström & Shavrukov).
$\mathrm{Th}_{\forall\exists}\, \mathscr{M}^* = \mathrm{Th}_{\forall\exists}\, \Sigma_1/T$ *with* Σ_1*-ill* T.

Non-standard elements of r.e. sets

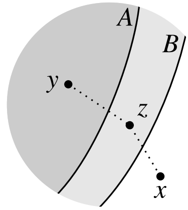
Pictures

Major subsets



- A subset A is called a *major* subset of B if they have the same recursive subsets.
- This is equivalent to A and B having the same footprint on the set of minimal points of the dual space. This follows from the Reduction Property.
- Now, curiously, that implies that their intersections with the set of maximal nodes also coincide. This can be seen as yet more evidence to support the thesis that $(\mathscr{E}^*)^\star$ is rather flat, for it suggests that the sets of minimal and of maximal points lie rather close to one another.
- This happens because if there were maximal nodes in B but not in A then A would not be simple within B, so by essentially the same argument we used to characterize simplicity there would be an infinite r.e. set in the difference $B - A$, and hence also an infinite recursive one.
- So this would be a more accurate picture of the situation.
- Maass & Stob showed that all intervals between an r.e. set and its majot subset are isomorphic as lattices.
- We call this isomorphism type $\mathscr{M}^*$, the major interval.
- Now we have been remarking that, compared to the lattice of Σ_1 sentences, the dual space of $\mathscr{E}^*$ looks flat and shallow because, for example, of minimax points. But the major intreval looks in fact very much like Σ_1/T with a Σ_1-ill theory T because the $\forall\exists$ theories of the two lattices coincide. Here Michael Stob is responsible for the left-hand side of the equality, and others, for the right-hand side. So $(\mathscr{E}^*)^\star$ contains an interval that does not really look all that flat.

Small subsets

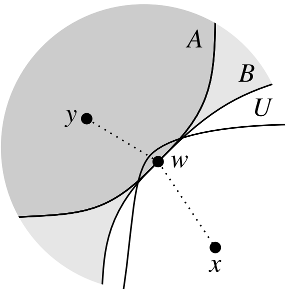


Definition. Let $A \subset B$ be r.e. A is *small in* B $(A \subset_s B)$

$$\iff \quad \forall \text{r.e. } U\,(B - A \subseteq U \;\Rightarrow\; U \cup \overline{B} \text{ r.e.}).$$

Proposition. *Suppose $A \leq B$ in any relatively normal L. Then $A \subset_s B \iff$ for all $x \leq y$ in $L^\star$ such that $B \notin x$ and $A \in y$ there is $z \in (x, y)$ with $z \in B^\star - A^\star$.*

Proof of ($\Rightarrow$). Suppose $x \leq y$ in $L^\star$ with no points from $B^\star - A^\star$ in between.



Let w be minimal in $[x, y] \cap B^\star$.

Then $w \in \min B^\star \cap \min A^\star$
(assumption + relative normality).

Hence $w \not\geq u$, all $u \in B^\star - A^\star$.

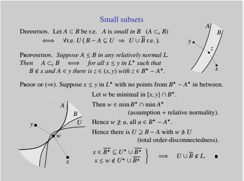
Hence there is $U \supseteq B - A$ with $w \not\ni U$
(total order-disconnectedness).

$$\left.\begin{array}{l} x \in \overline{B^\star} \subseteq U^\star \cup \overline{B^\star} \\ x \leq w \notin U^\star \cup \overline{B^\star} \end{array}\right\} \implies U \cup \overline{B} \notin L. \quad \blacksquare$$

Non-standard elements of r.e. sets

Pictures

Small subsets



- Small subsets were introduced by Lachlan. Small major subsets are helpful with the $\forall\exists$ theory of $\mathscr{E}^*$.
- Here is the original definition of small subsets …I found it difficult to understand. I understood it better once I translated it in terms of the Priestley dual: For every pair of points such that the larger one lies in $A^\star$ and the smaller one, outside $B^\star$ there is a point strictly in between that lies in the difference $B^\star - A^\star$.
- Note that both the Definition and the Proposition make sense in any distributive lattice. The statement that some Boolean combination is r.e. is translated by saying that the appropriate Boolean term is an element of the lattice, that is, corresponds to an upwards closed subset of the dual space.
- We sketch a proof of one of the directions in order to sample the fairly typical flavour of this kind of argument. It probably makes better sense just to note the ingredients rather than follow every detail of the proof.
- Suppose between x and y there were no points from the difference $B^\star - A^\star$.
- Choose a point w that is minimal among points in $B^\star$ lying between x and y — there must be a minimal point because this set is closed. Since the dual space is treelike, the interval $[x, y]$ is in fact a chain.
- Since we have assumed that between x and y there are no elements of the difference $B - A$, and also because the ordering is tree-like, w must be a minimal point of B as well as of A.
- Therefore in the light grey area $B - A$ there can be no point u that lies below w.
- This implies that there is an upwards closed clopen U that contains the difference of B and A but not the point w — this is a consequence of the total order-disconnectedness of the dual space.
- Now observe that x is a point in the complement of B, hence x lies in this union,
- but the higher point w lies outside that same union.
- Hence $U \cup \overline{B}$ is not an element of the lattice L because it corresponds to a subset of the dual space that is not upwards closed.
- Thus having assumed that the r.h.s. of the Proposition fails, we have shown that A is not small in B in the sense of the original definition.

Prime filters, models, extensions

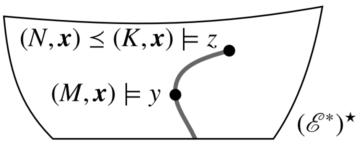
- $\mathrm{Th}_{\Sigma_1}(M,\boldsymbol{x})$ is a point in $(\mathscr{E}^*)^\star$ for any $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$.
- Every point of $(\mathscr{E}^*)^\star$ is of the form $\mathrm{Th}_{\Sigma_1}(M,\boldsymbol{x})$ for some $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$.
- Notation $(M,\boldsymbol{x})$ generally presupposes countable $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$.
- For y in $(\mathscr{E}^*)^\star$, write $(M,\boldsymbol{x}) \models y$ if $y = \mathrm{Th}_{\Sigma_1}(M,\boldsymbol{x})$.

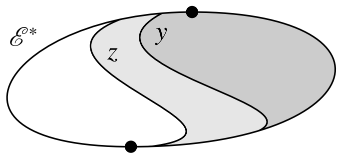
Equivalently, $(M,\boldsymbol{x}) \models \boldsymbol{x} \in X \Leftrightarrow X \in y$ for all r.e. X.

PROPOSITION. *For y, z in $(\mathscr{E}^*)^\star$, $y \leq z$*

$\Leftrightarrow$ *every model $(M,\boldsymbol{x}) \models y$ has an extension $(N,\boldsymbol{x}) \supseteq (M,\boldsymbol{x})$ s.t. $(N,\boldsymbol{x}) \models z$.*

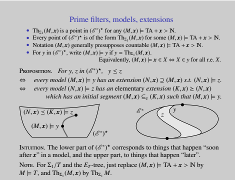
$\Leftrightarrow$ *every model $(N,\boldsymbol{x}) \models z$ has an* elementary *extension $(K,\boldsymbol{x}) \succeq (N,\boldsymbol{x})$ which has an initial segment $(M,\boldsymbol{x}) \subseteq_e (K,\boldsymbol{x})$ such that $(M,\boldsymbol{x}) \models y$.*





INTUITION. The lower part of $(\mathscr{E}^*)^\star$ corresponds to things that happen "soon after $\boldsymbol{x}$" in a model, and the upper part, to things that happen "later".

NOTE. For Σ_1/T and the E_T-tree, just replace $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$ by $M \models T$, and $\mathrm{Th}_{\Sigma_1}(M,\boldsymbol{x})$ by $\mathrm{Th}_{\Sigma_1} M$.

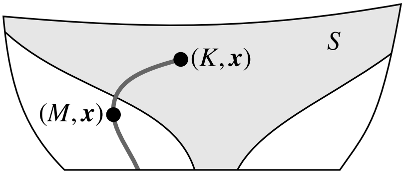


- We have already noted that the collection of r.e. sets to which a non-standard element of a model of True Arithmetic belongs forms a prime filter in $\mathscr{E}^*$, that is, a point in $(\mathscr{E}^*)^\star$.
- Also, every prime filter in $\mathscr{E}^*$ has that form for an appropriate model. This is a represetation of points of the dual space that we are going to use extensively.
- The notation $(M,\boldsymbol{x})$ will generally presuppose that M is a countable model of true arithmetic and $\boldsymbol{x}$ is non-standard.
- We write that such a model is a model of a prime filter y if y coincides with the Σ_1 theory of the model M with parameter $\boldsymbol{x}$.
- Equivalently, in M, the distinguished element $\boldsymbol{x}$ belongs to exactly those sets that are in the prime filter y.
- The following Proposition characterizes the ordering relation on the dual space in terms of model extensions. Thus $y \leq z$
- if and only if to each model M of the lower point y
- there is an extension N to a model of the higher point z. The extension is required to preserve the distinguished element $\boldsymbol{x}$.
- Equivalently, every model of the higher point z
- has an *elementary* extension K (alternatively, a cofinal extension would suffice — we only need to beef up the standard system)
- which can be cut down to a model of the lower point y.
- The intuition that the lower part of the dual space corresponds to things that happen "soon after $\boldsymbol{x}$" in a model, and the upper part, to things that happen "later".
- Exactly the same situation obtains in the E-tree of a theory T — except all models are now models of T and the parameter $\boldsymbol{x}$ goes: we are now dealing with Σ_1 *sentences*.

Model theory of simple sets

Corollary. *Let S be r.e. Then S is simple*

$\iff$ *each countable* $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$ *has an extension* $(M,\boldsymbol{x}) \subseteq (K,\boldsymbol{x}) \models \boldsymbol{x} \in S$.



Proposition (after Wilkie; J. Schmerl) *Let an r.e. S be simple. Then each* $(M,\boldsymbol{x}) \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$ *has an* end-*extension* $(M,\boldsymbol{x}) \subseteq_{\mathrm{e}} (K,\boldsymbol{x}) \models \boldsymbol{x} \in S$.

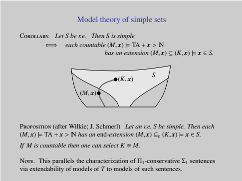
If M is countable then one can select $K \cong M$.

Note. This parallels the characterization of Π_1-conservative Σ_1 sentences via extendability of models of T to models of such sentences.

Non-standard elements of r.e. sets

Models of arithmetic

Model theory of simple sets

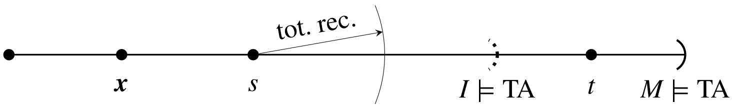

Model theory of simple sets
Corollary. Let S be r.e. Then S is simple ⟺ each countable (M,x) ⊨ TA + x > ℕ has an extension (M,x) ⊆ (K,x) ⊨ x ∈ S.
Proposition (after Wilkie; J. Schmerl) Let an r.e. S be simple. Then each (M,x) ⊨ TA + x > ℕ has an end-extension (M,x) ⊆e (K,x) ⊨ x ∈ S.
If M is countable then one can select K ≅ M.
Note. This parallels the characterization of Π1-conservative Σ1 sentences via extendability of models of T to models of such sentences.

- Recall our characterization of simple r.e. sets: A set is simple if and only if its picture covers all of the maximal points of the dual space.
- To illustrate the facts about extensions from the previous slide we can immediately conclude a model-theoretical characterization of simple r.e. sets: A set S is simple if and only if
- to each countable model M of true arithmetic with a non-standard element $\boldsymbol{x}$
- there is an extension K in which $\boldsymbol{x}$ becomes an element of S. Observe that the element $\boldsymbol{x}$ is fairly arbitrary, so not just every infinite r.e. set fails to avoid S, but also no non-standard number can feel safe from eventual membership in S in an appropriate extension.
- With a little extra work (most of it due to Wilkie and Schmerl) we can both drop the countability assumption and arrange the extension to be an end-extension.
- Furthermore, in the countable case, the extension can be chosen isomorphic to the original model. Of course, the isomorphism cannot preserve the distinguished element $\boldsymbol{x}$.
- In Σ_1/T, there is a similar characterization of Π_1-conservative Σ_1 sentences via extendability of any model of T to a model of such a sentence.

Recursive functions and hinges

LEMMA (Wilkie). *Let* $s < t \in M \models \mathrm{TA}$. *Then*

$$\exists I \subseteq_e M \text{ with } I \models \mathrm{TA} \text{ and } s \in I < t$$
$$\iff \quad M \models f(s) < t \text{ for all total recursive } f.$$



- $\boldsymbol{x} \in_s B$ means "$\boldsymbol{x}$ gets into B by stage s".
  $\boldsymbol{x} \in_{\mathrm{at}\ s} B$ means $(\boldsymbol{x} \in_s B \ \&\ \boldsymbol{x} \notin_{s-1} B)$.
- $\{\text{r.e. } A \mid \text{for some tot. rec. } f,\ (M,\boldsymbol{x}) \models \boldsymbol{x} \in_{f(s)} A\} = y_s \in (\mathscr{E}^*)^\star$, any $s \geq \boldsymbol{x}$.
- When $\boldsymbol{x} \in_{\mathrm{at}\ s} A$, the prime filter y_s is *hinged* (on A)

y is hinged on A

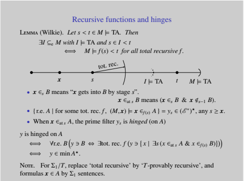
$$\iff \quad \forall \text{r.e. } B \Big(y \ni B \Leftrightarrow \exists \text{tot. rec. } f\,(y \ni \{x \mid \exists s\,(x \in_{\mathrm{at}\ s} A \ \&\ x \in_{f(s)} B)\})\Big)$$
$$\iff \quad y \in \min A^\star.$$

NOTE. For Σ_1/T, replace 'total recursive' by 'T-provably recursive', and formulas $\boldsymbol{x} \in A$ by Σ_1 sentences.

Non-standard elements of r.e. sets

Models of arithmetic

Recursive functions and hinges



- Suppose we've got two numbers in a non-standard model of True Arithmetic.
- Question: when can we find an initial segment containing the smaller but not the larger number and still have this segment model True Arithmetic?
- Answer: this happens exactly when you cannot get from the smaller number past the larger one by total recursive functions. Here we are talking usual total recursive functions with *standard* indices. This has been noted by Alex Wilkie and can be seen as a consequence of Friedman's theorem on self-embeddings of countable models of arithmetic.
- The following notation is read "$\boldsymbol{x}$ is enumerated into B by stage s". This refers to some fixed enumeration of all r.e. sets as increasing sequences of finite sets with the usual properties.
- The notation "at s" is used to say that $\boldsymbol{x}$ is enumerated at *exactly* stage s.
- It follows from Wilkie's Lemma that once you take a distinguished non-standard number $\boldsymbol{x}$ and all r.e. sets where $\boldsymbol{x}$ appears at stages that are less than some total recursive function of s, you get a prime filter, that is, a point in $(\mathscr{E}^*)^\star$.
- If in this situation there is something to distinguish that number, namely, the number s is the stage at which $\boldsymbol{x}$ enters some r.e. set A, we say that the point y *is hinged* (*on* A, if we have to be specific) — or that A is a *hinge* for y.
- A point y's being hinged on A is equivalent to y consisting of all the sets B that the distinguished element $\boldsymbol{x}$ of the model enters at most total recursively later than it enters A — which is confirmed by the membership in y of this r.e. set. This gives us a definition of hinges that quantifies away the number s and shows that it is independent of the choice of the model.
- Yet another equivalent definition says that, in the dual space, y is a minimal point of $A^\star$, for any prime filter smaller than y will in this situation fail to contain A.
- In the lattice of Σ_1 sentences modulo T, we have exactly the same situation with 'T-provably recursive' in place of 'total recursive' and Σ_1 sentences in place of formulas expressing membership of $\boldsymbol{x}$ in r.e. sets.

Minimal points, maximal points, and hinges

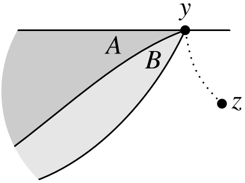
- Minimal points $y \in \min(\mathscr{E}^*)^\star$ hinge on any r.e. $X \in y$ as $y \in \min X^\star$.
- For $y \in \min(\mathscr{E}^*)^\star$ we have: $\quad (M, \boldsymbol{x}) \models y$
  $\iff \quad \forall \text{r.e. } X\,((M,\boldsymbol{x}) \models \boldsymbol{x} \in X \quad \Rightarrow \quad \exists \text{tot. rec. } f\ (M,\boldsymbol{x}) \models \boldsymbol{x} \in_{f(\boldsymbol{x})} X)$.
- The minimax points $y \in \min(\mathscr{E}^*)^\star \cap \max(\mathscr{E}^*)^\star$ are also hinged.

Proposition. $y \in \max(\mathscr{E}^*)^\star$ *and* y *is hinged* $\implies$ $y \in \min(\mathscr{E}^*)^\star$.

Proof. Suppose $y \in \max(\mathscr{E}^*)^\star$ hinges on B and $z < y$.
B is not recursive because $B \notin z < y \ni B$.

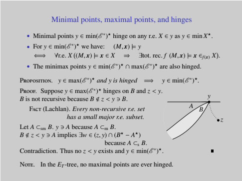
Fact (Lachlan). *Every non-recursive r.e. set has a small major r.e. subset.*

Let $A \subset_{\text{sm}} B$. $y \ni A$ because $A \subset_{\text{m}} B$.
$B \notin z < y \ni A$ implies $\exists w \in (z, y) \cap (B^\star - A^\star)$ because $A \subset_{\text{s}} B$.
Contradiction. Thus no $z < y$ exists and $y \in \min(\mathscr{E}^*)^\star$. ∎



Note. In the E_T-tree, no maximal points are ever hinged.

Non-standard elements of r.e. sets
Models of arithmetic
Minimal points, maximal points, and hinges


Minimal points, maximal points, and hinges
Minimal points $y \in \min(\mathscr{E}^*)^\star$ hinge on any r.e. $X \in y$ as $y \in \min X^\star$.
For $y \in \min(\mathscr{E}^*)^\star$ we have: $(M, \boldsymbol{x}) \models y$ $\iff \forall \text{r.e. } X\,((M,\boldsymbol{x}) \models \boldsymbol{x} \in X \Rightarrow \exists \text{tot. rec. } f\ (M,\boldsymbol{x}) \models \boldsymbol{x} \in_{f(\boldsymbol{x})} X)$.
The minimax points $y \in \min(\mathscr{E}^*)^\star \cap \max(\mathscr{E}^*)^\star$ are also hinged.
Proposition. $y \in \max(\mathscr{E}^*)^\star$ and y is hinged $\implies$ $y \in \min(\mathscr{E}^*)^\star$.
Proof. Suppose $y \in \max(\mathscr{E}^*)^\star$ hinges on B and $z < y$. B is not recursive because $B \notin z < y \ni B$.
Fact (Lachlan). Every non-recursive r.e. set has a small major r.e. subset.
Let $A \subset_{\text{sm}} B$. $y \ni A$ because $A \subset_{\text{m}} B$. $B \notin z < y \ni A$ implies $\exists w \in (z, y) \cap (B^\star - A^\star)$ because $A \subset_{\text{s}} B$.
Contradiction. Thus no $z < y$ exists and $y \in \min(\mathscr{E}^*)^\star$. ∎
Note. In the E_T-tree, no maximal points are ever hinged.

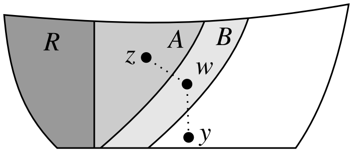
- Consider minimal points of the dual space. By virtue of being minimal, they are hinged — on any r.e. set that covers them.
- If M is a model of a minimal point
- then the distinguished element $\boldsymbol{x}$ enters any r.e. set it belongs to at a total recursive distance from $\boldsymbol{x}$. For otherwise by Wilkie's Lemma there would be an initial segment modelling True Arithmetic together with a smaller prime filter.
- So in particular the minimax points whose existence we discussed a while ago are also hinged. We are going to see that these minimax points are rather atypical because they are the only maximal points that are hinged.
- In other words, if a maximal point is hinged then it is also minimal.
- Let us see why. Suppose y is a maximal point hinging on B and there is a point $z < y$.
- Since y is in B but the smaller z is not, $B^\star$ is not downward closed, so B cannot be recursive.
- Now we recall a fact due to Lachlan: Every non-recursive r.e. set has a small major subset.
- So let A be a small major subset of B.
- A covers y because A is major in B, so the maximal points in $A^\star$ and $B^\star$ must be the same.
- Since A is also small in B, y lies in $A^\star$ and the smaller z lies outside $B^\star$, there must be a point between z and y that lies in the difference $B^\star - A^\star$.
- But between z and y there are no points other than y that lie in $B^\star$, so we have a contradiction
- which proves that $z < y$ cannot exist so y is minimal in $(\mathscr{E}^*)^\star$.
- In the E-tree, no maximal points are ever hinged.

Small subsets again

REMINDER. Suppose $A \subseteq B$. Then

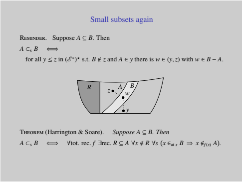
$A \subset_s B \iff$

for all $y \leq z$ in $(\mathscr{E}^*)^\star$ s.t. $B \notin z$ and $A \in y$ there is $w \in (y, z)$ with $w \in B - A$.



THEOREM (Harrington & Soare). *Suppose $A \subseteq B$. Then*

$A \subset_s B \iff \forall$tot. rec. f $\exists$rec. $R \subseteq A$ $\forall x \notin R$ $\forall s$ $(x \in_{\text{at } s} B \Rightarrow x \notin_{f(s)} A)$.

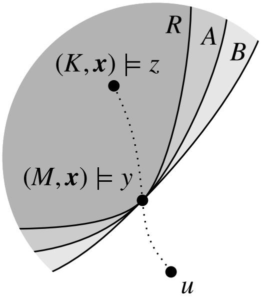


- We return to small subsets.
- Recall that we obtained the following equivalent of smallness in terms of the dual space (of any relatively normal lattice): ...
- The following Theorem of Harrington & Soare characterizes smallness as follows:
- Given any total recursive function f — you would typically think of a pretty fast growing function — there is a recursive subset R of A,
- which probably looks somewhat like this, such that any element outside R, when it is enumerated into B, has to wait at least f much time before it has a chance to get into A. (Recall that subscripts on the membership symbol refer to the first s steps of some fixed enumeration of all r.e. sets.)
- This is an example of what Soare calls a *dynamic* property of r.e. sets presumably because its form puts emphasis on the comparison between the stages at which elements are enumerated into different sets.
- We are going to give a proof of this theorem using our dual-space characterization of smallness together with representation of points in the dual space by non-standard models of True Arithmetic.

Proof of ($\Longrightarrow$)

THEOREM (Harrington & Soare). *Suppose $A \subseteq B$. Then*
$A \subset_s B \quad \Longleftrightarrow \quad \forall \text{tot. rec.}\, f\ \exists \text{rec.}\, R \subseteq A\ \forall x \notin R\ \forall s\ (x \in_{\text{at } s} B \Rightarrow x \notin_{f(s)} A)$.

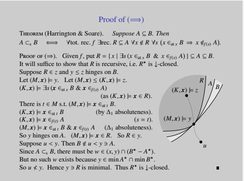
PROOF OF ($\Rightarrow$). Given f, put $R = \{\, x \mid \exists s\,(x \in_{\text{at } s} B\ \&\ x \in_{f(s)} A)\,\} \subseteq A \subseteq B$.
It will suffice to show that R is recursive, i.e. $R^\star$ is $\downarrow$-closed.
Suppose $R \in z$ and $y \le z$ hinges on B.
Let $(M, \boldsymbol{x}) \models y$. Let $(M, \boldsymbol{x}) \le (K, \boldsymbol{x}) \models z$.
$(K, \boldsymbol{x}) \models \exists s\,(\boldsymbol{x} \in_{\text{at } s} B\ \&\ \boldsymbol{x} \in_{f(s)} A)$
(as $(K, \boldsymbol{x}) \models \boldsymbol{x} \in R$).
There is $t \in M$ s.t. $(M, \boldsymbol{x}) \models \boldsymbol{x} \in_{\text{at } t} B$.
$(K, \boldsymbol{x}) \models \boldsymbol{x} \in_{\text{at } t} B$ (by Δ_1 absoluteness).
$(K, \boldsymbol{x}) \models \boldsymbol{x} \in_{f(t)} A$ ($s = t$).
$(M, \boldsymbol{x}) \models \boldsymbol{x} \in_{\text{at } t} B\ \&\ \boldsymbol{x} \in_{f(t)} A$ (Δ_1 absoluteness).
So y hinges on A. $(M, \boldsymbol{x}) \models \boldsymbol{x} \in R$. So $R \in y$.
Suppose $u < y$. Then $B \notin u < y \ni A$.
Since $A \subset_s B$, there must be $w \in (x, y) \cap (B^\star - A^\star)$.
But no such w exists because $y \in \min A^\star \cap \min B^\star$.
So $u \not< y$. Hence $y \ni R$ is minimal. Thus $R^\star$ is $\downarrow$-closed. ∎



Non-standard elements of r.e. sets

Dynamics

Harrington–Soare, proof of ($\Rightarrow$)



- We show that smallness implies the Harrington–Soare property.
- Given f, we define a set R which is clearly an r.e.
- subset of A as well as of B.
- It suffices to show that R is recursive because R explicitly disqualifies those x that fail to satisfy the implication in Harrington–Soare.
- We recall that recursiveness of an r.e. set is equivalent to being downward closed in the dual space, so downward closure is all we've got to show.
- So we assume that z lies in $R^\star$ and we will show that $R^\star$ covers everything below z. Since $R \subseteq B$ there must be $y \le z$ that hinges on B.
- Suppose y corresponds to some model M.
- There is then an extension of M to a model K of z that preserves the distinguished element $\boldsymbol{x}$.
- Since z lies in R, in the model K, $\boldsymbol{x}$ is an element of R which means that this formula holds in K.
- As y lies in B, we can fix a stage t in M at which $\boldsymbol{x}$ goes into B.
- By Δ_1 absoluteness for submodels, exactly the same thing happens in K. Note that this is the kind of formula whose truth is decided total recursively soon after $\boldsymbol{x}$ enters B.
- There can be at most one stage at which $\boldsymbol{x}$ gets into B, so, in K, the number s is equal to the number t, and we have that $\boldsymbol{x}$ gets into A by stage $f(t)$.
- As $t \in M$ and total recursive functions are absolute, the same picture obtains in the model M.
- This in particular means that y hinges on A because, in M, $\boldsymbol{x}$ is already in A and it could not be an element of A at any lower point because $A \subseteq B$.
- It also means that $\boldsymbol{x}$ is an element of R in M.
- So R in fact covers y.
- Now suppose there was an element u strictly below y.
- y lies in A and u is outside B,
- so by smallness of A in B there must be a point of $B - A$ in between — that's where we finally connect to the smallness assumption.
- But in the interval between u and y there is no space for a point like this for A and B are both hinges for y.
- Thus no $u < y$ can exist
- which proves that y is a minimal point of the dual space and the picture in fact looks like this.
- We have shown that $R^\star$ is downwards closed, i.o.w. R is recursive. This completes the proof of one direction.

Proof of ($\Longleftarrow$)

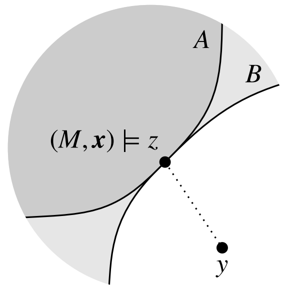
THEOREM (Harrington & Soare). *Suppose $A \subseteq B$. Then*

$A \subset_s B \iff \forall$tot. rec. f $\exists$rec. $R \subseteq A$ $\forall x \notin R$ $\forall s$ $(x \in_{\text{at } s} B \Rightarrow x \notin_{f(s)} A)$.

PROOF OF ($\Leftarrow$). Suppose $A \subseteq B$ and $A \not\subset_s B$.
There are $y < z$ with $A \in z$, $B \notin y$
and $[y, z] \cap (B^\star - A^\star) = \varnothing$.
W.l.o.g. $z \in \min A^\star \cap \min B^\star$, so z hinges on B.
Hence there is a total recursive f s.t.

$$(M, \boldsymbol{x}) \models \exists s\, (\boldsymbol{x} \in_{\text{at } s} B \;\&\; \boldsymbol{x} \in_{f(s)} A)$$

for any $(M, \boldsymbol{x}) \models z$.
Suppose $R \subseteq A$ is recursive.



$$\left.\begin{array}{l} y \not\ni A \Rightarrow y \not\ni R \\ \qquad\qquad z > y \end{array}\right\} \Rightarrow z \not\ni R \Rightarrow (M, \boldsymbol{x}) \models \boldsymbol{x} \notin R.$$

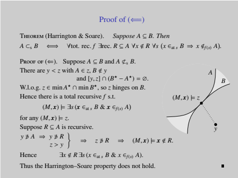
Hence $M \models \exists x \notin R\, \exists s\, (x \in_{\text{at } s} B \;\&\; x \in_{f(s)} A)$.

Thus the Harrington–Soare property does not hold. ∎

Non-standard elements of r.e. sets

Dynamics

Harrington–Soare, proof of ($\Leftarrow$)



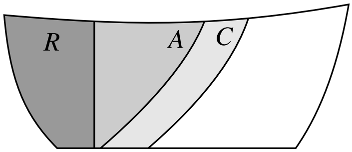
Proof of ($\Longleftarrow$)
THEOREM (Harrington & Soare). Suppose $A \subseteq B$. Then
$A \subset_s B \iff \forall$tot. rec. f $\exists$rec. $R \subseteq A$ $\forall x \notin R$ $\forall s$ $(x \in_{\text{at } s} B \Rightarrow x \notin_{f(s)} A)$.
PROOF OF ($\Leftarrow$). Suppose $A \subseteq B$ and $A \not\subset_s B$.
There are $y < z$ with $A \in z$, $B \notin y$
and $[y, z] \cap (B^\star - A^\star) = \varnothing$.
W.l.o.g. $z \in \min A^\star \cap \min B^\star$, so z hinges on B.
Hence there is a total recursive f s.t.
$(M, \boldsymbol{x}) \models \exists s\, (\boldsymbol{x} \in_{\text{at } s} B \;\&\; \boldsymbol{x} \in_{f(s)} A)$
for any $(M, \boldsymbol{x}) \models z$.
Suppose $R \subseteq A$ is recursive.
Hence $\exists x \notin R\, \exists s\, (x \in_{\text{at } s} B \;\&\; x \in_{f(s)} A)$.
Thus the Harrington–Soare property does not hold.

- Here we obtain smallness from the Harrington–Soare property.
- Suppose A is a subset, but not a small one, of B.
- Without loss of generality we may assume $z \in \min A^\star$ — we have seen this situation before in general relatively normal lattices.
- We have also seen that this implies $z \in \min B^\star$,
- so B (as well as, for that matter, A) is a hinge of z.
- There must be a model of true arithmetic together with a non-standard element $\boldsymbol{x}$ representing z. In that model the time of arrival of the distinguished element $\boldsymbol{x}$ into any r.e. set, in particular, into A, is bounded by a total recursive function of the stage of its arrival into B, so we call that recursive function f. The identity of f depends on the set A but does not really depend on the choice of the model M as long as it is a model of z.
- Suppose R were any recursive subset of A.
- A is not an element of y,
- so neither is its subset R.
- It follows that R is not an element of z either, for recursive sets are both upwards and downwards closed in the dual space.
- Hence in the model M, the distinguished element does not belong to R.
- Quantifying the distinguished element existentially, we get: …
- But M is a model of true arithmetic, so this statement is in fact true in the standard model.
- Thus we have shown that there is a total recursive f such that for any recursive subset R of A there are an x outside R and a stage s at which x is enumerated into B and it also gets enumerated into A not later than $f(s)$. This amounts to the negation of the Harrington–Soare property.
- To summarize, the Harrington–Soare property implies smallness.

Last time small subsets

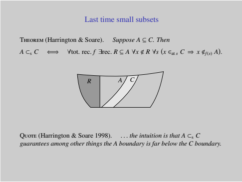
Theorem (Harrington & Soare). *Suppose $A \subseteq C$. Then*

$$A \subset_s C \iff \forall \text{tot. rec. } f\ \exists \text{rec. } R \subseteq A\ \forall x \notin R\ \forall s\ (x \in_{\text{at } s} C \Rightarrow x \notin_{f(s)} A).$$



Quote (Harrington & Soare 1998). *. . . the intuition is that $A \subset_s C$ guarantees among other things the A boundary is far below the C boundary.*

Non-standard elements of r.e. sets
Dynamics
Last time small subsets



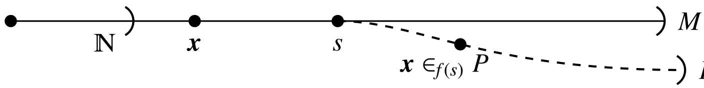
- While establishing their theorem, Harrington & Soare developed the following intuition: A is small in C impies that the 'A boundary' is far below the 'C boundary'.
- To our proof of the theorem, this is kind of more than mere intuition (although we seem to disagree on orientation). 'Far below', in our interpretation, means 'more than total recursively far away. Curiously, Harrington & Soare employed neither the dual space nor models of arithmetic in their proof of the theorem. Their original proof is neither long nor particularly difficult. For me, the advantage of the alternative proof lies in its visual character.

Model theory of prompt simplicity

DEFINITION (Maass). An r.e. P is *promptly simple*
$\iff$ there is a total recursive p s.t. for any infinite r.e. X
$\exists^\infty x\, \exists s\, (x \in_{\text{at } s} X \;\&\; x \in_{p(s)} P)$.

Prompt simplicity is *not* definable in $\mathscr{E}^*$ (Not even all maximal sets are promptly simple).

THEOREM. *P is promptly simple* $\iff$
there is a total recursive f s.t. for any countable $M \models$ TA and any $s > \boldsymbol{x} > \mathbb{N}$ in M there exists $K \models$ TA, $[0,s]_K \cong [0,s]_M$, and $K \models \boldsymbol{x} \in_{f(s)} P$.



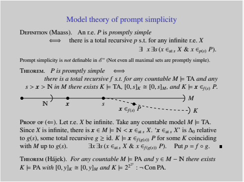
PROOF OF ($\Leftarrow$). Let r.e. X be infinite. Take any countable model $M \models$ TA. Since X is infinite, there is $\boldsymbol{x} \in M \models \mathbb{N} < \boldsymbol{x} \in_{\text{at } s} X$. '$\boldsymbol{x} \in_{\text{at } s} X$' is Δ_0 relative to $g(s)$, some total recursive $g \geq$ id. $K \models \boldsymbol{x} \in_{f(g(s))} P$ for some K coinciding with M up to $g(s)$. $K \models \exists x\, \exists s\, (x \in_{\text{at } s} X \;\&\; x \in_{f(g(s))} P)$. Put $p = f \circ g$. ∎

THEOREM (Hájek). *For any countable $M \models$ PA and $y \in M - \mathbb{N}$ there exists $K \models$ PA with $[0,y]_K \cong [0,y]_M$ and $K \models 2^{2^{2^y}} : \neg\,$Con PA.*

Non-standard elements of r.e. sets

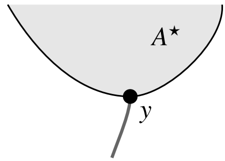
Dynamics

Model theory of prompt simplicity



- Recall that we have seen that a set is simple if and only if an appropriate (end-)extension can enter any non-standard element into that set. Now we are concerned with what happens if we require that any non-standard element can not only be entered into the set by some extension, but can be entered at reasonably short notice.
- Promptly simple sets were defined by Maass in connection with constructing automorphisms of $\mathscr{E}^*$. The definition of a *promptly simple* r.e. set is another example of dynamic property.
- The set P has to acquire an element of any infinite r.e. X not too much later than it appeared in X — the freshness of an element is measured by a total recursive function p of the stage of its arrival in X.
- This is known to be equivalent to requiring that there are infinitely many such elements x.
- While prompt simplicity is *not* definable in $\mathscr{E}^*$ — not even all maximal sets are promptly simple — we are going to show a characterization of prompt simplicity in terms of non-standard models:
- Namely, there must exist a total recursive f such that given any countable nonstandard model of true arithmetic with non-standard elements $\boldsymbol{x}$ and s,
- there is a model K coinciding with M up to and including s where $\boldsymbol{x}$ becomes an element of the promptly simple set P by stage $f(s)$.
- We are going to show the easy right-to-left direction to get a feeling for the connection. So let an infinite r.e. X be given.
- As X is infinite, there must be a nonstandard element $\boldsymbol{x}$ of X
- which gets into that set at stage s.
- Now we do not know too much about the complexity of enumeration of r.e. sets, but we do know that the formula '$\boldsymbol{x}$ gets into X at stage s' is Δ_0 relative to some number that is only recursively larger than s.
- By assumption, there is a model K that shares the initial segment up to $g(s)$ with M and where $\boldsymbol{x}$ is enumerated into P by stage $f(g(s))$.
- In that model, the following formula holds. The first conjunct is inherited from M because it is Δ_0 relative to $g(s)$.
- K being a model of true arthmetic, this statement is in fact true.
- This shows that we can just put the function p from the definition of prompt simplicity equal to the composition of f and g. This concludes the proof of the easy direction.
- This Theorem was inspired by a theorem of Hájek on models of Peano arithmetic: you can always arrange a proof of inconsistency of Peano at very short notice: Given any nonstandard y in a model of Peano, there is a model coinciding up to y with the original model such that in the new model, a proof of contradiction in Peano appears pretty soon after y.

Hinged points in relatively normal lattices

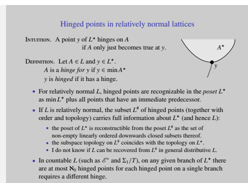


INTUITION. A point y of $L^\star$ hinges on A
if A only just becomes true at y.

DEFINITION. Let $A \in L$ and $y \in L^\star$.
A is a *hinge for* y if $y \in \min A^\star$
y is *hinged* if it has a hinge.

- For relatively normal L, hinged points are recognizable in the *poset* $L^\star$ as $\min L^\star$ plus all points that have an immediate predecessor.
- If L is relatively normal, the subset $L^\#$ of hinged points (together with order and topology) carries full information about $L^\star$ (and hence L):
  - the poset of $L^\star$ is reconstructible from the poset $L^\#$ as the set of non-empty linearly ordered downwards closed subsets thereof.
  - the subspace topology on $L^\#$ coincides with the topology on $L^\star$.
  - I do not know if L can be recovered from $L^\#$ in general distributive L.
- In countable L (such as $\mathscr{E}^*$ and Σ_1/T), on any given branch of $L^\star$ there are at most $\aleph_0$ hinged points for each hinged point on a single branch requires a different hinge.

Non-standard elements of r.e. sets
Degrees of index sets of prime filters
Hinged points in relatively normal lattices


Hinged points in relatively normal lattices
INTUITION. A point y of $L^\star$ hinges on A if A only just becomes true at y.
DEFINITION. Let $A \in L$ and $y \in L^\star$. A is a *hinge for* y if $y \in \min A^\star$ y is *hinged* if it has a hinge.
For relatively normal L, hinged points are recognizable in the *poset* $L^\star$ as $\min L^\star$ plus all points that have an immediate predecessor.
If L is relatively normal, the subset $L^\#$ of hinged points (together with order and topology) carries full information about $L^\star$ (and hence L):
the poset of $L^\star$ is reconstructible from the poset $L^\#$ as the set of non-empty linearly ordered downwards closed subsets thereof.
the subspace topology on $L^\#$ coincides with the topology on $L^\star$.
I do not know if L can be recovered from $L^\#$ in general distributive L.
In countable L (such as $\mathscr{E}^*$ and Σ_1/T), on any given branch of $L^\star$ there are at most $\aleph_0$ hinged points for each hinged point on a single branch requires a different hinge.

- So far we have largely been using the dual space as a kind of canvas to address traditional questions about r.e. sets. That perspective however also brings to the fore a motivation for its own questions, ones that one would not naturally come across within the traditional approach. An example we are going to consider are Turing degrees of index sets of hinged prime filters and their relation to the ordering of the dual space.
- Recall that in dual spaces of both $\mathscr{E}^*$ and Σ_1/T we had the intuition that a point y hinges on an r.e. set (or sentence) A when A only just becomes true at y.
- This is the picture.
- We also had the definition that works in any distributive lattice: y is a minimal point of $A^\star$.
- We say that a point is *hinged* if it has a hinge.
- Recall that a distributive lattice is called *relatively normal* if the ordering on the dual space is treelike. For such lattices, hinged points are recognizable from just the ordering on the dual space as minimal point plus all points that have an immediate predecessor.
- In relatively normal lattices, the subset $L^\#$ of hinged points carries, together with the inherited ordering and topology, full information needed to reconstruct the dual space and hence the lattice. Thus to understand a relatively normal lattice L it is in principle sufficient to understand the structure of $L^\#$.
- the set of points of $L^\star$ together with the ordering is the set of non-empty linearly ordered subsets of $L^\#$.
- the subspace topology on $L^\#$ essentially coincides with the topology of the full dual space.
- In general distributive lattices, I do not know if the subspace of hinged points is fully representative.
- Observe that in countable lattices (such as $\mathscr{E}^*$ or Σ_1/T) on any branch, that is, a maximal chain in the dual space, there can be at most countably many hinged points because each point in a chain requires a different hinge.

Jumping gaps

- Identify $y \in (\mathscr{E}^*)^\star$ with $\{\, e \mid W_e \in y \,\}$.

"JUMP THE GAP" LEMMA. *Suppose $y, z \in (\mathscr{E}^*)^\star$ are hinged. Then*

$$y \le z \quad\Longrightarrow\quad y \le_{\mathrm{T}} z$$

$$y \lessdot z \quad\Longrightarrow\quad y' \le_{\mathrm{T}} z$$

($\le_{\mathrm{T}}$ is Turing reducibility and $'$ is the Turing jump).

DEFINITION. A sequence $(x_n)_{n\in\omega}$ of subsets of ω is a *Steel sequence* if $\forall n\,(x'_{n+1} \le_{\mathrm{T}} x_n)$ and $\forall n\,(S(x_n, y) \leftrightarrow y = x_{n+1})$ for some arithmetical $S(\cdot,\cdot)$.

THEOREM (Steel). *No Steel sequences exist.*

COROLLARY. *There are no segments in $(\mathscr{E}^*)^\star$ order-isomorphic to ω^** (inverted ω).

NOTE. "Jump the Gap" Lemma holds verbatim in E_T-trees.

Non-standard elements of r.e. sets

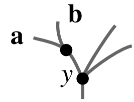
Degrees of index sets of prime filters

Jumping gaps

Jumping gaps
• Identify $y \in (\mathscr{E}^*)^\star$ with $\{\, e \mid W_e \in y \,\}$.
"JUMP THE GAP" LEMMA. Suppose $y, z \in (\mathscr{E}^*)^\star$ are hinged. Then
$y \le z \Longrightarrow y \le_{\mathrm{T}} z$
$y \lessdot z \Longrightarrow y' \le_{\mathrm{T}} z$
($\le_{\mathrm{T}}$ is Turing reducibility and $'$ is the Turing jump).
DEFINITION. A sequence $(x_n)_{n\in\omega}$ of subsets of ω is a Steel sequence if $\forall n\,(x'_{n+1} \le_{\mathrm{T}} x_n)$ and $\forall n\,(S(x_n, y) \leftrightarrow y = x_{n+1})$ for some arithmetical $S(\cdot,\cdot)$.
THEOREM (Steel). No Steel sequences exist.
COROLLARY. There are no segments in $(\mathscr{E}^*)^\star$ order-isomorphic to ω^* (inverted ω).
NOTE. "Jump the Gap" Lemma holds verbatim in E_T-trees.

- Recall that a prime filter of $\mathscr{E}^*$ is a collection of equivalence classes of r.e. sets modulo finite differences. They can therefore also be seen as certain collections of r.e. sets. Accordingly, one can ask questions about the index set of a prime filter, that is the collection of Gödelnumbers of r.e. sets that are elements of the prime filter.
- Remember we agreed to focus on the hinged points in the dual space. For these points we have the following Lemma:
- If y is smaller than z in the sense of the dual space then the index set of y is Turing-reducible to that of z.
- More importantly, if z is an immediate successor of y then the *Turing jump* of y is reducible to z.
- This Lemma places heavy restrictions on the ordering in $(\mathscr{E}^*)^\star$ due to a theorem of Steel which tells us about something called *Steel sequences*: A sequence of subsets of ω is a *Steel sequence* if the Turing jump of each next element is recursive in the previous one, and the sequence is definable in the following sense: each next element is uniformly arithmetical in the previous one.
- Steel's Theorem states that no such sequences exist.
- Here is a corollary: In $(\mathscr{E}^*)^\star$, there are no (convex) intervals of order type inverted-ω, for the index sets of points in that interval would form a Steel sequence.
- The same "Jump the Gap" Lemma holds as read in the E-tree of any r.e. theory — by a simpler proof. In fact, it was the sheer analogy with the E-tree version that motivated the present Lemma.

Germs at hinged points



Definition. Let $y \in (\mathscr{E}^*)^\star$ be non-maximal, and **a** and **b** be branches through $(\mathscr{E}^*)^\star$ with $y \in \mathbf{a}, \mathbf{b}$. Put

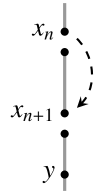
$$\mathbf{a} \simeq_y \mathbf{b} \quad \Longleftrightarrow \quad \text{there is } z > y \text{ with } z \in \mathbf{a}, \mathbf{b}.$$

$\simeq_y$ is an equivalence relation. Equivalence classes are called *germs at y*.

$\mathfrak{g}$ is a *successor* germ if y has an immediate successor on branches in $\mathfrak{g}$.

$\mathfrak{g}$ is a *dense* germ if on (any or all) branches $\mathbf{a} \in \mathfrak{g}$ there is $z > y$ such that the hinged nodes in $[y, z]$ are densely ordered.

Proposition. *Let y be a non-maximal hinged point. Then each germ at y is either a successor germ or a dense germ.*

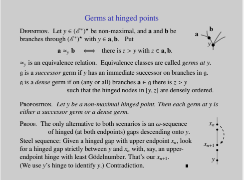


Proof. The only alternative to both scenarios is an ω-sequence of hinged (at both endpoints) gaps descending onto y.
Steel sequence: Given a hinged gap with upper endpoint x_n, look for a hinged gap strictly between y and x_n with, say, an upper-endpoint hinge with least Gödelnumber. That's our x_{n+1}.
(We use y's hinge to identify y.) Contradiction. ∎

Non-standard elements of r.e. sets

Degrees of index sets of prime filters

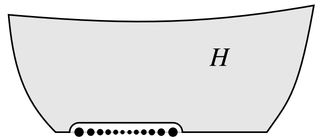
Germs at hinged points



Germs at hinged points
Definition. Let $y \in (\mathscr{E}^*)^\star$ be non-maximal, and **a** and **b** be branches through $(\mathscr{E}^*)^\star$ with $y \in \mathbf{a}, \mathbf{b}$. Put
$\mathbf{a} \simeq_y \mathbf{b} \iff$ there is $z > y$ with $z \in \mathbf{a}, \mathbf{b}$.
$\simeq_y$ is an equivalence relation. Equivalence classes are called *germs at y*.
$\mathfrak{g}$ is a *successor* germ if y has an immediate successor on branches in $\mathfrak{g}$.
$\mathfrak{g}$ is a *dense* germ if on (any or all) branches $\mathbf{a} \in \mathfrak{g}$ there is $z > y$ such that the hinged nodes in $[y, z]$ are densely ordered.
Proposition. *Let y be a non-maximal hinged point. Then each germ at y is either a successor germ or a dense germ.*
Proof. The only alternative to both scenarios is an ω-sequence of hinged (at both endpoints) gaps descending onto y.
Steel sequence: Given a hinged gap with upper endpoint x_n, look for a hinged gap strictly between y and x_n with, say, an upper-endpoint hinge with least Gödelnumber. That's our x_{n+1}.
(We use y's hinge to identify y.) Contradiction. ∎

- Suppose **a** and **b** are two branches (i.e. maximal chains) through a point y.
- We define an equivalence relation on branches through y by putting two branches into the same equivalence class if there is a point strictly above y that lies on both branches.
- So in the picture the branches **a** and **b** are equivalent at y while the other two branches are probably not equivalent. Equivalence classes are called *germs at y*.
- A germ is called a *successor* germ if on any or all branches from that germ the point y has an immediate successor.
- A germ is *dense* if there is some point on any or all branches of the germ such that the hinged points in the interval between y and that point are densely ordered. (Recall that there are at most countably many hinged points on any branch, so this defines what happens after y up to "initial isomorphism".) When you fill in all the non-hinged points, you get something order-isomorphic to a Cantor set.
- Proposition: At hinged nodes, all germs are either successor germs or dense germs.
- Here is why: the only alternative to both situations is an ω-sequence of hinged gaps (i.e. both endpoints hinged) descending onto y.
- Then the following arithmetical algorithm constructs a Steel sequence from that sequence. Given a hinged gap with upper endpoint x_n, look for a hinged gap strictly between y and x_n with, say, a hinge with least Gödelnumber. That's your x_{n+1}.
- Why is it important that the points in question, in particular y, be hinged? That's because we use y's hinge to identify the point y in our arithmetical algorithm, turning a 2nd order quantifier into a 1st order one. Without a hinge, this may be problematic.
- Now a Steel sequence is a contradiction all by itself. We have therefore excluded the third scenario, so any germ at any hinged point is either successor or dense.

Constructing dense germs



Suppose H is hyperhypersimple.

Each point outside $H^\star$ is minimal (hence hinged) and only has successor germs (successors are all hinged on H).

Proposition. *Let y be a non-minimal hinged (hence non-maximal) point. Then there is a dense germ at y.*

Comment. We construct a hinged node $z > y$ with $z \leq_T y$. By “Jump the Gap” Lemma, z determines a dense germ. ∎

- I do not know if there must be dense germs at minimal non-maximal points that are covered by every hyperhypersimple set.

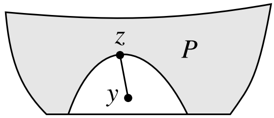
Note. In E_T-trees, there are dense germs at every hinged point.

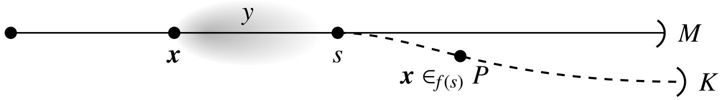
Constructing dense germs
Suppose H is hyperhypersimple.
Each point outside $H^\star$ is minimal (hence hinged) and only has successor germs (successors are all hinged on H).
Proposition. Let y be a non-minimal hinged (hence non-maximal) point. Then there is a dense germ at y.
Comment. We construct a hinged node $z > y$ with $z \leq_T y$. By “Jump the Gap” Lemma, z determines a dense germ. ∎
I do not know if there must be dense germs at minimal non-maximal points that are covered by every hyperhypersimple set.
Note. In E_T-trees, there are dense germs at every hinged point.

- Recall hyperhypersimple sets (the complements are antichains).
- Each point outside $H^\star$ only has successor germs for on every branch through one of these points, the first thing that happens after the initial point is the distinguished element entering the hyperhypersimple set H.
- We are going to see that these minimal points are pretty unique in this respect
- because every non-minimal hinged point (which as we recall also has to be non-maximal) sports a dense germ.
- The strategy of this construction is to produce a larger hinged point that is recursive in y. This already suggests that Turing complexity of hinged points has some influence on the ordering of $(\mathscr{E}^*)^\star$.
- By “Jump the Gap” Lemma, there can be no hinged gaps between y and z — that would lead to a Turing jump in complexity, so we are dealing with a dense germ. The construction appears to require more-than-recursive distance between the distinguished element $\boldsymbol{x}$ in a model of arithmetic and the stage at which $\boldsymbol{x}$ enters the r.e. set that is y's hinge. This distance gives one just enough elbow room, and that's not something we have at minimal points.
- I do not know if there are minimal points that only have successor germs for any reason other than the one we have described, namely lying outside some hyperhypersimple set.
- In the lattice of Σ_1 sentences of any r.e. theory, there are no analogues of hyperhypersimple sets. Also, there are dense germs at each and every hinged point.

Constructing successor germs

Let P be promptly simple.
Let $y \not\ni P$. Let $(M, \boldsymbol{x}) \models y$ (hence $(M, \boldsymbol{x}) \models \boldsymbol{x} \notin P$).
Let $s \in M$ be such that $(M, \boldsymbol{x}) \models \boldsymbol{x} \in_s Y$ for all $Y \in y$.





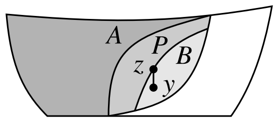
Let $z = \{\text{r.e. } Z \mid \exists\text{tot. rec. } g\, ((K, \boldsymbol{x}) \models \boldsymbol{x} \in_{g(s)} Z)\} \in (\mathscr{E}^*)^\star$. Then $y \lessdot z$.

Definition (Maass). Let $A \subseteq P \subseteq B$. P is *promptly simple* in $[A, B]$
$\iff$ there is a total recursive p s.t. for any r.e. X with $X \cap (B - A)$ infinite
$\exists^\infty x\, \exists s\, (x \in_{\text{at } s} X \ \&\ x \in_{p(s)} P \ \&\ x \notin A)$.

Fact. *Let $A \subset_{\text{m}} B$. Then there is $P \neq^* B$ promptly simple in $[A, B]$.*

Proposition. *Let $y \in B^\star - P^\star$ for P promptly simple in $[A, B]$.*
Then $y \lessdot z$ with some $z \in P^\star - A^\star$ and $z \leq_{\text{T}} y'$.

Note. Successor germs exist in E_T-trees at points not containing appropriate variants of $\neg\,\text{Con}\, T$, in particular at each hinged point.



- We also have a construction for successor germs. The main tool of our construction are the promptly simple sets that we have seen before.
- So let P be promptly simple and y be a point outside $P^\star$.
- Let M be a model of y. (Recall this means the distinguished element $\boldsymbol{x}$ lies precisely in those r.e. sets that are in y. In particular, in M, $\boldsymbol{x} \notin P$.)
- Select the number $s \in M$ so that by stage s, the distinguished element $\boldsymbol{x}$ has already entered all r.e. sets that it is ever going to enter within M.
- By our model-theoretic characterization of promptly simple sets, there is a model K coinciding with M up to and including s where $\boldsymbol{x}$ becomes an element of P at a total recursive distance from s.
- In the model K, we take all r.e. sets that $\boldsymbol{x}$ enters at stages that are total recursively far away from s. This is a prime filter z hinging on P.
- It is also clear that z is an imediate successor of y.
- We now relativize this construction to a more general situation. Maass defined prompt simplicity relative to an interval $[A, B]$ in $\mathscr{E}^*$. (We need not really go into this definition.)
- Important for us is the fact that relative to any major interval there is a promptly simple set. It probably looks somewhat like this.
- Taking any point in the difference $B - P$,
- we can construct an immediate successor to that point lying in $A - P$. Better still, the prime filter z is recursive in the jump of y. In view of "Jump the Gap" Lemma, this is the best one can generally hope to do as regards complexity.
- In this setup the set A appears to be a bit of a red herring. It is basically only used to guarantee that most of the difference $B - P$ lies well away from the minimal points. We could probably (hope to) achieve a similar effect with a definition of something like "promptly major subset", but we don't currently have one.
- In the E-tree, the role of (generalizations of) promptly simple sets is played by (appropriate versions of) inconsistency statements. In fact they are sufficiently ubiquitous to ensure that every hinged point in the E-tree enjoys a successor.
- So this has been another contribution to the question, how is a promptly simple set like an inconsistency statement?
- That's it. This completes our story for today. Thank you for your attention.