Tropical eigenspaces and their reachability by matrix orbits

Peter Butkovic

Credits

(Chronological order)
R.A.Cuninghame-Green
N.N.Vorobyev
M.Gondran
G.Cohen
E.Wagneur
S.Gaubert
M.Joswig
S.Sergeev
...

1. Introduction

Max-plus and variants

- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$
- $a \oplus b = \max(a, b)$
- $a \otimes b = a + b$
- $(\overline{\mathbb{R}}, \oplus, \otimes)$ ... idempotent, commutative semiring

Max-plus and variants

- $\mathcal{G} = (G, \otimes, \leq)$ ... linearly ordered commutative group
- $a \oplus b = \max(a, b)$
- $\varepsilon \leq a$ for all $a \in G$ (adjoined)
- $(G \cup \{\varepsilon\}, \oplus, \otimes)$ ... commutative idempotent semiring
- $\mathcal{G}_0 = (\mathbb{R}, +, \leq)$ ... max-plus
- $\mathcal{G}_1 = (\mathbb{R}, +, \geq)$ ... min-plus $(x \longrightarrow -x)$
- $\mathcal{G}_2 = (\mathbb{R}^+, \cdot, \leq)$ ... max-times $(x \longrightarrow e^x)$
- $\mathcal{G}_3 = (\mathbb{Z}, +, \leq)$
- ...
- In what follows: $\mathcal{G}_0$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\varepsilon = -\infty\}$

Extension to matrices and vectors

- $A \oplus B = (a_{ij} \oplus b_{ij})$
- $A \otimes B = \left(\sum_k^{\oplus} a_{ik} \otimes b_{kj}\right)$
- $\alpha \otimes A = (\alpha \otimes a_{ij})$

Some basic properties

- A^{-1} exists $\Longleftrightarrow$ A is a generalised permutation matrix

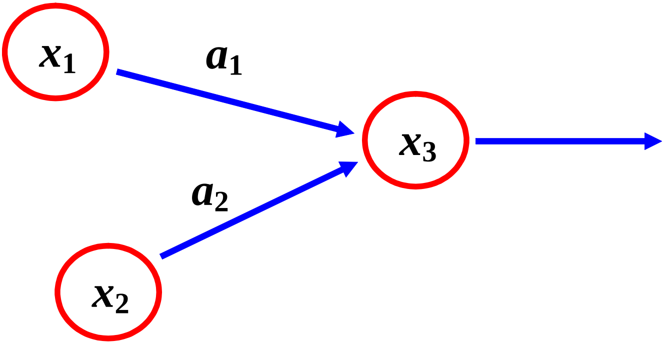
- $a \oplus a = a$
- $(a \oplus b)^k = a^k \oplus b^k$, if $a, b \geq 0$
- $(A \oplus B)^k \neq A^k \oplus B^k$
- $(I \oplus A)^k = I \oplus A \oplus A^2 \oplus ... \oplus A^k$

Associated digraph

- $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n} \longrightarrow D_A = (N, \{(i,j); a_{ij} > -\infty\}, (a_{ij}))$

  ... *associated digraph*
- A is *irreducible* iff D_A strongly connected

Tropical linear algebra: Non-linear problems treated as linear



$$x_3 = \max(x_1 + a_1, x_2 + a_2)$$
$$= a_1 \otimes x_1 \oplus a_2 \otimes x_2 = (a_1, a_2) \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

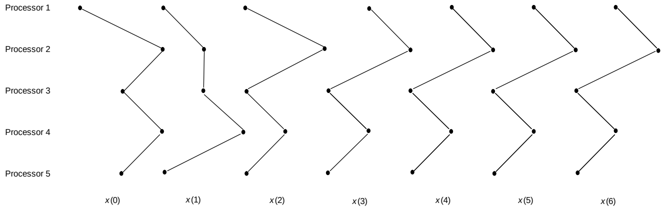
The eigenproblem

Given $A \in \overline{\mathbb{R}}^{n \times n}$, find $\lambda \in \overline{\mathbb{R}}$ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$

MULTI-PROCESSOR INTERACTIVE SYSTEM (MPIS)

(R.A.Cuninghame-Green)

- Processors $P_1, ..., P_n$ work interactively and in stages
- $x_i(r)$ ... starting time of the r^{th} stage on processor P_i $(i = 1, \ldots, n; r = 0, 1, ...)$
- a_{ij} ... time P_j needs to prepare the component for P_i
- $x_i(r+1) = max(x_1(r) + a_{i1}, \ldots, x_n(r) + a_{in})$ $(i = 1, \ldots, n; r = 0, 1, ...)$
- $x_i(r+1) = \sum_k^{\oplus} a_{ik} \otimes x_k(r) \quad (i = 1, \ldots, n; r = 0, 1, ...)$
- $x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$
- $A : x(0) \rightarrow x(1) \rightarrow x(2) \rightarrow \ldots$


Processor 1
Processor 2
Processor 3
Processor 4
Processor 5
x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)

MPIS: STEADY REGIME

- Given $x(0)$, will the MPIS reach a *steady regime* (that is, will it move forward in regular steps)?
- Equivalently, is there a λ and an r_0 such that

$$x(r+1) = \lambda \otimes x(r) \quad (r \geq r_0)?$$

-

$$x(r+1) = A \otimes x(r) \quad (r = 0, 1, \ldots)$$

- Steady regime is reached if and only if for some λ and r, $x(r)$ is a solution to

$$A \otimes x = \lambda \otimes x$$

- Since

$$x(r) = A \otimes x(r-1) = A^{(2)} \otimes x(r-2) = \ldots = A^{(r)} \otimes x(0),$$

a steady regime is reached if and only if $A^{(r)} \otimes x(0)$ "hits" an eigenvector of A for some r.

Two problems

- **Problem 1 (Eigenproblem):** Given $A \in \overline{\mathbb{R}}^{n \times n}$, find $\lambda \in \overline{\mathbb{R}}$ and $x \neq \varepsilon$ such that $A \otimes x = \lambda \otimes x$
- **Problem 2 (Reachability of an eigenspace):** Given $A \in \overline{\mathbb{R}}^{n \times n}$ and an $x \in \overline{\mathbb{R}}^{n}$, $x \neq \varepsilon$, is there a k such that $A^{(k)} \otimes x$ is an eigenvector of A?

2. TROPICAL EIGENPROBLEM

Maximum cycle mean

- Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$, the *mean of a cycle* $\sigma = (i_1, \ldots, i_k)$:

$$\mu(\sigma, A) = \frac{a_{i_1 i_2} + a_{i_2 i_3} + \ldots + a_{i_k i_1}}{k}$$

- *Maximum cycle mean of* $A \in \overline{\mathbb{R}}^{n \times n}$:

$$\lambda(A) = \max\{\mu(\sigma, A); \sigma \text{ cycle}\}$$

- $\mu(\sigma, A) = \lambda(A)$ ... σ is *critical*
- If

$$A = \begin{pmatrix} -2 & \boxed{1} & -3 \\ 3 & 0 & \boxed{3} \\ \boxed{5} & 2 & 1 \end{pmatrix}$$

- then

$$\begin{aligned} \lambda(A) &= \max\{-2, 0, 1, 2, 1, 5/2, 3, 2/3\} = 3 \\ \sigma &= (1, 2, 3) \text{ is critical} \end{aligned}$$

Maximum cycle mean is the principal eigenvalue

- For any A, $\lambda(A)$ is
  - an eigenvalue of A
  - the greatest (*principal*) eigenvalue of A
  - the only eigenvalue of A whose corresponding eigenvectors may be finite
  - the unique eigenvalue if A is irreducible
- Every eigenvalue of A is the maximum cycle mean of some principal submatrix

Definite matrices

- A is *definite* if $\lambda(A) = 0$
- $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$
- In particular: $\lambda\left((\lambda(A))^{-1} \otimes A\right) = (\lambda(A))^{-1} \otimes \lambda(A) = 0$
- $A \longrightarrow A_\lambda = (\lambda(A))^{-1} \otimes A$ (transition to a definite matrix)

Maximum cycle mean

- Many algorithms for the computation of $\lambda(A)$ (Karp's is $O(n^3)$)
- $\lambda(A) = \varepsilon$ if and only if D_A acyclic
- The eigenproblem for $\lambda(A) = \varepsilon$ treated separately

Transitive closures

- For $A \in \overline{\mathbb{R}}^{n \times n}$ we define:
- $A^+ = \quad A \oplus A^2 \oplus A^3 \oplus \ldots$ (*metric matrix/weak transitive closure*)
- $A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus \ldots$ (*Kleene star/strong transitive closure*)
- If A is definite:
  - $A^+ = \quad A \oplus A^2 \oplus \ldots \oplus A^{n-1} \oplus A^n$
  - $A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$

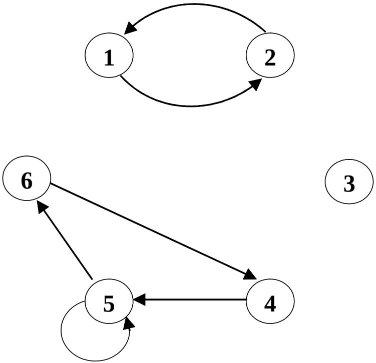
Critical graph

- $\mu(\sigma, A) = \lambda(A)$ ... σ is *critical*
- *Critical graph* of A: $C_A = (N, E_c)$ where E_c is the set of arcs of all critical cycles
- N_c ... the set of nodes of critical cycles
- $i \sim j$ (*equivalent* nodes) ... i and j belong to the same critical cycle

Eigenproblem: The principal eigenvalue and eigenvectors

$$A = \begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}, \quad \lambda(A) = 8$$

Eigenproblem: The principal eigenvalue and eigenvectors



- Critical cycles: $(1,2,1)$, $(5,5)$, $(4,5,6,4)$
- Node sets of all strongly connected components: $\{1,2\}, \{3\}, \{4,5,6\}$
- $N_c = \{1,2,4,5,6\}$

Eigenspaces

- $V(A,\lambda) = \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes x, x \neq \varepsilon\}, \lambda \in \overline{\mathbb{R}}$
- $V(A,\lambda) \cup \{\varepsilon\}$ is a tropical subspace: for $x, y \in V(A,\lambda)$ and $\alpha \in \overline{\mathbb{R}}$:
  - $x \oplus y \in V(A,\lambda)$ and
  - $\alpha \otimes x \in V(A,\lambda)$
- $V(A) = \bigcup_{\lambda \in \Lambda(A)} V(A,\lambda)$
- $\Lambda(A) = \{\lambda \in \overline{\mathbb{R}}; V(A,\lambda) \neq \emptyset\}$ ... *spectrum* of A

Principal eigenspace

- $\lambda(A)$ is an eigenvalue for any matrix $A \in \overline{\mathbb{R}}^{n\times n}$ (*principal eigenvalue*)
- $A \longrightarrow A_\lambda \longrightarrow (A_\lambda)^+$ (briefly A_λ^+)
- If $\lambda(A) > \varepsilon$ then every column of A_λ^+ with zero diagonal entry is an eigenvector of A with corresponding eigenvalue $\lambda(A)$ (*principal eigenvector*)
- An essentially unique basis of $V(A, \lambda(A))$ (*principal eigenspace*) can be obtained by taking exactly one principal eigenvector of A for each equivalence class in $(N_c, \sim)$
- If $A_\lambda^+ = (g_1, ..., g_n)$ then $i \sim j$ if and only if $g_i = \alpha \otimes g_j, \alpha \in \mathbb{R}$
- If A is irreducible then $V(A) = V(A, \lambda(A))$ and $V(A) \subseteq \mathbb{R}^n$

$$\underbrace{\begin{pmatrix} 7 & \boxed{9} & 5 & 5 & 3 & 7 \\ \boxed{7} & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & \boxed{9} & 5 \\ 4 & 2 & 6 & 6 & 8 & \boxed{8} \\ 3 & 0 & 5 & \boxed{7} & 1 & 2 \end{pmatrix}}_{A} \xrightarrow{-8} \underbrace{\begin{pmatrix} -1 & \boxed{1} & -3 & -3 & -5 & -1 \\ \boxed{-1} & -3 & -6 & -1 & -8 & -4 \\ 0 & -8 & -5 & -5 & 0 & -8 \\ -1 & -6 & -3 & -1 & \boxed{1} & -3 \\ -4 & -6 & -2 & -2 & 0 & \boxed{0} \\ -5 & -8 & -5 & \boxed{-1} & -7 & -6 \end{pmatrix}}_{A_\lambda}$$

$$\underbrace{\begin{pmatrix} \boxed{0} & 1 & -1 & 0 & 1 & 1 \\ -1 & \boxed{0} & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & \boxed{0} & 1 & 1 \\ -2 & -1 & -2 & -1 & \boxed{0} & 0 \\ -2 & -1 & -2 & -1 & 0 & \boxed{0} \end{pmatrix}}_{A_\lambda^+} \longrightarrow \begin{pmatrix} 0 & . & . & 0 & . & . \\ -1 & . & . & -1 & . & . \\ 0 & . & . & 0 & . & . \\ -1 & . & . & 0 & . & . \\ -2 & . & . & -1 & . & . \\ -2 & . & . & -1 & . & . \end{pmatrix}$$

- $A = \begin{pmatrix} 0 & 3 & & \\ 1 & -1 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}$, blank $= \varepsilon$
- $\lambda(A) = 2$
- $N_c = \{1, 2, 3\}$
- $1 \sim 2$
- $\dim(A) = 2$
- $A_\lambda^+ = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$
- A basis of the principal eigenspace is e.g. $\left\{ g_2 = (1, 0, \varepsilon, \varepsilon)^T, g_3 = (\varepsilon, \varepsilon, 0, \varepsilon)^T \right\}$

Link to nonnegative matrices

"Passage Theorem" (Friedland 1986)

- A ... an irreducible nonnegative matrix
- $\rho(A)$ ... the Perron root of A
- $\{A^k\}_{k=1}^{\infty}$ ... sequence of *Hadamard (Schur)* powers
- Then $\left(\rho\left(A^k\right)\right)^{1/k} \longrightarrow \lambda(A)$ (in max-times) and

$$\lambda(A) \leq \rho(A) \leq n\lambda(A)$$

- This is based on $\left(a^k + b^k\right)^{1/k} \longrightarrow \max(a, b)$ for $k \longrightarrow \infty$
- Similarly we have

$$\left(per\left(A^k\right)\right)^{1/k} \longrightarrow \sum_{\pi}^{\oplus} \prod_{i}^{\otimes} a_{i,\pi(i)} = \max_{\sigma} \sum_{i} a_{i,\sigma(i)}$$

Finding all eigenvalues: Reduced digraph

- $A \sim B$ for matrices A and B : A can be obtained from B by a simultaneous permutation of rows and columns
- If $A \sim B$ then $\Lambda(A) = \Lambda(B)$ and there is a bijection between $V(A)$ and $V(B)$
- *Frobenius Normal Form* (FNF):

$$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, \ A_{11}, ..., A_{rr} \text{ irreducible}$$

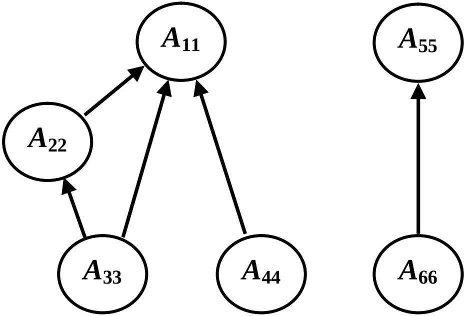
- The corresponding partition of N : $N_1, ..., N_r$ ... *classes (of A)*
- *Reduced digraph* (partially ordered set):

$$R_A = (\{N_1, ..., N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\})$$

- $N_i \longrightarrow N_j$ means: there is a directed path from N_i to N_j in R_A

Finding all eigenvalues: Reduced digraph

- $$\begin{pmatrix} A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\ * & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66} \end{pmatrix} \quad (* = \text{finite})$$



- *Initial classes*: no incoming arcs
- *Final classes:* no outgoing arcs

Finding all eigenvalues: Spectral Theorem

- A in an FNF:

$$\begin{pmatrix} A_{11} & & & & \\ A_{21} & A_{22} & & \varepsilon & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ A_{r1} & A_{r2} & \cdots & \cdots & A_{rr} \end{pmatrix}, A_{11}, \ldots, A_{rr} \text{ irreducible}$$

- **Spectral Theorem (Gaubert, Bapat, 1992):**

$$\Lambda(A) = \{\lambda(A_{ii}); \lambda(A_{ii}) \geq \lambda(A_{jj}) \text{ if } j \longrightarrow i\}$$

- i is called *spectral* if $\lambda(A_{ii}) \geq \lambda(A_{jj})$ whenever $j \longrightarrow i$

Finding all eigenvalues

- $A = \begin{pmatrix} 0 & 3 & & & & \\ 5 & 1 & & & & \\ & & 4 & & & \\ & & 0 & 3 & 1 & \\ & & & -1 & 2 & \\ & & & 1 & & 5 \end{pmatrix}$ (blank $= \varepsilon$)
- $\lambda(A_{11}) = 4, \lambda(A_{22}) = 4, \lambda(A_{33}) = 3, \lambda(A_{44}) = 5, r = 4$
- $\lambda(A) = 5$
- $\Lambda(A) = \{4, 5\}$
- N_1, N_4 are spectral (N_2 is not)

Finding all eigenvectors

- Let $A \in \overline{\mathbb{R}}^{n \times n}$ be in an FNF, $N_1, ..., N_r$ be the classes of A and $R = \{1, ..., r\}$
- Let $\lambda \in \Lambda(A), \lambda > \varepsilon$ and

$$I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i \text{ spectral}\}$$

- $A \longrightarrow \lambda^{-1} \otimes A \longrightarrow (\lambda^{-1} \otimes A)^+ = (g_{ij}) = (g_1, ..., g_n)$
- $N_c(\lambda) = \bigcup_{i \in I(\lambda)} N_c(A_{ii}) = \{j \in N; g_{jj} = 0, j \in \bigcup_{i \in I(\lambda)} N_i\}$
- $i, j \in N_c(\lambda)$ are called $\lambda-$ *equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle of cycle mean λ

Finding all eigenvectors

Theorem

Let $A \in \overline{\mathbb{R}}^{n \times n}$ *and* $\lambda \in \Lambda(A), \lambda > \varepsilon$.

- *For each* $\lambda \in \Lambda(A)$ *we have*

  $$V(A, \lambda) = \{(\lambda^{-1} \otimes A)^{+} \otimes z; z \in \overline{\mathbb{R}}^{n}, z_j = \varepsilon \text{ for all } j \notin N_c(\lambda)\}$$

- *A basis of* $V(A, \lambda)$ *can be obtained by taking one* g_j *for each* $\lambda-$ *equivalence class*

- The spectrum and bases of all eigenspaces for $A \in \overline{\mathbb{R}}^{n \times n}$ ... $O(n^3)$

3. Reachability of eigenspaces by matrix orbits

Attraction set

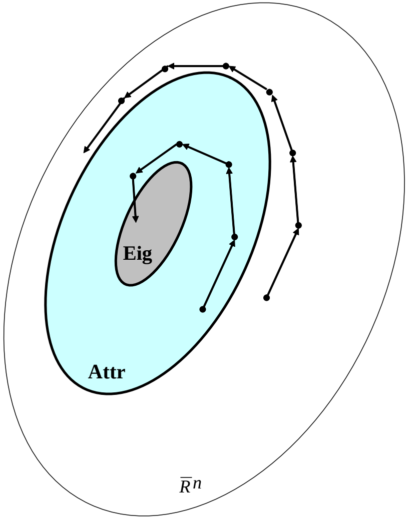
Problem 2 (Reachability of an eigenspace): Given A and an $x \neq \varepsilon$, is there a k such that $A^k \otimes x$ is an eigenvector of A?

- *Matrix orbit* with starting vector x : $A \otimes x, A^2 \otimes x, \ldots, A^k \otimes x, \ldots$
- *Attraction set*:

$$Attr\left(A\right) = \left\{x; \left(\exists k\right) A^k \otimes x \in V(A)\right\}$$

-

$$V(A) \subseteq attr\left(A\right) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$$


Eig
Attr
$\overline{R}^n$

Cyclicity of a matrix

- *Cyclicity* of a strongly connected digraph = g.c.d. of the lengths of its cycles
- *Cyclicity* of a digraph = l.c.m. of cyclicities of its SCC
- Let $A \in \overline{\mathbb{R}}^{n \times n}$
  - C_A... *critical digraph* of A
  - *Cyclicity of a matrix* A : $\sigma(A)$ = cyclicity of C_A
  - A is *primitive* if $\sigma(A) = 1$

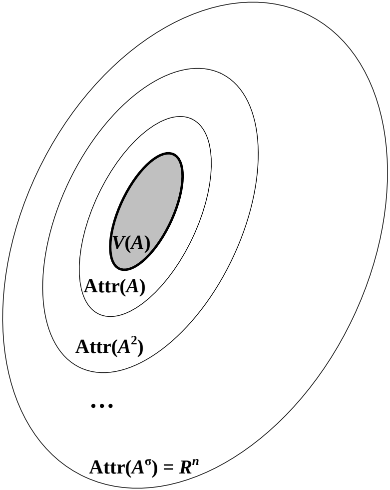
Cyclicity Theorem

- **Cyclicity Theorem (Cohen et al 1985)**
  Every irreducible matrix A is *ultimately periodic with period* $\sigma = \sigma(A)$:

$$A^{k+\sigma} = (\lambda(A))^{\sigma} \otimes A^{k} \text{ for all } k \geq k_0$$

- **Corollary**
  If A is irreducible:
  $(\forall x \neq \varepsilon)\, A^k \otimes x \in V(A^s)$ for some k and $s \leq \sigma(A)$

- Given A irreducible and x, find the smallest s for which $(\exists k)\ A^k \otimes x \in V(A^s)$
- $O\left(n^3 \log n\right)$ algorithm (Sergeev 2009)



Strongly and weakly stable matrices

- $V(A) \subseteq attr(A) \subseteq \overline{\mathbb{R}}^n - \{\varepsilon\}$
- Two extremes:
  - $attr(A) = \overline{\mathbb{R}}^n - \{\varepsilon\}$ ... *A strongly stable* (robust)
  - $attr(A) = V(A)$ ... *A weakly stable*

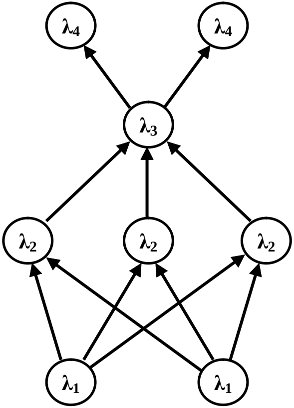
Strong stability (robustness)

- If A is irreducible and primitive then by the Cyclicity Theorem:
  - $A^{k+1} = \lambda(A) \otimes A^k$ for k large
  - $A^{k+1} \otimes x = \lambda(A) \otimes A^k \otimes x$ for k large and any $x \in \overline{\mathbb{R}}^n$
- A irreducible: A is robust $\iff$ A is primitive
- **Robustness criterion for reducible matrices** (PB & Gaubert & RACG 2009):
  A with FNF classes $N_1, \ldots N_r$ and no ε column is robust if and only if
  - All nontrivial classes are primitive and spectral
  - $(\forall i, j)$ If N_i, N_j are non-trivial, $N_i \not\rightarrow N_j$ and $N_j \not\rightarrow N_i$ then

$$\lambda(A_{ii}) = \lambda(A_{jj})$$

Strongly stable (robust) matrices

Reduced digraph of a robust matrix with $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$:

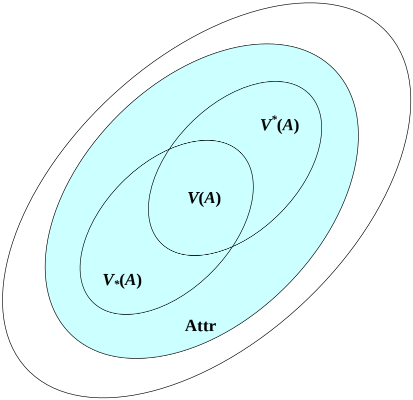


Weakly stable matrices

- A weakly stable: $attr\,(A) = V(A)$
- Let A be irreducible
- $V\,(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda\,(A) \otimes x, x \neq \varepsilon\}$ ... *eigenvectors*
- $V_*\,(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x \leq \lambda\,(A) \otimes x, x \neq \varepsilon\}$ ... *subeigenvectors*
- $V^*\,(A) = \{x \in \overline{\mathbb{R}}^n; A \otimes x \geq \lambda\,(A) \otimes x, x \neq \varepsilon\}$ ... *supereigenvectors*

Weakly stable matrices

- $V(A) \subseteq V_*(A) \subseteq Attr(A)$
- $V(A) \subseteq V^*(A) \subseteq Attr(A)$
- A weakly stable $\Longrightarrow V(A) = V^*(A) = V_*(A) = Attr(A)$



Weakly stable matrices

Theorem (PB+Sergeev, 2011)

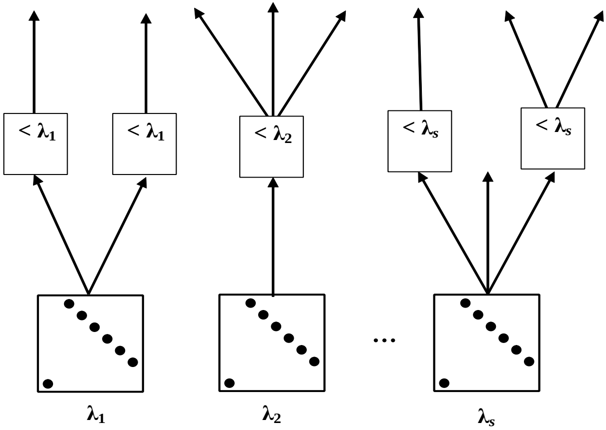
Let A be irreducible.
A is weakly stable $\Longleftrightarrow$ C_A is a Hamilton cycle in D_A.

$$\begin{pmatrix} & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \\ * & & & & \end{pmatrix}$$

Weakly stable matrices

Theorem (PB+Sergeev, 2011)

A (reducible) is weakly stable if and only if every spectral class is initial and weakly stable



THANK YOU

P. Butkovic: Max-linear Systems: Theory and Algorithms (Springer Monographs in Mathematics, Springer-Verlag 2010)