Palindromic pencils, orbits, and the solution of the equation $XA + AX^T = 0$

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Outline

1. Motivation
   - Palindromic pencils
   - Orbits and the computation of canonical forms
   - Congruence orbits

2. Solution of $XA + AX^T = 0$
   - Dimension of the congruence orbits
   - Generic canonical structure

3. Orbits of palindromic pencils

Congruence and palindromic pencils

Given $A, B \in \mathbb{C}^{n\times n}$

$A+\lambda B$: matrix pencil

Eigenstructure: Invariants under strict equivalence.

$A'+\lambda B' = P(A+\lambda B)Q,$ P, Q nonsingular (strict equivalence)

Canonical Form: **Kronecker Canonical Form**

Particular cases:

- Generalized Eigenvalue Problem (eigenvalues and eigenvectors)
- $B=-I \longrightarrow A-\lambda I \Longrightarrow Q=P^{-1}$ (similarity, Jordan Canonical Form)
  Standard Eigenvalue Problem

$A+\lambda A^T$: palindromic pencil

To preserve the structure: $P(A+\lambda A^T)P^T = (PA)+\lambda(PA)^T$ (congruence)

Palindromic pencils

Applications:

- Quadratic palindromic polynomials $\lambda^2 A + \lambda B + A^T$ (with $B^T = B$)
  - Rail traffic noise of high speed trains.
  - Surface Acoustic Waves (SAW) filters.
  - Discrete Optimal Control of higher order difference equations.
- Eigenstructure: comprises relevant (physical) information of the system.
- Palindromic **pencils**: useful in the numerical solution of eigenvalue problems of quadratic matrix polynomials (through linearizations).

Some interesting questions

Due to roundoff errors, uncertainty in the data, etc., usually we **do not** compute the **exact** canonical form.

- Which are the **nearby** canonical structures (JCF, KCF) to a given one?
- Which is the **generic** canonical structure?

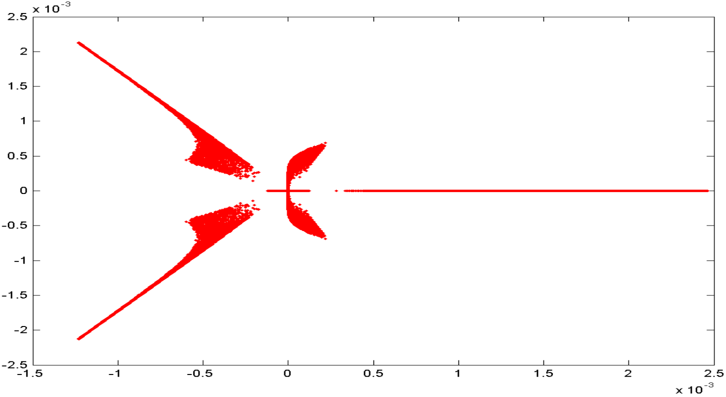
Same question for matrices/matrix pencils in a particular subset (low-rank, palindromic, symmetric,...)

► The **theory of orbits** provides a theoretical framework.

An illustrative example

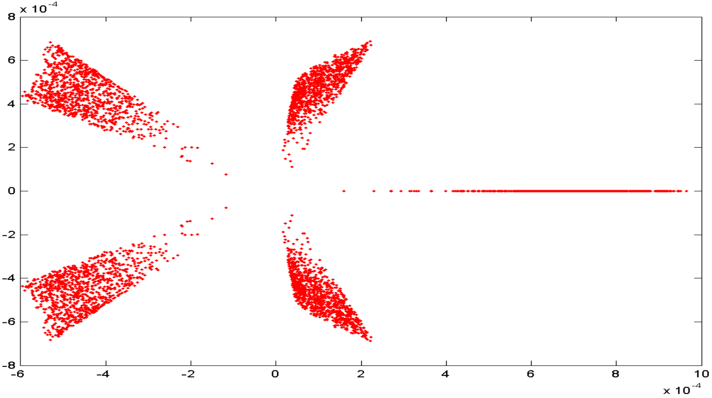
$A = J_3(0) \oplus J_2(0)$

10000 random perturbations with norm $\sim \sqrt{2^{-52}}$ (positive entries)



An illustrative example

$A = J_3(0) \oplus J_2(0)$
50000 (filtered) random perturbations with norm $\sim \sqrt{2^{-52}}$ (positive entries)



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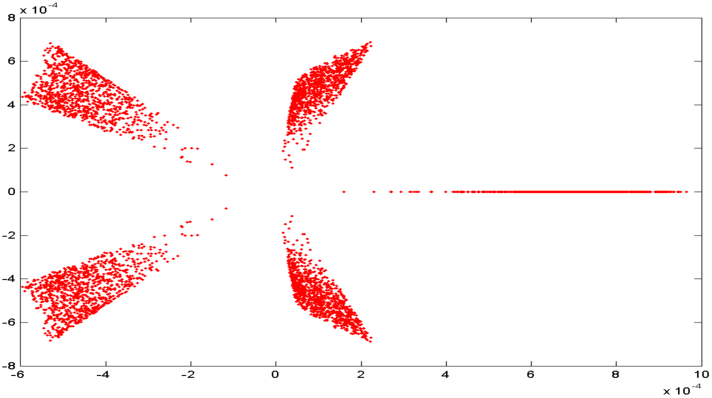

$J_5(0)$ is “close" to $J_3(0) \oplus J_2(0)$

An illustrative example

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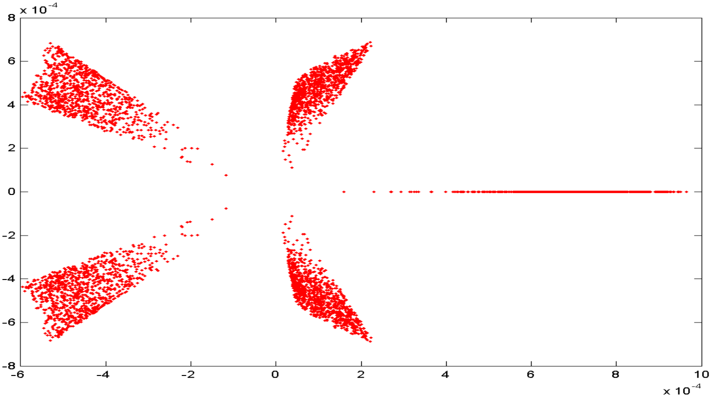

$J_3(0) \oplus J_2(0)$ is in the **closure** of the **orbit** of $J_5(0)$

Congruence, equivalence and similarity. Orbits

Given $A, B \in \mathbb{C}^{n\times n}$

$$\mathscr{O}(A) = \{PAP^T : P \text{ nonsingular}\} \quad \text{Congruence orbit of } A$$
$$\mathscr{O}_s(A) = \{PAP^{-1} : P \text{ nonsingular}\} \quad \text{Similarity orbit of } A$$
$$\mathscr{O}_e(A+\lambda B) = \{P(A+\lambda B)Q : P, Q \text{ nonsingular}\} \quad \text{Equivalency orbit of } A+\lambda B$$

Similarity/equivalency orbits:

- Have been widely studied: Arnold (1971), Demmel-Edelman (1995), Edelman-Elmroth-Kågström (1997, 1999), Johansson (2006), ...
- Correspond to matrices with the same Jordan Canonical Form (JCF) / Pencils with the same Kronecker Canonical Form (KCF).
- The **dimension** of these manifolds gives us an idea of their “size".
- The description of the hierarchy between closures of different orbits allows to identify nearby Jordan/Kronecker structures and may be useful in the design and analysis of algorithms to compute the JCF/KCF.

Congruence orbits?

Codimension of the tangent space

$$T_{\mathcal{O}(A)}(A) = \{XA + AX^T \,:\, X \in \mathbb{C}^{n\times n}\}$$ Tangent space of $\mathcal{O}(A)$ at A

$$T_{\mathcal{O}_s(A)}(A) = \{XA - AX \,:\, X \in \mathbb{C}^{n\times n}\}$$ Tangent space of $\mathcal{O}_s(A)$ at A

Then:

(a) codim $\mathcal{O}(A) =$ codim $T_{\mathcal{O}(A)}(A) =$ dim(solution space of $XA + AX^T = 0$)

(b) codim $\mathcal{O}_s(A) =$ codim $T_{\mathcal{O}_s(A)}(A) =$ dim(solution space of $XA - AX = 0$)

Solution of $XA - AX = 0$: known since the 1950's (Gantmacher). Depends on the JCF of A.

Goal: Solve $XA + AX^T = 0$

(We are mainly interested in the **dimension** of the solution space, but we are able also to give the solution!)

Change of basis

Notation: $\mathscr{S}_A = \{X \in \mathbb{C}^{n\times n} : XA + AX^T = 0\}$ (solution space)

Set $B = PAP^T$ (P nonsingular) then $\mathscr{S}_A = P^{-1}\mathscr{S}_B P$

In particular: $\boxed{\dim \mathscr{S}_A = \dim \mathscr{S}_B}$

Procedure to solve $XA + AX^T = 0$:

1. Set $C_A = PAP^T$, the canonical form of A (for congruence)
2. Solve $YC_A + C_AY^T = 0$
3. Undo the change: $X = P^{-1}YP$

The canonical form for congruence

Theorem (Canonical form for congruence [Horn & Sergeichuk, 2006])

Each matrix $A \in \mathbb{C}^{n\times n}$ is **congruent** to a direct sum (uniquely determined up to permutation) of blocks of types 0, I and II.

$$(\textit{Type } 0) \quad J_k(0) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}_{k\times k}$$

$$(\textit{Type I}) \quad \Gamma_k = \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}_{k\times k}, \quad \Gamma_1 = [1]$$

$$(\textit{Type II}) \quad H_{2k}(\mu) = \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}_{2k\times 2k}, \quad H_2(\mu) = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} \quad (0 \neq \mu \neq (-1)^{k+1})$$

- Another canonical form for congruence: [Turnbull & Aitken, 1932], **Six types of blocks**

Partition on canonical blocks

Set $C_A = D_1 \oplus \cdots \oplus D_s$, $\quad D_i = J_k(0),\ \Gamma_k,$ or $H_{2k}(\mu)$ (Canonical form of A)

Partition $X = \begin{bmatrix} X_{11} & \dots & X_{1s} \\ \vdots & & \vdots \\ X_{s1} & \dots & X_{ss} \end{bmatrix}$ conformally with C_A.

Equating the (i,j) and (j,i) blocks of $XC_A + C_A X^T = 0$, we get:

- $i = j$: $X_{ii} D_i + D_i X_{ii}^T = 0 \qquad \to$ codim D_i (codimension)
- $i \neq j$: $\begin{array}{ll} (i,j) & X_{ij} D_j + D_i^T X_{ji}^T = 0 \\ (j,i) & X_{ji} D_i + D_j^T X_{ij}^T = 0 \end{array} \qquad \to$ inter (D_i, D_j) (interaction)

Then:

$$\dim \mathscr{S}_A = \text{ codim } \mathscr{O}(A) = \sum_i \text{codim } D_i + \sum_{i,j} \text{inter}\,(D_i, D_j)$$

Partition on canonical blocks (ctd)

The problem reduces to solve:

(a) $XD+DX^T=0$

(b) $$\begin{array}{l} XD_1+D_2Y^T=0 \\ YD_1+D_2X^T=0 \end{array}$$

With $D, D_1, D_2 = J_k(0)$ (type 0), Γ_k (type I), or $H_{2k}(\mu)$ (type II)

Codimension of individual blocks

Type	Equation	Codimension
0	$XJ_k(0) + J_k(0)X^T = 0$	$c_0 = \lceil \frac{k}{2} \rceil$
I	$X\Gamma_k + \Gamma_k X^T = 0$	$c_1 = \lfloor \frac{k}{2} \rfloor$
II	$XH_{2k}(\mu) + H_{2k}(\mu)X^T = 0$	$c_2 = \begin{cases} k\,, & \text{if } \mu \neq (-1)^k \\ k + 2\lceil \frac{k}{2} \rceil\,, & \text{if } \mu = (-1)^k \end{cases}$

- Explicit solution for types 0, I available.
- Algorithm for computing solution for type II.
- Solution of $X\Gamma_k + \Gamma_k X^T = 0$ (type I):

$$X = \begin{bmatrix} 0 & & & & & & 0 \\ x_1 & 0 & & & & & \\ 0 & x_1 & 0 & & & & \\ x_2 & 0 & x_1 & 0 & & & \\ 0 & x_2 & 0 & x_1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ x_{\frac{k}{2}} & \dots & 0 & x_2 & 0 & x_1 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & & & & & & & & 0 \\ x_1 & 0 & & & & & & & \\ 0 & x_1 & 0 & & & & & & \\ x_2 & 0 & x_1 & 0 & & & & & \\ 0 & x_2 & 0 & x_1 & 0 & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ x_{\frac{k-1}{2}} & \dots & 0 & x_2 & 0 & x_1 & 0 & & \\ 0 & x_{\frac{k-1}{2}} & \dots & 0 & x_2 & 0 & x_1 & 0 & \end{bmatrix}$$

(k even) (k odd)

Interaction between canonical blocks

Blocks of the same type:

Type	Equation	Interaction
0-0	$XJ_k(0)+J_\ell(0)Y^T=0$, $YJ_\ell(0)+J_k(0)X^T=0$ $(k\geq\ell)$	$i_{00}=\begin{cases}\ell, & \ell \text{ even}\\ k, & \ell \text{ odd and } k\neq\ell\\ k+1, & \ell \text{ odd and } k=\ell\end{cases}$
I-I	$X\Gamma_k+\Gamma_\ell Y^T=0$, $Y\Gamma_\ell+\Gamma_k X^T=0$	$i_{11}=\begin{cases}0, & k,\ell \text{ different parity}\\ \min\{k,\ell\}, & k,\ell \text{ same parity}\end{cases}$
II-II	$XH_{2k}(\mu)+H_{2\ell}(\widetilde{\mu})Y^T=0$, $YH_{2\ell}(\widetilde{\mu})+H_{2k}(\mu)X^T=0$	$i_{22}=\begin{cases}4\min\{k,\ell\}, & \mu=\widetilde{\mu}=\pm1\\ 2\min\{k,\ell\}, & \mu=\widetilde{\mu}\neq\pm1\\ 2\min\{k,\ell\}, & \mu\neq\widetilde{\mu},\ \mu\widetilde{\mu}=1\\ 0, & \mu\neq\widetilde{\mu},\ \mu\widetilde{\mu}\neq1\end{cases}$

Interaction between canonical blocks (ctd)

Blocks of different type:

Type	Equation	Interaction
0-I	$XJ_k(0)+\Gamma_\ell Y^T=0$ $Y\Gamma_\ell+J_k(0)X^T=0$	$i_{01}=\begin{cases}0, & k \text{ even}\\ \ell, & k \text{ odd}\end{cases}$
0-II	$XJ_k(0)+H_{2\ell}(\mu)Y^T=0$ $YH_{2\ell}(\mu)+J_k(0)X^T=0$	$i_{02}=\begin{cases}0, & k \text{ even}\\ 2\ell, & k \text{ odd}\end{cases}$
I-II	$X\Gamma_k+H_{2\ell}(\mu)Y^T=0$ $YH_{2\ell}(\mu)+\Gamma_k X^T=0$	$i_{12}=\begin{cases}2\min\{k,\ell\}, & \mu=(-1)^{k+1}\\ 0, & \mu\neq(-1)^{k+1}\end{cases}$

- Explicit solution available (for all cases).

The codimension formula

Theorem

Let $A \in \mathbb{C}^{n\times n}$ be a matrix with canonical form for congruence

$$\begin{aligned} C_A = & J_{p_1}(0) \oplus J_{p_2}(0) \oplus \cdots \oplus J_{p_a}(0) \\ & \oplus \Gamma_{q_1} \oplus \Gamma_{q_2} \oplus \cdots \oplus \Gamma_{q_b} \\ & \oplus H_{2r_1}(\mu_1) \oplus H_{2r_2}(\mu_2) \oplus \cdots \oplus H_{2r_c}(\mu_c). \end{aligned}$$

Then the codimension of the orbit of A for the action of congruence, i.e., the dimension of the solution space of $XA+AX^T=0$, depends only on C_A. It can be computed as the sum

$$c_{\text{Total}} = c_0 + c_1 + c_2 + i_{00} + i_{11} + i_{22} + i_{01} + i_{02} + i_{12}\,.$$

Application: Generic canonical form for congruence

Generic = codimension zero

Theorem

The minimal codimension for a congruence orbit in $\mathbb{C}^{n\times n}$ is $\lfloor n/2 \rfloor$.

No generic canonical form for congruence!!

Similarity orbits (JCF): There is no generic JCF (with fixed eigenvalues)

- The **generic** Jordan structure is $J_1(\lambda_1)\oplus\cdots\oplus J_1(\lambda_n)$, with $\lambda_1,\ldots,\lambda_n$ different (not fixed)

Bundles

For similarity (Arnold, 1971):

Given $A \in \mathbb{C}^{n\times n}$, with

$$J_A = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_d}\,,$$

where

$$J_{\lambda_i} = J_{n_{i,1}}(\lambda_i) \oplus \cdots \oplus J_{n_{i,q_i}}(\lambda_i), \qquad \text{for } i = 1, \ldots, d,$$

the **similarity bundle** of A is

$$\mathscr{B}_s(A) = \bigcup_{\substack{\lambda_i' \in \mathbb{C},\, i=1,\ldots,d \\ \lambda_i' \neq \lambda_j'}} J_{\lambda_1'} \oplus \cdots \oplus J_{\lambda_d'}$$

Given A with $C_A = \bigoplus_{i=1}^{a} J_{p_i}(0) \oplus \bigoplus_{i=1}^{b} \Gamma_{q_i} \oplus \bigoplus_{i=1}^{t} \mathscr{H}(\mu_i)$, $\mu_i \neq \mu_j$, $\mu_i \neq 1/\mu_j$ if $i \neq j$,

Definition: Congruence bundle of A:

$$\mathscr{B}(A) = \bigcup_{\substack{\mu_i' \in \mathbb{C},\, i=1,\ldots,t \\ \mu_i' \neq \mu_j',\, \mu_i'\mu_j' \neq 1,\, i \neq j}} \mathscr{O}\left(\bigoplus_{i=1}^{a} J_{p_i}(0) \oplus \bigoplus_{i=1}^{b} \Gamma_{q_i} \oplus \bigoplus_{i=1}^{t} \mathscr{H}(\mu_i') \right).$$

(same structure as C_A but unfixed complex values μ in type II blocks)

The generic structure

$\mathrm{codim}(\mathscr{B}(A)) = \mathrm{codim}(\mathscr{O}(A)) - t\,.$
(t=number of different $\mu' s$ appearing in type II blocks of C_A)

Theorem

The following bundles in $\mathbb{C}^{n\times n}$ have codimension zero

1. n even

$$G_n = H_2(\mu_1)\oplus H_2(\mu_2)\oplus\cdots\oplus H_2(\mu_{n/2}),$$

with $\mu_i \neq \pm 1$, $i=1,\ldots,n/2$, $\mu_i\neq\mu_j$ and $\mu_i\neq 1/\mu_j$ if $i\neq j$.

2. n odd

$$G_n = H_2(\mu_1)\oplus H_2(\mu_2)\oplus\cdots\oplus H_2(\mu_{(n-1)/2})\oplus\Gamma_1,$$

with $\mu_i \neq \pm 1$, $i=1,\ldots,(n-1)/2$, $\mu_i\neq\mu_j$ and $\mu_i\neq 1/\mu_j$ if $i\neq j$.

Then G_n is the **generic** canonical structure for congruence in $\mathbb{C}^{n\times n}$ (with unspecified values $\mu_1,\mu_2,\ldots,\mu_{\lfloor n/2\rfloor}$).

Congruence vs equivalence

Congruence orbit of $A+\lambda A^T$: $\mathscr{O}(A+\lambda A^T) = \{P(A+\lambda A^T)P^T : \det P \neq 0\}$

A, B are congruent iff $A+\lambda A^T$, $B+\lambda B^T$ are congruent.

There is a bijection $\mathscr{O}(A) \longrightarrow \mathscr{O}(A+\lambda A^T)$

The generic canonical form for congruence of $A+\lambda A^T$ is $G_n+\lambda G_n^T$

The KCF of $A+\lambda A^T$ is congruent to $A+\lambda A^T$

Canonical form for **congruence** of palindromics: **KCF !!!**

We can determine:

- **dimension of** $\mathscr{O}(A+\lambda A^T)$
- generic KCF of palindromic pencils

Generic KCF of palindromic pencils

Theorem

The generic **KCF** of palindromic pencils in $\mathbb{C}^{n\times n}$ is

1. If n is even:

$$(\lambda+\mu_1)\oplus(\lambda+1/\mu_1)\oplus(\lambda+\mu_2)\oplus(\lambda+1/\mu_2)\oplus\cdots\oplus(\lambda+\mu_{n/2})\oplus(\lambda+1/\mu_{n/2}),$$

   where $\mu_1,\ldots,\mu_{n/2}$ are unspecified complex numbers such that $0\neq\mu_i\neq\pm1$, $i=1,\ldots,n/2$, $\mu_i\neq\mu_j$ and $\mu_i\neq 1/\mu_j$ if $i\neq j$.

2. If n is odd:

$$\begin{array}{c}(\lambda+\mu_1)\oplus(\lambda+1/\mu_1)\oplus(\lambda+\mu_2)\oplus(\lambda+1/\mu_2)\oplus\cdots\oplus\\(\lambda+\mu_{(n-1)/2})\oplus(\lambda+1/\mu_{(n-1)/2})\oplus(\lambda+1)\end{array},$$

   where $\mu_1,\ldots,\mu_{(n-1)/2}$ are unspecified complex numbers such that $0\neq\mu_i\neq\pm1$, $i=1,\ldots,(n-1)/2$, $\mu_i\neq\mu_j$ and $\mu_i\neq 1/\mu_j$ if $i\neq j$.

Conclusions

- We have solved the matrix equation $XA + AX^T = 0$, for $A \in \mathbb{C}^{n\times n}$.
- As a consequence, we have computed the dimension of the congruence orbit of A in terms of the canonical form by congruence of A.
- We have determined the generic canonical structure for congruence in $\mathbb{C}^{n\times n}$ and also the generic KCF of palindromic pencils.

Related and future work

- Solve the matrix equation $XA + AX^* = 0$ (done, to appear in ELA).
- Other related equations: $XA + AX^T = C$, $XA + BX^T = C$, $A, B, C \in \mathbb{C}^{n \times n}$.
- Describe the hierarchy between closures of congruence orbits.

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