A Hierarchy of Ramified Theories around PRA

Elliott J. Spoors and Stanley S. Wainer[1] (Leeds UK)

Logic Colloquium 2012, Manchester.

[1]Later parts of this work were done while the second author visited the Isaac Newton Institute, supported by the programme "Syntax and Semantics; the legacy of A. Turing".

§1. Input–Output Theories.

- $EA(I;O)$ is a 2-sorted theory with elementary strength.
- $EA(I;O) \subset EA(I;O)^+ \vdash \mathcal{E}^3(x)\downarrow$.
- $EA(I_1;O)^+(I_2)^+ \vdash \mathcal{E}^4(x^2)\downarrow$.
- $EA(I_1, I_2, \ldots, I_k; O))^+ \vdash \mathcal{E}^{k+2}(x^k)\downarrow$.
- $EA(I_1, I_2, \ldots, I_\omega; O))^+ \vdash \mathcal{E}^\omega(x^\omega)\downarrow$.

The Main Principles:
(1) Inputs govern induction-length.
(2) If a value is computable from inputs only, then it may be used as an input.

§2. $EA(I;O)$ – Leivant (1995), Ostrin-Wainer (2005)

- Quantified "output" variables $a, b, c, \dots$.
- Unquantified "input" variables $x, y, z, \dots$ (constants).
- Terms $0, Succ, +, \times, \pi, \pi_0, \pi_1, \dots$ with usual axioms.
- "Predicative Induction" up to x:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow A(x)$$

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow \forall a \leq x A(a).$$

Theorem

Define $f(x){\downarrow} \equiv \exists a C_f(x,a)$ for some Σ_1 formula C_f. Then

$$EA(I;O) \vdash f(x){\downarrow} \text{ if and only if } f \text{ is an elementary function.}$$

Gentzen iterated exponentials

With formula $A(a)$ associate

$$A'(b) \;\equiv\; \forall a(A(a) \rightarrow a + 2^b{\downarrow} \wedge A(a + 2^b))$$

Then in $EA(I;O)$

$$\vdash\; \text{Prog}A(a) \rightarrow \text{Prog}A'(b).$$

Therefore

$$\vdash\; \text{Prog}A(a) \rightarrow A'(x)$$

and hence

$$\vdash\; \text{Prog}A(a) \rightarrow A(2^x)$$

$$\vdash\; \text{Prog}A(a) \rightarrow A(2^{2^x})$$

etcetera.
Hence all elementary functions are provably defined.

§3. $EA(I;O)^+$.

$EA(I;O)$ is not "user-friendly" since composition of functions $f : I \to O$ cannot be proved straightforwardly – however Wirz (2005) developed a variety of derived rules showing this.
To remedy this, add a Σ_1-"Reflection Rule" as in Cantini (2002):

$$\frac{\Sigma(\vec{x}), \exists a A(a, \vec{x})}{\Sigma(\vec{x}), \exists y A(y, \vec{x})}$$

where the only free parameters are inputs $\vec{x}$. And add I-quantifiers:

$$\frac{\Gamma, A(x)}{\Gamma, \forall y A(y)} \qquad \frac{\Gamma, A(t(\vec{x}))}{\Gamma, \exists y A(y)}.$$

Note: the inductions are still restricted to $EA(I;O)$ formulas only.

Then if $\vdash f(x)\downarrow$ and $\vdash g(x)\downarrow$ we can directly prove $\forall y f(y)\downarrow$ and (by reflection) $\exists y(g(x) = y)$. Therefore $EA(I;O)^+ \vdash f(g(x))\downarrow$.

Upper Bounds via $EA(I;O)^+_\infty$.

The infinitary system $n : I; m : O \vdash^\alpha \Gamma$ has rules, where $\beta \prec_n \alpha$:

$$(\exists O)\frac{n; m \vdash^\beta_C k \quad n; m \vdash^\beta \Gamma, A(k)}{n; m \vdash^\alpha \Gamma, \exists a A(a)} \qquad (\exists I)\frac{n; - \vdash^\beta_C k \quad n; m \vdash^\beta \Gamma, A(k)}{n; m \vdash^\alpha \Gamma, \exists x A(x)}$$

$$(\forall O)\frac{\{n; \max(m, i) \vdash^\beta \Gamma, A(i)\}_i}{n; m \vdash^\alpha \Gamma, \forall a A(a)} \qquad (\forall I)\frac{\{\max(n, i); m \vdash^\beta \Gamma, A(i)\}_i}{n; m \vdash^\alpha \Gamma, \forall x A(x)}$$

and $(\vee), (\wedge)$ and (Cut) as usual, together with Computation Rules:

$$(Ax)\ n; m \vdash^\alpha_C k \ \text{ if } \ k \leq q(m) \qquad (C)\frac{n; m \vdash^\beta_C m' \quad n; m' \vdash^\beta_C k}{n; m \vdash^\alpha_C k}$$

where q is some quadratic majorising the term constructors.

Bounding Functions.

The ordinal assignmnet is “slow growing”, i.e. $|\{\beta : \beta \prec_n \alpha\}| = G_\alpha(n)$.

Lemma

$n; m \vdash^\alpha_C k$ if and only if $k \le q^r(m)$ where $r = G_{2^\alpha}(n)$.
This is elementary if $\alpha \prec \varepsilon_0$.

Theorem

By embedding and cut-reduction, if $EA(I;O)^+ \vdash f(x)\!\downarrow$ then there is an $\alpha \prec \varepsilon_0$ such that for every n,

$$n; - \vdash^\alpha \exists a C_f(n,a)$$

with, at worst, Σ_1 cuts.
Therefore f is definable by a bounded formula with elementary bounds, so $f \in \mathcal{E}^3$.

§4. $EA(I_1; O)^+(I_2)^+$.

Add to $EA(I_1; O)^+$ new I_2–inputs $u, v, \ldots$ and a new level of inductions:

$$A(0) \wedge \forall a(A(a) \to A(a+1)) \to A(u)$$

where A is now any $EA(I_1; O)^+$ formula. Then:

- $EA(I_1; O) \vdash 2^x\downarrow$
- $EA(I_1; O)^+ \vdash \forall x \exists y(2^x = y)$
- $EA(I_1; O)^+ \vdash \exists y(2^x_a = y) \to \exists y(2^x_{a+1} = y)$
- $EA(I_1; O)^+(I_2) \vdash \forall x \exists y(2^x_u = y)$

Then add I_2–quantifier rules and a Σ_1–reflection rule for I_2. This allows compositions of the superexponential etc., so

$$EA(I_1; O)^+(I_2)^+ \vdash \mathcal{E}^4(u)\downarrow \ .$$

Layered infinitary system $EA(I_1;O)^+(I_2)^+_\infty$.

Tait-style sequents are now $n_2 : I_2; n_1 : I_1; m : O \vdash^{\alpha,\gamma} \Gamma$.
Ordinal assignment is governed by $\beta \prec_{n_2} \alpha$ with γ a parameter.

New $(\exists I_2)$ and $(\forall I_2)$ rules are added on top of $EA(I_1;O)^+_\infty$.
The layering axiom is $n_2; n_1; m \vdash^{\alpha,\gamma} \Gamma$ if $n_1; m \vdash^{\gamma} \Gamma$.

The new computation rule is

$$\frac{n_2; n_1; - \vdash^{\beta,\gamma}_C n' \qquad n_2; n'; m \vdash^{\beta,\gamma}_C k}{n_2; n_1; m \vdash^{\alpha,\gamma}_C k} \,.$$

Lemma ($\mathcal{E}^4$ Bounding)

Let $B_\gamma(n_1) = q^{G_{2^\gamma}(n_1)}(0)$ be the bounding function at level 1. Then

$$n_2; n_1; - \vdash^{\alpha,\gamma}_C k \;\; \textit{iff} \;\; k \leq B_\gamma^{G_{2^\alpha}(n_2)}(n_1)\,.$$

§5. Theorem.

By embedding into $EA(I_1; O)^+(I_2)^+_\infty$, reducing cut-rank, and using the above bounding lemma, every function provably defined in $EA(I_1; O)^+(I_2)^+$ is $\mathcal{E}^4$-definable.
This extends similarly to higher levels:

$$EA(I_1; O)^+ \vdash \mathcal{E}^3(I_1) \downarrow$$

$$EA(I_1; O)^+(I_2)^+ \vdash \mathcal{E}^4(I_2) \downarrow$$

$$EA(I_1; O)^+(I_2)^+(I_3)^+ \vdash \mathcal{E}^5(I_3) \downarrow$$

$$EA(I_1, I_2, I_3, \ldots, I_k; O)^+ \vdash \mathcal{E}^{k+2}(I_k) \downarrow.$$

$$EA(I_1, I_2, I_3, \ldots, I_\omega; O)^+ \vdash \mathcal{E}^\omega(I_\omega) \downarrow.$$

§6. Level ω – Ackermann.

A version of Ackermann: $F_0(n) = n + 1$ and $F_{r+1}(n) = F_r^n(n)$.
Suppressing ordinal bounds, $EA(I_1, I_2, \ldots, I_\omega; O)_\infty^+$ proves:

$$n : I_r \vdash \forall x^r \exists y^r (F_r(x) = y) \to \exists y^r (F_r^a(n) = y) \to \exists y^r (F_r^{a+1}(n) = y)$$

Hence by induction on a, using repeated cuts:

$$k : I_{r+1}; n : I_r \vdash \forall x^r \exists y^r (F_r(x) = y) \to \exists y^r (F_r^k(n) = y)$$

Hence with $n := k$ and a Cut on $\forall x^r \exists y^r (F_r(x) = y)$:

$$k : I_{r+1} \vdash \exists y^r (F_{r+1}(k) = y)$$

Then by Reflection, $k : I_{r+1} \vdash \exists y^{r+1} (F_{r+1}(k) = y)$ and so:

$$\vdash \forall x^{r+1} \exists y^{r+1} (F_{r+1}(x) = y)$$

Therefore by induction on r: $\vdash \forall r^\omega \forall x^r \exists y^r (F_r(x) = y)$.

Bounding in $EA(I_1, I_2, \ldots, I_\omega; O)^+_\infty$.

- Cut rank ρ may now be infinite, so apply predicative C-E:

$$\vdash^\alpha_\rho \Gamma \;\Rightarrow\; \vdash^{\varphi_\rho(\alpha)}_0 \Gamma$$

where $\varphi_0(\alpha) = \alpha + 1$, $\varphi_{\rho+1}(\alpha) = \varphi^\alpha_\rho(\alpha)$,$\varphi_\omega(\alpha) = \sup \varphi_r(\alpha)$.

- The slow-growing G-collapse of φ_ω is the Ackermann F.
- The computation rules are, with $\beta \prec_r \alpha$:

$$r : I_\omega; n : I_p \vdash^\alpha_C k : I_p \quad \text{if} \quad r : I_\omega; n : I_p; 0 : I_{p-1} \vdash^\alpha_C k : I_{p-1}$$

and

$$\frac{r : I_\omega; n : I_p \vdash^\beta_C n' : I_p \qquad r : I_\omega; n' : I_p \vdash^\beta_C k : I_p}{r : I_\omega; n : I_p \vdash^\alpha_C k : I_p}$$

- Then $r : I_\omega; n : I_p \vdash^\alpha_C k : I_p \;\Rightarrow\; k \leq B_p(\alpha, r, n)$ where

$B_p(\alpha, r, n) =$ the $G_{2^\alpha}(r)$ iterate of $B_{p-1}(\alpha, -, 0)$ on $\max(r, n)$.

References

1) **S. Bellantoni & S. Cook**: "A new recursion theoretic characterization of the polytime functions". Computational Complexity Vol. 2 (1992) 97-110.
2) **A. Cantini**: "Polytime, combinatory logic and positive safe induction". Archive for Math. Logic Vol. 41 (2002) 169-189.
3) **D. Leivant**: "Intrinsic theories and computational complexity". In D. Leivant (Ed) LCC'94, LNCS Vol. 960 (1995) 177-194.
4) **G. Ostrin & S. Wainer**: "Elementary arithmetic". Annals of Pure and Applied Logic Vol. 133 (2005) 275-292.
5) **H. Schwichtenberg & S. Wainer**: "Proofs and Computations". ASL Perspectives in Logic, CUP (2012) 465 + xiii.
6) **H. Simmons**: " The realm of primitive recursion". Archive for Math. Logic Vol. 27 (1988) 177-188.
7) **E. Spoors & S. Wainer**: "A hierarchy of ramified theories below PRA". To appear in a volume in honour of Helmut Schwichtenberg, Ontos Math. Logic (2012).