Uncertainty Quantification: Does it need efficient linear algebra?

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Yes.

Saddle Point Problems

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (S)$$

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A finite dimensional discretization of the following infinite-dimensional problem: find $(u, p) \in V \times W$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \qquad \forall v \in V, \\ b(u,q) &= g(q) \qquad \forall q \in W. \end{aligned} \qquad (V)$$

Where, V and W represent Hilbert spaces; $a : V \times V \to \mathbb{R}$ and $b : V \times W \to \mathbb{R}$ are bounded bilinear forms and $f : V \to \mathbb{R}$ and $g : W \to \mathbb{R}$ are linear functionals.

Saddle Point Problems II

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad (S)$$

That is, for given approximation spaces $V_h \subset V$ and $W_h \subset W$, we want to compute $(u_h, p_h) \in V_h \times W_h$ such that

$$\begin{aligned} a(u_h, v) + b(v, p_h) &= f(v) \qquad \forall v \in V_h, \\ b(u_h, q) &= g(q) \qquad \forall q \in W_h. \end{aligned}$$

There is a natural energy norm for measuring the quality of approximation for functions in the space $V \times W$,

$$\|(u,p)\|_{V\times W} = \|u\|_V + \|p\|_W \,.$$

Our goal is to construct an optimal iterative solver for (S)...

There is a natural energy norm for measuring the quality of approximation for functions in the space $V \times W$,

$$\|(u,p)\|_{V\times W} = \|u\|_V + \|p\|_W \, .$$

Our goal is to construct an optimal iterative solver for (S)...

that is, we would like to construct a sequence of rapidly converging iterates $(u_h^{(1)}, p_h^{(1)}), (u_h^{(2)}, p_h^{(2)}), (u_h^{(3)}, p_h^{(3)}), \ldots$ with the property that the iteration is terminated once the energy norm of the algebraic error $(u_h - u_h^{(m)}, p_h - p_h^{(m)})$ is commensurate with the discretization error:

$$\|u_h - u_h^{(m)}\|_V + \|p_h - p_h^{(m)}\|_W \sim \|u - u_h^{(m)}\|_V + \|p - p_h^{(m)}\|_W .$$

The deterministic case is sorted in case where the bilinear form is symmetric:

- David Silvester & Valeria Simoncini.
  EST_MINRES: An optimal iterative solver for symmetric indefinite systems stemming from mixed approximation
  ACM Trans. Math. Softw., vol. 37 no. 4, 2010.

Working title ....

Does the optimal solver concept extend to stochastic (possibly non-symmetric) saddle point problems?

Fluid flow with random data

Deterministic incompressible flow models:

- Potential flow
- Stokes flow
- Navier–Stokes flow

Solution schemes require the following data:

- the spatial domain (geometry)
- boundary conditions
- source terms
- coefficients (e.g. permeability, viscosity, ...)

any, or all of which, may be uncertain.

Navier–Stokes problem

Find $\boldsymbol{u}(\boldsymbol{x},\omega)$ and $p(\boldsymbol{x},\omega)$ such that $\mathbb{P}$-a.s.,

$$
\begin{aligned}
-\nu(\omega)\nabla^2\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } D \subset \mathbb{R}^d \ (d=2,3) \\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } D, \\
\boldsymbol{u} &= \boldsymbol{g} \quad \text{on } \partial D_D, \\
\nu(\omega)\frac{\partial \boldsymbol{u}}{\partial n} - \boldsymbol{n}p &= \boldsymbol{0} \quad \text{on } \partial D_N.
\end{aligned}
$$

If the viscosity is uncertain, we might model it via

$$\boxed{\nu(\omega) = \mu + \sigma\xi(\omega).}$$

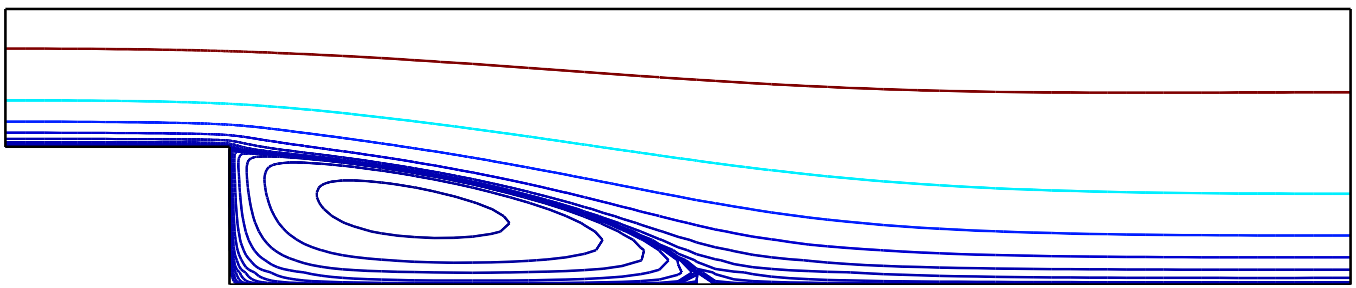
If $\xi \sim U(-\sqrt{3},\sqrt{3})$, then ν is a uniform random variable with

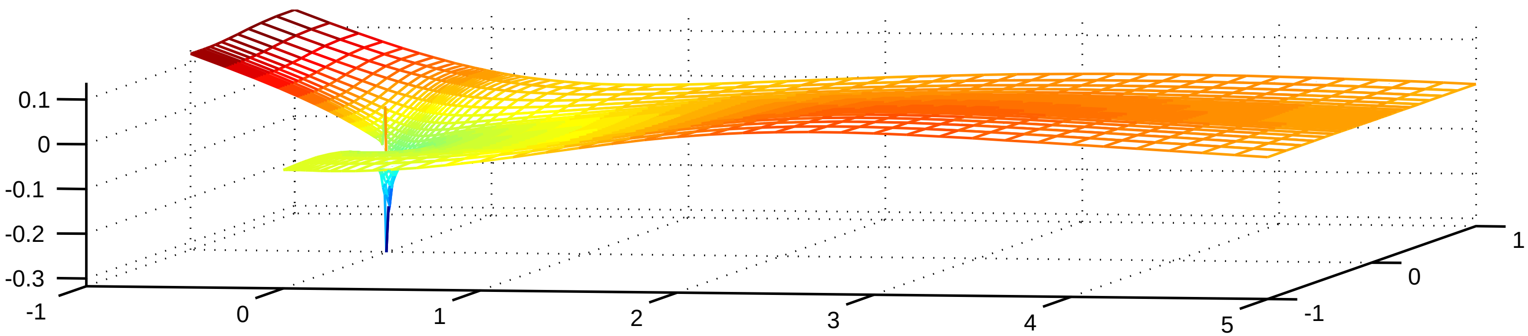
$$\mathbb{E}[\nu] = \mu, \qquad \text{Var}[\nu] = \sigma^2.$$

N–S example I: flow over a step

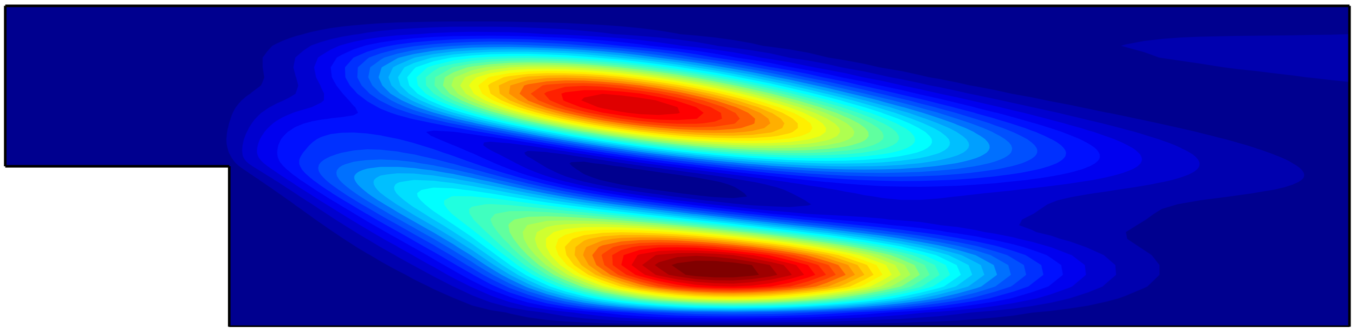
Streamlines of the mean flow field (top) and plot of the mean pressure field (bottom):

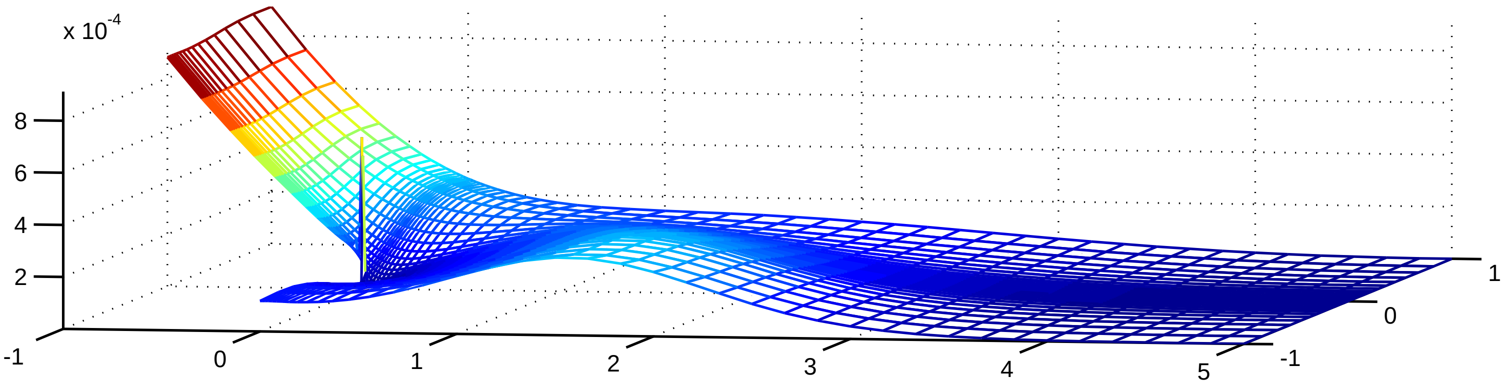
$$\mu = 1/50, \quad \sigma = \mu/10$$





Variance of the magnitude of flow field (top) and variance of the pressure (bottom)

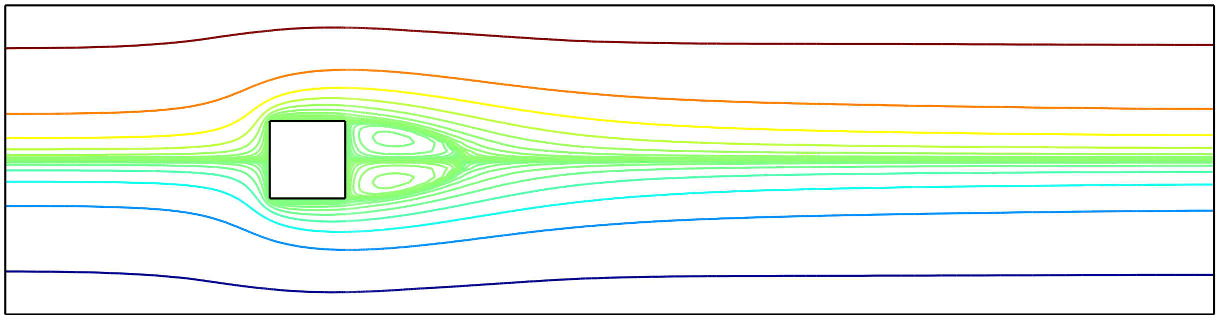


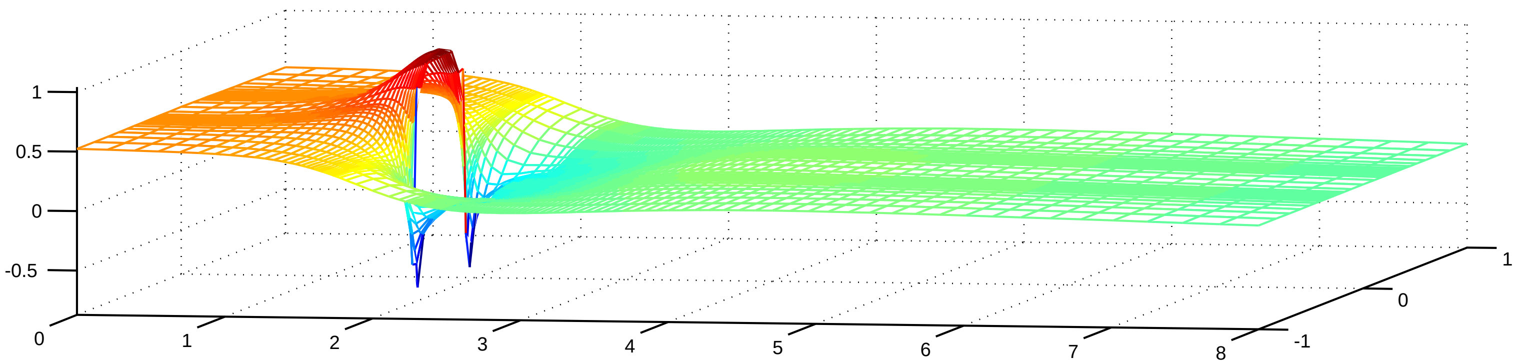


N–S example II: flow around an obstacle

Streamlines of the mean flow field (top) and plot of the mean pressure field (bottom):

$$\mu = 1/100, \quad \sigma = 3\mu/10$$





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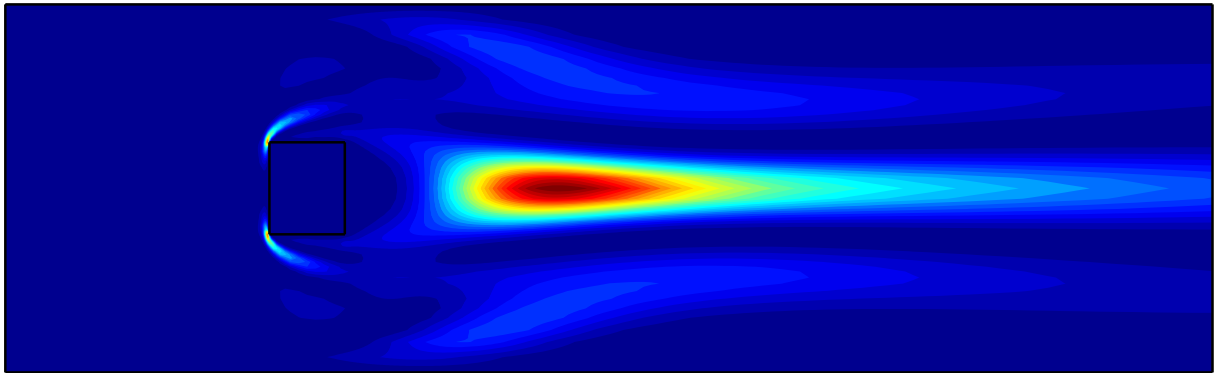

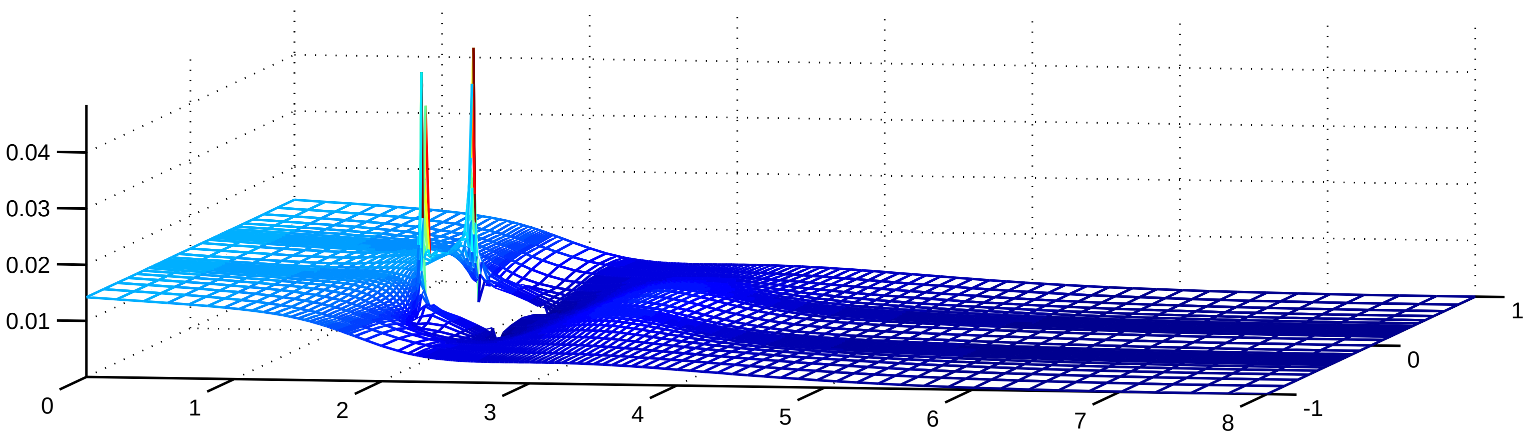

Stochastic discretisation methods

- Monte Carlo Methods
- Perturbation Methods
- Stochastic Galerkin Methods
- Stochastic Collocation Methods
- Stochastic Reduced Basis Methods
- ...

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Key points

- If the number of random variables describing the input data is small then Stochastic Galerkin and Stochastic Collocation methods can outperform Monte Carlo.
- If software for the deterministic problem is to be useful for Stochastic Galerkin approximation then specialised solvers need to be developed.

Potential flow problem

$$
\begin{aligned}
-A(\cdot,\omega)\,\nabla p &= \boldsymbol{u} && \text{in } D,\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } D\subset\mathbb{R}^d\ (d=2,3)\\
p &= g && \text{on } \partial D_{\mathrm{D}},\\
\boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial D_{\mathrm{N}}.
\end{aligned}
$$

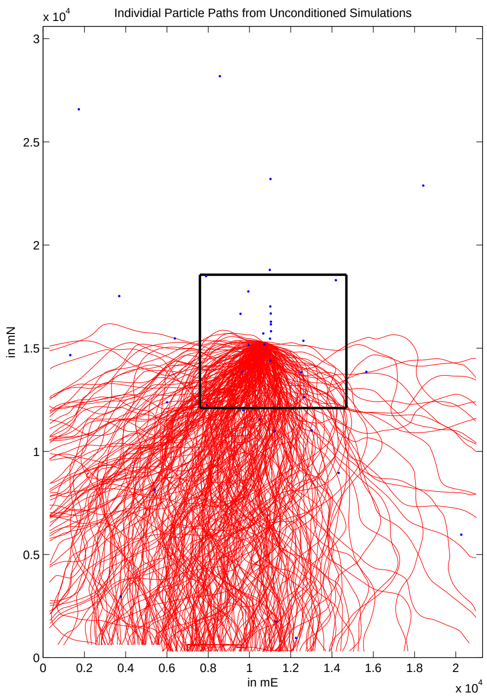
Solution variables:

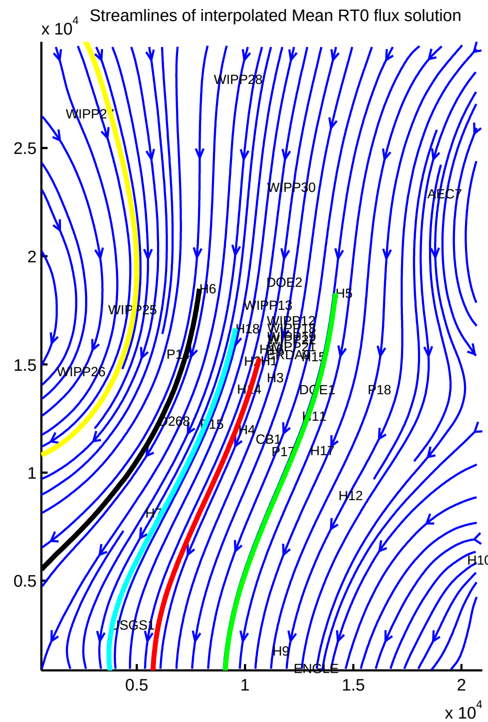
$$
\begin{aligned}
p &= p(\boldsymbol{x},\omega) && \text{hydraulic head (pressure)}\\
\boldsymbol{u} &= \boldsymbol{u}(\boldsymbol{x},\omega) && \text{velocity field}
\end{aligned}
$$

Data:

T	$=$	$A^{-1}(\boldsymbol{x},\omega)$	inverse permeabiliy
g	$=$	$g(\boldsymbol{x})$	boundary data
D	$\subset$	$\mathbb{R}^d$	spatial domain

- $T(\boldsymbol{x}, \omega)$ is a random field with a given mean $\mu(\boldsymbol{x})$ and covariance function $C(\boldsymbol{x}, \boldsymbol{x})$
- Individual realisations of particle paths (left), mean velocity field (right):





Rest of the talk ...

- Linear algebra/solver issues
  - Navier–Stokes equations
  - potential flow (linear in stochastic parameters)
- Open questions
  - potential flow (nonlinear in stochastic parameters)
  - ...

Steady-state flow with random data

Problem statement

$$\vec{u} \cdot \nabla \vec{u} - \nu \nabla^2 \vec{u} + \nabla p = 0 \qquad \text{in } \Omega$$
$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \Omega$$
$$\vec{u} = \vec{g} \quad \text{on } \Gamma_D$$
$$\nu \nabla \vec{u} \cdot \vec{n} - p \vec{n} = \vec{0} \quad \text{on } \Gamma_N.$$

We model uncertainty in the viscosity as

$$\nu(\omega) = \mu + \sigma \xi(\omega).$$

If $\xi \sim U(-\sqrt{3}, \sqrt{3})$, then ν is a uniform random variable with

$$\mathbb{E}[\nu(\omega)] = \mu, \qquad \text{Var}[\nu(\omega)] = \sigma^2.$$

Stochastic Galerkin discretisation I

Ingredients

- Picard iteration;
- standard finite element spaces $\mathbf{X}_E^h$ and M^h;
- a suitable finite-dimensional subspace $S^k \subset L^2_\rho(\Lambda)$, where $\Lambda := \xi(\Xi)$, $\Lambda \ni y$.

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Discrete formulation

Find $\vec{u}_{hk}^{n+1} \in \mathbf{X}_E^h \otimes S^k$ and $p_{hk}^{n+1} \in M^h \otimes S^k$ satisfying:

$$\mathbb{E}\Big[\nu(y)\big(\nabla \vec{u}_{hk}^{n+1}, \nabla \vec{v}\big)\Big] + \mathbb{E}\Big[\big(\vec{u}_{hk}^{n} \cdot \nabla \vec{u}_{hk}^{n+1}, \vec{v}\big)\Big] - \mathbb{E}\Big[\big(p_{hk}^{n+1}, \nabla \cdot \vec{v}\big)\Big] = 0$$

$$\mathbb{E}\Big[\big(\nabla \cdot \vec{u}_{hk}^{n+1}, q\big)\Big] = 0$$

for all $\vec{v} \in \mathbf{X}_0^h \otimes S^k$ and $q \in M^h \otimes S^k$.

Stochastic Galerkin discretisation II

Discrete formulation

Find $\vec{u}_{hk}^{n+1} \in \mathbf{X}_E^h \otimes S^k$ and $p_{hk}^{n+1} \in M^h \otimes S^k$ satisfying:

$$\mathbb{E}\Big[\nu(y)\big(\nabla \vec{u}_{hk}^{n+1}, \nabla \vec{v}\big)\Big] + \mathbb{E}\Big[\big(\vec{u}_{hk}^{n} \cdot \nabla \vec{u}_{hk}^{n+1}, \vec{v}\big)\Big] - \mathbb{E}\Big[\big(p_{hk}^{n+1}, \nabla \cdot \vec{v}\big)\Big] = 0$$

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Sets of basis functions

$\mathbf{X}_0^h = \mathtt{span}\big\{(\phi_i(\vec{x}), 0), (0, \phi_i(\vec{x}))\big\}_{i=1}^{n_u}$; $M^h = \mathtt{span}\,\{\psi_j(\vec{x})\}_{j=1}^{n_p}$;

$S^k = \mathtt{span}\,\{\varphi_\ell(y)\}_{\ell=0}^{k}$.

Stochastic Galerkin discretisation III

The linear system at the $(n+1)$st Picard iteration is

$$\begin{pmatrix} \mathbb{F}_\nu^n & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^n \\ \boldsymbol{\beta}^n \end{pmatrix} = \begin{pmatrix} \mathbf{f}^n \\ \mathbf{g}^n \end{pmatrix}$$

with

$$\mathbb{F}_\nu^n = \begin{pmatrix} F_\nu^n & 0 \\ 0 & F_\nu^n \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} G_0 \otimes B_{x_1} & G_0 \otimes B_{x_2} \end{pmatrix}$$

and

$$F_\nu^n := (\mu G_0 + \sigma G_1) \otimes A + \sum_{\ell=0}^{k} H_\ell \otimes N_\ell,$$

B_{x_1}, B_{x_2} are discrete representations of the first derivatives.

The system dimension is: $(n_u + n_p)(k+1) \times (n_u + n_p)(k+1)$.

(1-1) block: $F_\nu^n := (\mu G_0 + \sigma G_1) \otimes A + \sum_{\ell=0}^k H_\ell \otimes N_\ell$.

- F_ν^n is a non-symmetric matrix.
- convection matrices N_ℓ $(\ell = 0, \ldots, k)$ are given by

$$[N_\ell]_{ij} = (\,\vec{u}_{h\ell}^n(\vec{x}) \cdot \nabla\phi_i, \phi_j) \quad i, j = 0, \ldots, n_u.$$

where $\vec{u}_{h\ell}^n$ are the ‘spatial coefficients’ in the expansion of the lagged velocity field,

$$\vec{u}_{hk}^n(\vec{x}, y) = \sum_{\ell=0}^{k} \left(\underbrace{\sum_{i=1}^{n_u} \vec{u}_{i\ell}^n \, \phi_i(\vec{x})}_{\vec{u}_{h\ell}^n(\vec{x})} \right) \varphi_\ell(y).$$

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$$[N_\ell]_{ij} = (\, \vec{u}_{h\ell}^{\,n}(\vec{x}) \cdot \nabla \phi_i, \phi_j) \quad i, j = 0, \ldots, n_u.$$

- G_0, G_1 and H_ℓ are all $(k+1) \times (k+1)$ matrices:

$$\begin{aligned} G_0 := [G_0]_{\ell s} &= \mathbb{E}\left[\varphi_s(y)\,\varphi_\ell(y)\right], \\ G_1 := [G_1]_{\ell s} &= \mathbb{E}\left[y\,\varphi_s(y)\,\varphi_\ell(y)\right], \\ H_\ell := [H_\ell]_{ms} &= \mathbb{E}\left[\varphi_\ell(y)\,\varphi_s(y)\,\varphi_m(y)\right]. \end{aligned}$$

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If $\{\varphi_\ell(y)\}_{\ell=0}^{k}$ are scaled Legendre polynomials on Λ, then

- $G_0 = H_0 = I$, $G_1 = H_1$ is sparse (2 non-zeros per row);
- H_ℓ is dense for $\ell \geq 2$.

Ideal preconditioning

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \mathcal{P}^{-1} \quad \mathcal{P} \begin{pmatrix} \boldsymbol{\alpha}^u \\ \boldsymbol{\alpha}^p \end{pmatrix} = \begin{pmatrix} \mathbf{f}^u \\ \mathbf{f}^p \end{pmatrix}$$

An ideal preconditioner is given by

$$\begin{pmatrix} F & B^T \\ B & 0 \end{pmatrix} \underbrace{\begin{pmatrix} F^{-1} & F^{-1}B^T S^{-1} \\ 0 & -S^{-1} \end{pmatrix}}_{\mathcal{P}^{-1}} = \begin{pmatrix} I & 0 \\ BF^{-1} & I \end{pmatrix}.$$

For an efficient preconditioner we need to construct a sparse approximation to the "exact" Schur complement

$$S^{-1} = (BF^{-1}B^T)^{-1}$$

Preconditioning I

Rearrange the (1-1) block:

$$\begin{aligned} F^n_\nu &= (\mu G_0 + \sigma G_1) \otimes A + \textstyle\sum_{\ell=0}^{k} H_\ell \otimes N_\ell \\ &= I \otimes (\mu A_0 + N_0) + \sigma G_1 \otimes A + \textstyle\sum_{\ell=1}^{k} H_\ell \otimes N_\ell \end{aligned}$$

and define

$$F_0 := (\mu A_0 + N_0).$$

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and define

$$F_0 := (\mu A_0 + N_0).$$

A natural candidate for $\mathbb{P}_F$ is the block-diagonal mean-based approximation:

$$\mathbb{P}_F = \mathbb{F}_0 := \begin{pmatrix} I \otimes F_0 & 0 \\ 0 & I \otimes F_0 \end{pmatrix}.$$

This is a good approximation when $\frac{\sigma}{\mu}$ is not too large.

Preconditioning II

Replacing $\mathbb{F}_\nu^n$ by $\mathbb{F}_0$ in the Schur-complement gives

$$\begin{aligned}
\mathbb{S} &\approx \mathbb{B}\mathbb{F}_0^{-1}\mathbb{B}^T \\
&= (I \otimes B_{x_1})(I \otimes F_0^{-1})(I \otimes B_{x_1}^T) + (I \otimes B_{x_2})(I \otimes F_0^{-1})(I \otimes B_{x_2}^T) \\
&= I \otimes (B_{x_1}, B_{x_2}) F_0^{-1} (B_{x_1}, B_{x_2})^T =: I \otimes S_0 =: \mathbb{S}_0 = \mathbb{P}_S.
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S_0 is the Schur-complement corresponding to the deterministic problem with

- viscosity μ
- convection coefficient $\vec{u}_{hk}^0$ (the mean component of velocity at the previous Picard step)

Preconditioning III

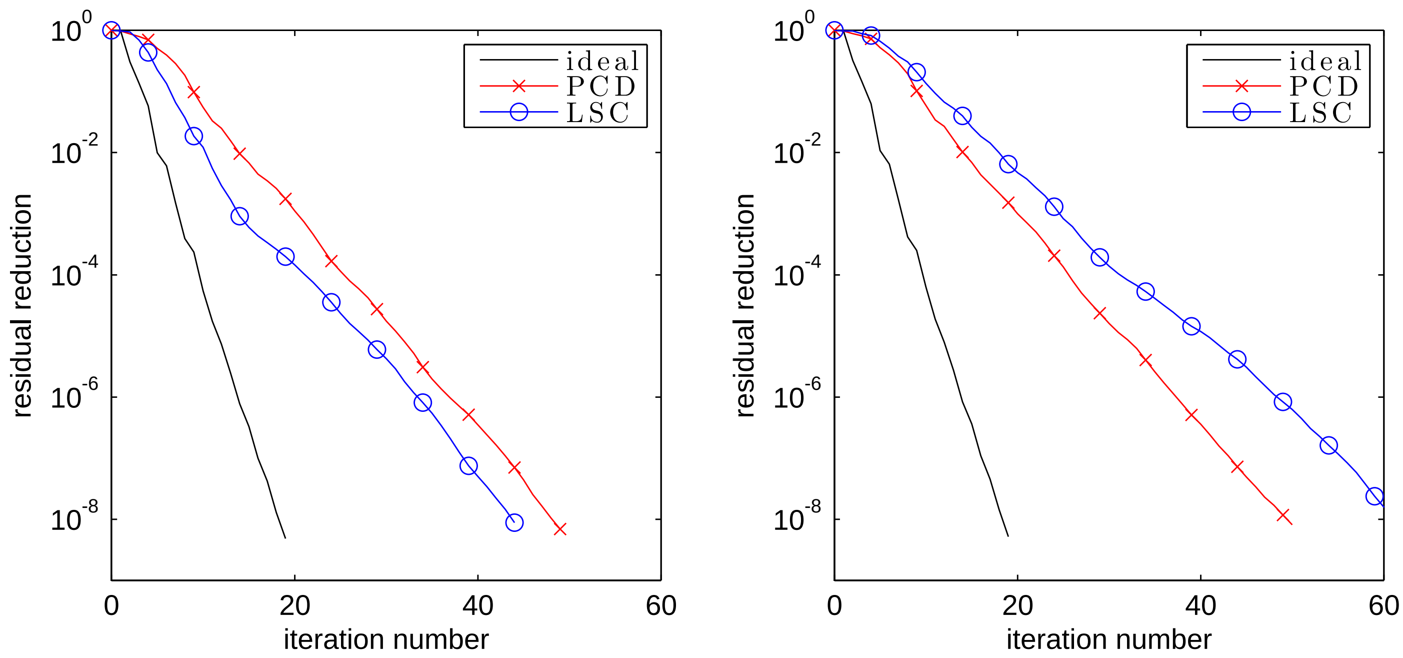
Replacing $\mathbb{F}_\nu^n$ by $\mathbb{F}_0$ in the Schur-complement gives

$$\begin{aligned}\mathbb{S} &\approx \mathbb{B}\mathbb{F}_0^{-1}\mathbb{B}^T \\ &= (I \otimes B_{x_1})(I \otimes F_0^{-1})(I \otimes B_{x_1}^T) + (I \otimes B_{x_2})(I \otimes F_0^{-1})(I \otimes B_{x_2}^T) \\ &= I \otimes (B_{x_1}, B_{x_2}) F_0^{-1} (B_{x_1}, B_{x_2})^T =: I \otimes S_0 =: \mathbb{S}_0 = \mathbb{P}_S.\end{aligned}$$

To apply $\mathbb{P}_S^{-1}$ in each GMRES iteration requires $(k+1)$ solves with S_0. This can be done

- exactly (ideal preconditioner); or
- inexactly with the deterministic approaches:
  – pressure convection–diffusion approximation (PCD)
  – least–squares commutator approximation (LSC).

Flow over a step



GMRES convergence for a coarsened grid (left) and for a reference grid (right) ($\mu = 1/50$; $\sigma = 2\mu/10$).

Typical GMRES iteration counts

		$\mathbb{E}[Re]$	Coarse grid $k=2$	4	6	Fine grid $k=2$	4	6
Ideal	$\sigma = \mu/10$	67	14	14	14	14	14	15
	$\sigma = 2\mu/10$	70	18	20	21	14	20	21
	$\sigma = 3\mu/10$	74	25	28	29	25	28	29
PCD	$\sigma = \mu/10$	67	37	38	39	37	39	39
	$\sigma = 2\mu/10$	70	43	44	50	44	48	50
	$\sigma = 3\mu/10$	74	53	56	61	54	58	62
LSC	$\sigma = \mu/10$	67	25	26	27	43	49	52
	$\sigma = 2\mu/10$	70	31	34	36	48	58	63
	$\sigma = 3\mu/10$	74	35	45	48	51	68	77

For further details, see

- David Silvester & Alex Bespalov & Catherine Powell
A framework for the development of implicit solvers for incompressible flow problems. Discrete and Continuous Dynamical Systems — Series S, vol. 5, 1195–1221, 2012.
- Catherine Powell & David Silvester
Preconditioning steady-state Navier–Stokes equations with random data. SIAM J. Scientific Computing, vol. 34, A2482–A2506, 2012.

Potential flow | linear stochastic formulation

Let $D \subset \mathbb{R}^d$ ($d = 2,\, 3$), and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Suppose that the input $A^{-1}(\boldsymbol{x}, \omega) : D \times \Omega \to \mathbb{R}$ is a second-order correlated random field.

We seek random fields $p(\boldsymbol{x}, \omega)$, $\boldsymbol{u}(\boldsymbol{x}, \omega)$ such that $\mathbb{P}$-almost everywhere in Ω:

$$
\begin{aligned}
A^{-1}(\boldsymbol{x}, \omega)\, \boldsymbol{u}(\boldsymbol{x}, \omega) - \nabla p(\boldsymbol{x}, \omega) &= 0, & & \\
\nabla \cdot \boldsymbol{u}(\boldsymbol{x}, \omega) &= 0 & &\boldsymbol{x} \text{ in } D, \\
p(\boldsymbol{x}, \omega) &= g(\boldsymbol{x}) & &\boldsymbol{x} \text{ on } \partial D_{\mathrm{D}}, \\
\boldsymbol{u}(\boldsymbol{x}, \omega) \cdot \boldsymbol{n} &= 0 & &\boldsymbol{x} \text{ on } \partial D_{\mathrm{N}}.
\end{aligned}
$$

$$
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p(\boldsymbol{x},\omega) &= g(\boldsymbol{x}) \qquad \boldsymbol{x} \text{ on } \partial D_{\mathrm{D}}, \\
\boldsymbol{u}(\boldsymbol{x},\omega)\cdot\boldsymbol{n} &= 0 \qquad \boldsymbol{x} \text{ on } \partial D_{\mathrm{N}}.
\end{aligned}
$$

Weak formulation

Find $\boldsymbol{u}(\boldsymbol{x},\omega) \in \mathcal{V} := L^2_{\mathbb{P}}(\Omega, H_0(div, D))$ and $p(\boldsymbol{x},\omega) \in \mathcal{W} := L^2_{\mathbb{P}}(\Omega, L^2(D))$ such that for all $\boldsymbol{v}(\boldsymbol{x},\omega) \in \mathcal{V}$ and $w(\boldsymbol{x},\omega) \in \mathcal{W}$:

$$
\begin{aligned}
\left\langle \left(A^{-1}\boldsymbol{u},\boldsymbol{v}\right)\right\rangle + \langle(p,\nabla\cdot\boldsymbol{v})\rangle &= \left\langle (g,\boldsymbol{v}\cdot\boldsymbol{n})_{\partial D_{\mathrm{D}}}\right\rangle, \\
\langle(w,\nabla\cdot\boldsymbol{u})\rangle &= 0.
\end{aligned}
$$

Discretisation strategy

Three levels of approximation

- Approximation of random data: the permeability $A^{-1}(\boldsymbol{x}, \omega)$ is approximated by a function $A_M^{-1}(\boldsymbol{x}, \boldsymbol{\xi}(\omega))$ of M random variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_M)$ taking values in $\Gamma \subset \mathbb{R}^M$. Note that $A_M(\boldsymbol{x}, \boldsymbol{\xi})$ could be a linear or a nonlinear function in $\boldsymbol{\xi}$.
- Spatial discretisation on D: e.g., lowest-order mixed FEM with mesh-size h;
- Approximation on Γ: e.g., orthogonal polynomials of total degree $\leq k$.

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- $(M + d)$–dimensional deterministic PDE to solve;
- Three discretisation parameters (M, h, k), hence, three separate sources of error ...

.... see the definitive reference:

- Alex Bespalov & Catherine Powell & David Silvester.
  A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data,
  SIAM J. Numerical Analysis, vol. 50, 2039–2063, 2012.

Linearity assumption

$$A^{-1}(\boldsymbol{x},\omega) \approx A_M^{-1}(\boldsymbol{x},\omega) = E[A^{-1}](\boldsymbol{x}) + \sum_{n=1}^{M} \sqrt{\lambda_n}\varphi_n(\boldsymbol{x})\,\xi_n(\omega),$$

where

- $\{\lambda_n, \varphi_n\}, n = 1, 2, \ldots$ are eigenvalues and eigenfunctions of the operator associated with the covariance $C(\boldsymbol{x}, \boldsymbol{x}')$ of $A^{-1}(\boldsymbol{x},\omega)$; for example,

$$C[a](\boldsymbol{x}, \boldsymbol{x}') = \exp\left(-\tfrac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}'\|_{\ell_1}\right), \quad \boldsymbol{x},\, \boldsymbol{x}' \in [-1, 1]^2.$$

- $\xi_1, \xi_2, \ldots$ are independent (uniform) random variables with mean zero and unit variance.

Linear algebra system

$$V_h = \mathtt{span}\,\{\boldsymbol{\varphi}_i\}_{i=1}^{n_u}\,, W_h = \mathtt{span}\,\{\phi_j\}_{j=1}^{n_p}\,, S_p = \mathtt{span}\,\{\psi_k(\mathbf{y})\}_{k=1}^{n_\xi}$$

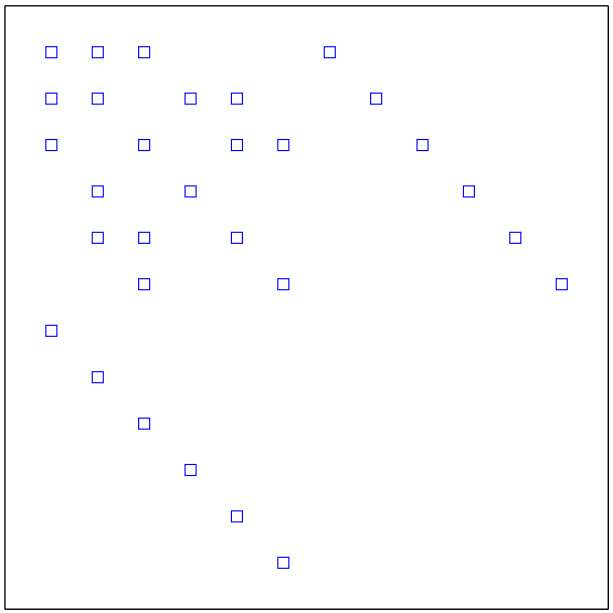
$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix} \qquad (S)$$

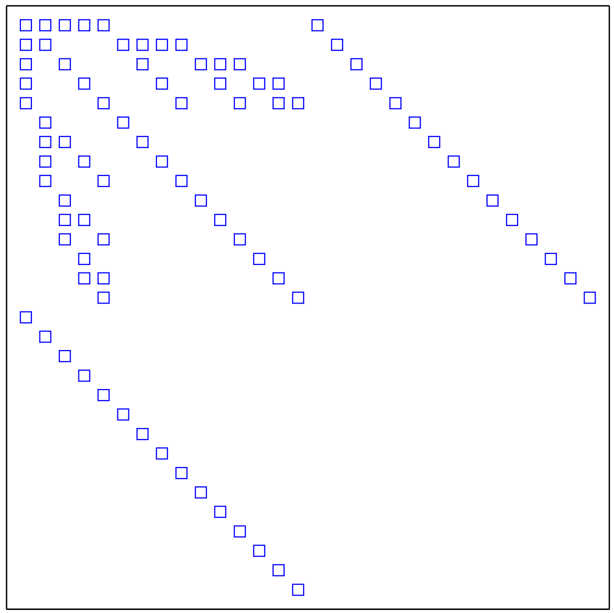
Properties

- The system dimension is $n_x n_\xi$ where $n_x = n_u + n_p$.
- $\begin{pmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{pmatrix} = \begin{pmatrix} I \otimes A_0 + \sum_{k=1}^{M} G_k \otimes A_k & I \otimes B^T \\ I \otimes B & 0 \end{pmatrix}$
- $[A_0]_{ij} = \int_D \mu(\boldsymbol{x})\, \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \, d\boldsymbol{x},$
  $[A_k]_{ij} = \int_D \sqrt{\lambda_k}\varphi_k(\boldsymbol{x})\, \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j \, d\boldsymbol{x},$
  $[B]_{is} = -\int_D \phi_s \nabla \cdot \boldsymbol{\varphi}_i \, d\boldsymbol{x}, \qquad [G_k]_{rs} = \langle\, y_k\, \psi_r(\boldsymbol{y})\psi_s(\boldsymbol{y})\, \rangle\,.$

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{pmatrix} = \begin{pmatrix} I \otimes A_0 + \sum_{k=1}^{M} G_k \otimes A_k & I \otimes B^T \\ I \otimes B & 0 \end{pmatrix}$$

Sparsity structure: $M = 2, k = 2$ (left) and $M = 4, k = 2$ (right)





Schur complement preconditioner

Approximate $\mathcal{A} \approx I \otimes \texttt{diag}(A_0)$. An efficient preconditioner is

$$P = \begin{pmatrix} I \otimes \texttt{diag}(A_0) & 0 \\ 0 & \mathcal{B}\left(I \otimes \texttt{diag}(A_0)\right)^{-1} \mathcal{B}^T \end{pmatrix}$$

$$= \begin{pmatrix} I \otimes \texttt{diag}(A_0) & 0 \\ 0 & I \otimes \left(B\,\texttt{diag}(A_0)^{-1} B^T\right) \end{pmatrix}.$$

Properties

- $B\,\texttt{diag}(A_0)^{-1}B^T \approx \nabla \cdot \mu(\boldsymbol{x})\nabla$ and optimal elliptic PDE solvers (based on AMG) can be utilised for the Schur complement solves (exactly as for the deterministic case).
- The cost of computing $P^{-1}\mathbf{r}$ is $O(n_\xi \times (n_u + n_p))$.

HSL	HSL_MI20
PACKAGE SPECIFICATION	HSL 2007

1 SUMMARY

Given an $n \times n$ sparse matrix $\mathbf{A}$ and an $n-$vector $\mathbf{z}$, `HSL_MI20` computes the vector $\mathbf{x} = \mathbf{Mz}$, where $\mathbf{M}$ is an algebraic multigrid (AMG) v-cycle preconditioner for $\mathbf{A}$. A classical AMG method is used, as described in [1] (see also Section 5 below for a brief description of the algorithm). The matrix $\mathbf{A}$ must have positive diagonal entries and (most of) the off-diagonal entries must be negative (the diagonal should be large compared to the sum of the off-diagonals). During the multigrid coarsening process, positive off-diagonal entries are ignored and, when calculating the interpolation weights, positive off-diagonal entries are added to the diagonal.

Reference

[1] K. Stüben. *An Introduction to Algebraic Multigrid*. In U. Trottenberg, C. Oosterlee, A. Schüller, eds, 'Multigrid', Academic Press, 2001, pp 413-532.

ATTRIBUTES — Version: 1.1.0 **Types:** Real (single, double). **Uses:** `HSL_MA48`, `HSL_MC65`, `HSL_ZD11`, and the `LAPACK` routines `_GETRF` and `_GETRS`. **Date:** September 2006. **Origin:** J. W. Boyle, University of Manchester and J. A. Scott, Rutherford Appleton Laboratory. **Language:** Fortran 95, plus allocatable dummy arguments and allocatable components of derived types. **Remark:** The development of `HSL_MI20` was funded by EPSRC grants EP/C000528/1 and GR/S42170.

Sample results ...

Bessel covariance function for random input with $\mu(\boldsymbol{x}) = 1$ and $M = 6$ random variables $\longrightarrow$ capture 98% of the total variance.

	k	2	3	4	5
	$n_\xi(n_u + n_p)$	344,064	1,032,192	2,580,480	5,677,056
$\frac{\sigma}{\mu} = 0.1$	# MINRES itns	45	46	48	48
	# V-cycles	1,260	3,864	10,080	22,176
	total solve time	14.0s	45.35s	119.01s	262.04s
$\frac{\sigma}{\mu} = 0.2$	# MINRES itns	55	59	62	63
	# V-cycles	1,540	4,956	13,020	29,106
	total solve time	17.18s	58.51s	154.82s	379.01
$\frac{\sigma}{\mu} = 0.3$	# MINRES itns	66	74	80	86
	# V-cycles	1,848	6,216	16,800	39,732
	total solve time	20.66s	72.97s	199.75s	486.74

Full details are in the references ...

- O. Ernst & C. Powell & D. Silvester & E. Ullmann, Efficient solvers for a linear stochastic Galerkin mixed formulation of the steady-state diffusion equation SIAM J. Sci. Comput., 31, 1424–1447, 2009.
- H. Elman & D. Furnival & C.Powell, $H(div)$ preconditioning for a mixed finite element formulation of the stochastic diffusion equation. Math. Comput. 79, 733–760, 2010.
- C. Powell & E. Ullmann, Preconditioning stochastic Galerkin saddle point systems. SIAM J. Matrix. Anal., 31, 2813–2840, 2010.
- A. Gordon & C. Powell, On solving stochastic collocation systems with algebraic multigrid. IMA J. Numer. Anal., 32, 1051–1070, 2012.