Pseudo-analytic structures: model theory and algebraic geometry

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Strongly minimal structures. Examples

(1) **Trivial**
G-set $\mathbf{M} = \{M, g\cdot\}_{g \in G}$, for G a group acting on M "in a nice way". E.g. the upper half plane $\mathcal{H}$ with the action of $\mathrm{GL}^+(2, \mathbb{Q})$.

(2) **Linear**
Abelian divisible torsion-free groups; Abelian groups of prime exponent; Vector spaces over a given division ring K.

(3) **Algebraically closed fields** in the language $(+, \cdot, =)$

Dimension notions

for finite $X \subset \mathbf{M}$:
(1) Trivial structures: **the number of "generic" G-orbits in** $G \cdot X$
(2) Linear structures:
the linear dimension $\mathrm{lin.d}_{\mathrm{Q}}(X)$ of $\langle X \rangle$
(3) Algebraically closed fields:
the transcendence degree $\mathrm{tr.d}(X)$ over the prime subfield.
Dual notion: the **dimension of an algebraic variety** V over F

$$\dim V = \max\{ \mathrm{tr.d}_F(x_1, \ldots, x_n) \,|(x_1, \ldots, x_n) \in V\}.$$

Three basic geometries of stability theory

(1) **Trivial geometry**

(2) **Linear geometry**

(3) **Algebraic geometry.**

Trichotomy Conjecture
Any uncountably categorical structure is reducible to (1) - (3)?

Hrushovski's construction of new stable structures

Given a class of structures **M** with a dimension notions d_1, and d_0 we want to consider a *new function* f on **M** and extend the dimension theory.

On $(\mathbf{M}, \mathrm{f})$ introduce a **predimension**

$$\delta(x_1, \ldots, x_n) =$$
$$\mathrm{d}_1(x_1, \ldots, x_n, \mathrm{f}(x_1), \ldots, \mathrm{f}(x_n)) - \mathrm{d}_0(x_1, \ldots, x_n).$$

We must assume

$$\delta(X) \geq 0, \text{ for all finite } X \subset M$$

(Hrushovski inequality).

Use the Fraisse amalgamation procedure in the class $(\mathbf{M}, \mathrm{f})$ respecting the predimension δ.
Under certain **tameness** assumptions on **M**, d_1 and d_0 this gives rise to a complete theory of **generic** structure, which is stable and even strongly minimal with a geometry distinct from (1)-(3).

Variations (two-sorted fusion)

$$\begin{array}{c} (M_1; L_1) \\ \downarrow f \\ (M_2; L_2) \end{array}$$

$$\delta(X) = \mathrm{d}_1(X) + \mathrm{d}_2(f(X)) - \mathrm{d}_0(X)$$

$\mathrm{d}_1 =$ dimension in $\mathbf{M}_1$, $\mathrm{d}_2 =$ dimension in $\mathbf{M}_2$, $\mathrm{d}_0 =$ dimension for the f-invariant part of both structures.

Example. (Hrushovski, 1992)

$$\begin{array}{c} (F_1; +, \cdot) \\ \downarrow f \\ (F_2; +, \cdot) \end{array}$$

$$\mathrm{d}_1(X) = \mathrm{tr.d}\, F_1(X),\ \mathrm{d}_2(Y) = \mathrm{tr.d}\, F_2(Y),\ \ \mathrm{d}_0(X) = |X|,\ f \text{ bijection}$$

Can be seen as a **fusion** of two pregeometries with dimensions d_1 and d_2, preserving a common part corresponding to predimension d_0.

Are Hrushovski structures mathematical pathologies?

Observation: If **M** is a field of characteristic 0 and we want $\mathrm{f} = \mathrm{ex}$ to be a group homomorphism:

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2),$$

then the corresponding predimension must be

$$\delta_{\mathrm{exp}}(X) = \mathrm{tr.d}(X \cup \mathrm{ex}(X)) - \mathrm{lin.d}_{\mathbb{Q}}(X) \geq 0.$$

The Hrushovski inequality, in the case of the complex numbers and $\mathrm{ex} = \exp$, is equivalent to

$$\mathrm{tr.d}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n,$$

assuming that $x_1, \ldots, x_n$ are linearly independent.
This is **the Schanuel conjecture.**

Can we carry out Hrushovski construction for δ_{exp}?

Issues:

(i) not enough tameness in δ_{exp}

(ii) the natural prototype $\mathbb{C}_{\text{exp}}$ has the ring $\mathbb{Z}$ as a definable substructrure.

Solution. Treat this case in a non-elementary setting.

Theorem (2003) The amended Hrushovski construction for fields with pseudo-exponentiation produces an $L_{\omega_1,\omega}(Q)$-theory T_{exp} of a field with pseudo-exponentiation, categorical in all uncountable powers.

Q is a quantifier "there exists uncountably many".

Axioms of $T_{\exp}$

The language $(+, \cdot, \mathrm{ex}, 0, 1)$

ACF_0 algebraically closed fields of characteristic 0;

EXP1: $\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$;

EXP2: $\ker \mathrm{ex} = \pi\mathbb{Z}$;

SCH: for any finite X

$$\delta(X) = \mathrm{tr.d}(X \cup \mathrm{ex}(X)) - \mathrm{lin.d}_{\mathrm{Q}}(X) \geq 0$$

this is $L_{\omega_1,\omega}$.

Axioms of T_{exp}, continued

As a result of Fraisse amalgamation models of T_{exp} are **existentially closed** with respect to embedding respecting δ_{exp}.

EC: For any *rotund* system of polynomial equations

$$P(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$$

there exists a (generic) solution satisfying

$$y_i = \mathrm{ex}(x_i) \quad i = 1, \ldots, n.$$

(this is basically first order, but "generic" requires $L_{\omega_1,\omega}$.)

And
Countable closure property

CC: For maximal rotund systems of equations the set of solutions is at most countable. $L_{\omega_1,\omega}(Q)$

Reformulation

Theorem *Given an uncountable cardinal λ, there is a unique model of axioms* $\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$ *of cardinality λ.*

This is a consequence of Theorems A and B:

Theorem A *The $L_{\omega_1,\omega}(Q)$-sentence*

$$\mathrm{ACF}_0 + \mathrm{EXP} + \mathrm{SCH} + \mathrm{EC} + \mathrm{CC}$$

is axiomatising a **quasi-minimal excellent abstract elementary class** *(AEC)*.

Theorem B (Essentially S.Shelah 1983, see also J.Baldwin 2010) *A quasi-minimal excellent AEC has a unique model in any uncountable cardinality*.

Remark. "Excellence" is essential. The earlier Kiesler's theory of homogeneous $L_{\omega_1,\omega}$-categoricity is not applicable here.

Theorem A

The proof reduces to the following arithmetic-algebraic facts:

(i) the action of $\mathrm{Gal}(\tilde{\mathbb{Q}} : \mathbb{Q})$ on Tors is maximal possible (Dedekind);

(ii) given k, a finitely generated extension of (1) $\mathbb{Q}(\mathrm{Tors})$ or of (2) $\tilde{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, and given a finitely generated subgroup A of the multiplicative group $G_m(k) := k^\times$, the group

$$\mathrm{Hull}_k(A)/T(k) \cap \mathrm{Hull}_k(A), \text{ where } T(k) = \begin{cases} Tors, & \text{if } (1) \\ \tilde{\mathbb{Q}}, & \text{if } (2) \end{cases}$$

is free. (Follows from Kummer theory)

(iii) similar to (ii) but for $k =$ composite of finite *independent system* of algebraically closed fields (Bays and Z.)

In fact, (i)-(iii) is **equivalent to categoricity** of $T_{\exp}$ modulo model theory.

Theorem A for other transcendental functions

We need to know a "complete system of functional equations" and the "Schanuel conjecture" for the function(s).

The Weierstrass function $\mathfrak{P}(\tau, z)$ (as a function of z) and the structure on the elliptic curve $E_{j(\tau)}$:

$$\langle \mathfrak{P}(\tau, z), \mathfrak{P}(\tau, z)' \rangle : \mathbb{C} \to E_{j(\tau)} \setminus \{\infty\} \subset \mathbb{C}^2$$

Since

$$(\mathfrak{P}')^2 = 4\mathfrak{P}^3 - g_2 \cdot \mathfrak{P} - g_3(\tau), \quad g_2 = g_2(\tau),\ g_3 = g_3(\tau),$$

the problem reduces to the structure

$$(\mathbb{C}, +, \cdot, \mathfrak{P}(z))$$

The Schanuel-type conjecture was deduced from the André conjecture on 1-motives by C.Bertolin.

Theorem A for $\mathfrak{P}(\tau, z)$

M.Gavrilovich, M.Bays, J.Kirby, B.Hardt (published and work in progress):

(i) the action of $\mathrm{Gal}(\tilde{\mathbb{Q}}(j(\tau)) : \mathbb{Q}(j(\tau))$ on $E_{j(\tau)}(\mathrm{Tors})$ is maximal possible (essentially, the *hard theorem* of J.-P. Serre);

(ii) given k, a finitely generated extension of (1) $\mathbb{Q}(E_{j(\tau)}(\mathrm{Tors}))$ or of (2) $\tilde{\mathbb{Q}}(j(\tau))$, the algebraic closure of $\mathbb{Q}(j(\tau))$, and given a finitely generated subgroup A of the group $E_{j(\tau)}(k)$, the group

$$\mathrm{Hull}_k(A)/T(k) \cap \mathrm{Hull}_k(A), \text{ where } T(k) = \begin{cases} E_{j(\tau)}(\mathrm{Tors}) & \text{if (1)} \\ \tilde{\mathbb{Q}}(j(\tau)), & \text{if (2)} \end{cases}$$

is free. (Mordell-Weil, Ribet)

(iii) follows from (ii) in general for commutative algebraic groups (Bays–Hardt, using Shelah's techniques)

Theorem A for other transcendental functions

Weierstrass function $\mathfrak{P}(\tau, z)$ as function of τ and z still poorly understood, even at the level of functional equations and Schanuel-type conjecture.

Work on function $j(\tau)$ (modular invariant) in progress, A.Harris:

(i) adelic Mumford-Tate conjecture for Abelian varieties = product of elliptic curves. Theorem of Serre.

(ii) Shimura reciprocity and other elements of the theory of j-invariant.

(iii) Bays-Hardt as above.

Further transcendental functions are of interest. First of all the uniformising functions for (mixed) Shimura varieties (includes semi-abelian varieties).

Is T_{exp} the actual theory of exp?

Conjecture. $\mathbb{C}_{\text{exp}}$ is the unique model of T_{exp} of cardinality continuum.

This is equivalent to

Conjecture. $\mathbb{C}_{\text{exp}}$ satisfies SCH and EC.

Work on comparative analysis of properties of $\mathbb{C}_{\text{exp}}$ and T_{exp}. A.Macintyre, A.Wilkie, D.Marker, P. D'Aquino, G.Terzo, A.Shkop, V.Mantova, B.Z. and others.

Conclusion so far. Hrushovski's construction is behind classical analytic-algebraic geometry.

First order framework

Recall the issues with the first order treatment:

(i) not enough tameness in δ_{exp}

(ii) the natural prototype $\mathbb{C}_{\text{exp}}$ has the ring $\mathbb{Z}$ as a definable substructrure.

Solution for (ii): Work out first order axioms for the pseudo-exponentiation **modulo the complete arithmetic.**

The analysis of (i) lead to the possible remedy

Conjecture on Intersection with Tori (CIT), 2001.
(Formulation in model-theoretic form, using a 2-sorted predimension)

Let ${}^*\mathbb{C}$ and ${}^*\mathbb{Q}$ be nonstandard models of complex and rational number fields. Then for any finite $X \subset {}^*\mathbb{C}$,

$$\delta(X) := \mathrm{tr.d}(\exp(X)/\mathbb{C}) + \mathrm{lin.d.}_{{}^*\mathbb{Q}}(X/\mathbb{C}) - \mathrm{lin.d}_{\mathrm{Q}}(X/\ker) \geq 0.$$

First order framework

Recall T_{exp} : ACF_0 + EXP + SCH + EC + CC

Theorem (Kirby, Z., 2011) The axiom SCH (Schanuel condition) is first order axiomatisable (over the kernel) iff CIT is true.
In this case the complete system of first order axioms of pseudo-exponentiation can be written down explicitly modulo the complete arithmetic.

In effect, one can say that the models of the first order theory "split" into two mutually "orthogonal" components: the kernel (arithmetic) and an ω-stable part.

The theory is ω-stable over the arithmetic.

CIT and Pink's conjecture

CIT can be reformulated in an equivalent algebro-geometric form. Also, in the form applicable to semi-abelian varieties and indeed to any context where Schanuel-type conjecture makes sense.

Proposition. CIT implies Mordell-Lang (and Manin-Mumford) conjectures.

Later an equivalent of CIT was formulated by Bombieri, Masser and Zanier.

General form of CIT for mixed Shimura varieties was formulated by R.Pink in 2005. This includes André-Oort conjecture about special points on Shimura varieties. This is now referred to as Z.-Pink conjecture.

o-minimal attack on the Z.-Pink conjecture

J.Pila's idea of extending the Bombieri-Pila method of counting rational points on transcendental ovals to o-minimal context.

Pila-Wilkie's theorem (2005) establishes an upper bound for the number of rational points on the transcendental part of sets definable in o-minimal expansions of the reals.

Pila and Zannier (2009) showed how to solve Manin-Mumford-type problem using Pila-Wilkie and weak CIT (Ax's theorem, "Ax-Schanuel").

This method developed into the solution of a number of cases of André-Oort and ZP.