Purely Algebraic Results in Spectral Theory

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The most abstract formulation

We want to study spectral properties of elements of a non-commutative algebra $\mathcal{A}$ with identity e and scalars in a field $\mathbf{F}$.

We say that $\lambda \in \mathbf{F}$ does not lie in the spectrum of $a \in \mathcal{A}$ if the element $\lambda e - a$ is invertible, i.e. there exists $b \in \mathcal{A}$ such that

$$(a - \lambda e)b = b(a - \lambda e) = e.$$

Clearly $\mathrm{spec}(a) \subseteq \mathbf{F}$.

We might even want the scalars to lie in a commutative ring.

The most concrete example

We choose $\mathcal{A}$ to be the algebra of $n \times n$ matrices whose entries lie in the ring $\mathbf{F}$ of all polynomials.

A matrix a is invertible in the algebra $\mathcal{A}$ if and only if its determinant is an invertible element of $\mathbf{F}$, i.e. a non-zero constant.

A point in the spectrum of an $n \times n$ matrix is a polynomial, not a complex number.

Another concrete example

We choose $\mathcal{A}$ to be the algebra of $n \times n$ matrices.

and choose $\mathbf{F}$ to be the ring of all polynomials with degree at most N.

$\mathbf{F}$ is not an integral domain, and an element of $\mathbf{F}$ is invertible iff its constant coefficient is non-zero, e.g. $1 + 3x - 7x^2$.

A matrix a is invertible in the algebra $\mathcal{A}$ if and only if its determinant has a non-zero constant coefficient.

Problem 1. The spectrum of ab and ba

Theorem

If $a, b \in \mathcal{A}$ *and* $\mathbf{F}$ *is a field then*

$$\mathrm{spec}(ab)\backslash\{0\} = \mathrm{spec}(ba)\backslash\{0\}.$$

Proof.

If $\lambda \neq 0$ and $\lambda e - ab$ is invertible then a direct calculation shows that

$$c(\lambda e - ba) = (\lambda e - ba)c = e$$

where

$$c = \lambda^{-1}(e + b(\lambda e - ab)^{-1}a).$$

□

Extension

Theorem

If $a, b \in \mathcal{A}$ *and* $\mathcal{A}$ *is finite-dimensional then*

$$\mathrm{spec}(ab) = \mathrm{spec}(ba).$$

Proof.

In addition to the last theorem, we need to prove that ab is invertible if and only if ba is invertible.

If $\mathcal{A}$ is a matrix algebra this reduces to proving that $\det(ab)$ is invertible if and only if $\det(ba)$ is invertible. But each of these equals $\det(a)\det(b)$ because scalars commutate.

□

Problem 2. Sylvester's equation

The problem is to find conditions under which

$$ax - xb = c$$

has a solution $x \in \mathcal{A}$, given that $a, b, c \in \mathcal{A}$. Here we assume that $\mathcal{A}$ is a non-commutative algebra and that $\mathbf{F}$ is an arbitrary field of scalars, not necessarily algebraically closed.

As usual one could consider $\mathcal{A}$ to be the algebra of all $n \times n$ matrices with entries in $\mathbf{F}$.

The key condition for the existence of a solution is that a and b should be spectrally disjoint, which means more than just

$$\mathrm{spec}(a) \cap \mathrm{spec}(b) = \emptyset.$$

Spectral disjointness

We say that a and b are spectrally disjoint if they have minimum polynomials p and q that are relatively prime.

So we want

$$
\begin{aligned}
0 &= p(a) = \sum_{r=0}^{m} \alpha_r a^r \\
0 &= q(b) = \sum_{r=0}^{n} \beta_r b^r \\
1 &= ph + qk
\end{aligned}
$$

where $\alpha_m = 1$, $\beta_n = 1$ and p, q, h, k are polynomials with entries in $\mathbf{F}$.

Algebraically closure

Lemma

If $\mathbf{F}^*$ is the algebraic closure of $\mathbf{F}$ then a, b are spectrally disjoint in $\mathcal{A}$ if and only if their spectra are disjoint when calculated in $\mathcal{A}^* = \mathcal{A} \otimes \mathbf{F}^*$.

Sylvester's Theorem

Theorem

If $a, b \in \mathcal{A}$ are spectrally disjoint then the equation

$$ax - xb = c$$

has a unique solution $x \in \mathcal{A}$ for all $c \in \mathcal{A}$.

Proof.

We consider instead the problem $L_a(x) - R_b(x) = c$ where x, c are elements of $\mathcal{A}$ regarded as a vector space, and L_a, R_b are the linear operators on $\mathcal{A}$ defined by

$$L_a(x) = ax, \qquad R_b(x) = xb.$$

Purely algebraic methods are now used to prove that $L_a - R_b$ is an invertible operator on $\mathcal{A}$. □

Problem 3. The essential spectrum

If $\mathcal{K}$ is the set of all compact operators in $\mathcal{A} = \mathcal{L}(\mathcal{H})$ then $\mathcal{K}$ is a two-sided ideal.

One says that $a \in \mathcal{A}$ is a Fredholm operator if there exist $b, c \in \mathcal{A}$ such that

$$ab = e + k_1, \qquad ca = e + k_2$$

and $k_i \in \mathcal{K}$.

The essential spectrum of a is the set of $\lambda \in \mathbf{C}$ for which $a - \lambda e$ is not Fredholm. This is a subset of the spectrum.

The algebraic version of essential spectrum

Let $\mathcal{A}$ be an algebra over a field $\mathbf{F}$ and let $\mathcal{J}$ be a two-sided ideal in $\mathcal{A}$. Let $\pi_{\mathcal{J}} : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the quotient map.

Then the $\mathcal{J}$-essential spectrum $\mathrm{spec}_{\mathcal{J}}(a)$ of $a \in \mathcal{A}$ is defined to be the set of $\lambda \in \mathbf{F}$ for which $\pi_{\mathcal{J}}(\lambda e - a)$ is not invertible in $\mathcal{A}/\mathcal{J}$.

This enables one to classify the ordinary essential spectrum into parts, if $\mathcal{H} = L^2(\mathbf{R}^n)$ and there is a two-sided ideal corresponding to every direction at infinity in $\mathbf{R}^n$.

Dealing with unbounded operators

If $\mathcal{A}$ is an algebra over the field $\mathbf{F}$ then one might associate an 'operator not in $\mathcal{A}$' with a resolvent family (r, S). By this we mean a subset S of $\mathbf{F}$ and a map $r : S \to \mathcal{A}$ such that

$$r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu$$

for all $\lambda,\ \mu \in S$.

If $a \in \mathcal{A}$ one may put $r_\lambda = (\lambda e - a)^{-1}$ where S is the set of all $\lambda \in \mathbf{F}$ for which the inverse exists in $\mathcal{A}$.

More general resolvent families exist. Every such (r, S) has a unique extension $(\tilde{r}, \tilde{S})$ with $S \subseteq \tilde{S}$ which is maximal in the obvious sense. Then $\mathbf{F} \backslash \tilde{S}$ is called the spectrum of the resolvent family.

The essential spectrum of unbounded operators

Let (r, S) be a maximal resolvent family in $\mathcal{A}$ and let $\mathcal{J}$ be a two-sided ideal in $\mathcal{A}$ and let $\pi_{\mathcal{J}} : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the natural quotient map. Then one can define the resolvent family $(r_{\mathcal{J}}, S)$ in $\mathcal{A}/\mathcal{J}$ by $r_{\mathcal{J},\lambda} = \pi_{\mathcal{J}}(r_\lambda)$.

However $(r_{\mathcal{J}}, S)$ may not be maximal, so one should construct its maximal extension $(\tilde{r}_{\mathcal{J}}, \tilde{S})$. This has $S \subseteq \tilde{S}$. Therefore the $\mathcal{J}$-spectrum of the resolvent is contained within the ordinary spectrum of the resolvent.

This is exactly compatible with the definition of the spectrum and essential spectrum of an unbounded operator a if $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and $\mathcal{J}$ is the ideal of all compact operators.