Tropical aspects of eigenvalue problems

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Exercise

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

Eigenvalues ? $\epsilon \to 0$

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

Solution in this talk

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $= 5$
- “$5/2$” $= 3$
- “2^3” $=$ “$2 \times 2 \times 2$” $= 6$
- “$\sqrt{-1}$” $= -0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \text{”} \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The notation $a \oplus b := \max(a, b)$, $a \otimes b := a + b$, $\mathbb{0} := -\infty$, $\mathbb{1} := 0$ is also used in the tropical/max-plus litterature

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

- $\text{“}2 + 3\text{”} = 2$
- $\text{“}2 \times 3\text{”} = 5$

These algebras were invented by various schools in the world

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼65 . . . Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- . . . Kolokoltsov, Litvinov, Samborskii, Shpiz. . . Quasi-classic analysis, variations calculus
- Simon ∼ 78- . . . Hashiguchi, Leung, Pin, Krob, . . . Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- . . . Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- Optimal control
- Puhalskii ∼99- idempotent probabilities (large deviations)

and now in tropical geometry, after Viro, Mikhalkin, Passare, Sturmfels and many

(see the program of the workshops of the tropical semester at MSRI).

Tropical / max-plus algebra $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ equipped with

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}ab\text{”} = a + b$$

The max-plus spectral problem

Given $A = (A_{ij}) \in (\mathbb{R} \cup \{-\infty\})^{n \times n}$, find $v \in \mathbb{R} \cup \{-\infty\}^n$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}$, such that

$$\max_j A_{ij} + v_j = \lambda + v_i$$

$$\text{“}Av = \lambda v\text{”}$$

Among the oldest max-plus result: combinatorial (polynomial-time) characterization of the eigenvalues and eigenvectors.

Motivation 1: dynamic programming, one player

Set of nodes $[d] := \{1, \ldots, d\}$, arc (i, j) with weight A_{ij}

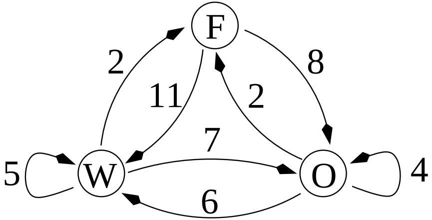
$$A^k_{ij} = \sum_{m_1,\ldots,m_{k-1} \in [d]} A_{im_1} A_{m_1 m_2} \cdots A_{m_{k-1} j}$$

Motivation 1: dynamic programming, one player

Set of nodes $[d] := \{1, \ldots, d\}$, arc (i, j) with weight A_{ij}

$$\begin{aligned} A^k_{ij} &= \max_{m_1,\ldots,m_{k-1}\in[d]} A_{im_1} + A_{m_1 m_2} + \cdots + A_{m_{k-1}j} \\ &= \text{max weight path } i \to j \text{ length } k \end{aligned}$$

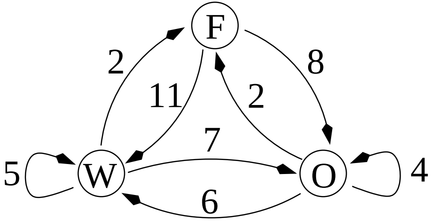
Crop rotation



A_{ij} = reward of the year if crop j follows crop i
F=fallow (no crop), W=wheat, O=oat,

$$(A^k v)_i = \sum_{j \in [d]} A^k_{ij} v_j$$

Crop rotation



A_{ij} = reward of the year if crop j follows crop i
F=fallow (no crop), W=wheat, O=oat,

$$(A^k v)_i = \max_{j \in [d]} A^k_{ij} + v_j$$

= reward in k years, init. crop i; v_j term. reward

Eigenvector

Find $v \in \mathbb{R}_{\max}^d$, $v \not\equiv 0$, $\lambda \in \mathbb{R}_{\max}$, such that

$$Av = \lambda v$$

$$A^k v = \lambda^k v$$

Eigenvector

Find $v \in \mathbb{R}_{\max}^d$, $v \not\equiv -\infty$, $\lambda \in \mathbb{R}_{\max}$, such that

$$\max_{j \in [d]} A_{ij} + v_j = \lambda + v_i$$

$$A^k v = k\lambda + v$$

λ = reward per time unit, eigenvector v prevents the Mme de Pompadour effect ("After me the diluvium")

Theorem (Max-plus spectral theorem, Cuninghame-Green, 61, Gondran & Minoux 77, Cohen et al. 83)

Assume $G(A)$ is strongly connected. Then

- *the eigenvalue is unique:*

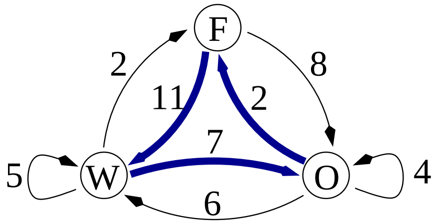
$$\rho_{\max}(A) := \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k}$$

- *Assume WLOG $\rho_{\max}(A) = 0$, then, $\exists \alpha_i \in \mathbb{R} \cup \{-\infty\}$,*

$$u = \max_{j \in \text{maximizing circuits}} \alpha_j + A^*_{\cdot,j}$$

$A^*_{ij} :=$ *max weight path arbitrary length $i \to j$.*

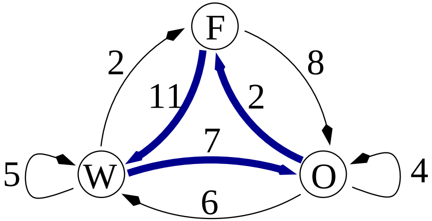
- "$A^{N+c} = \rho_{\max}(A)^c A^N$", $\exists N, c$



F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$

N. Bacaer, C.R. Acad. d'Agriculture de France, 03

Non-compact case: critical circuits are replaced by points of the horofunction boundary [Gromov, Rieffel], see Akian, SG, Walsh, Doc. Math. 09.



F=fallow (no crop), W=wheat, O=oat, $\rho_{\max}(A) = 20/3$
Actually, Bacaer showed that a memory of two years is needed to recover the different historical rotations

Non-compact case: critical circuits are replaced by points of the horofunction boundary [Gromov, Rieffel], see Akian, SG, Walsh, Doc. Math. 09.

Compare with the Perron-Frobenius theorem

Let $A \in \mathbb{R}^{n\times n}$, with $A_{ij} \geq 0 \ \forall i,j$ and assume that $G(A) := \{(i,j) \mid A_{ij} > 0\}$ is strongly connected. Then,

- $\exists u \in \mathbb{R}^n$, $u_i > 0 \ \forall i$, $Au = \rho(A)u$, with $\rho(A) := \max\{|\lambda| \mid \lambda \text{ eigenval. of } A\}$.
- The eigenvalue $\rho(A)$ is algebraically simple (in particular, the eigenvector is unique up to a scaling)
- Let c be the cyclicity of G (=gcd lengths of circuits), then $\rho_{\max}(A)^{-kc}A^{kc}$ converges as k tends to ∞.

2. Zero-sum two player games

v_i^k value in horizon k starting from state i.

$$v^{k+1} = f(v^k), \qquad f : \mathbb{R}^n \to \mathbb{R}^n, \quad n = \sharp \text{states}$$

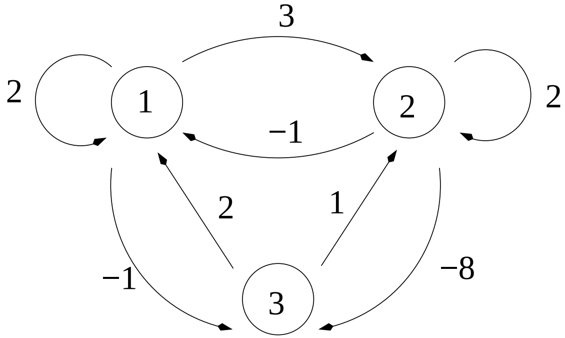
where the **one day operator** f satisfies

$$(M) \qquad x \leq y \implies f(x) \leq f(y)$$
$$(AH) \qquad f(\alpha + x) = \alpha + f(x), \qquad \forall \alpha \in \mathbb{R}$$

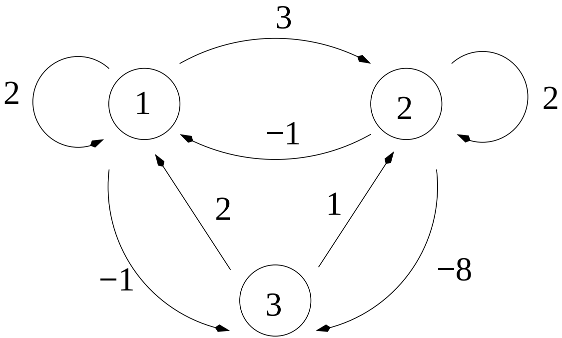
Find $u \in \mathbb{R}^n, \lambda \in \mathbb{R}$,

$$f(u) = \lambda + u \ ?$$

Max and Min flip a coin to decide who makes the move. Min always pay.



$$v_i^{k+1} = \frac{1}{2}(\max_{j:\, i\to j}(c_{ij} + v_j^k) + \min_{j:\, i\to j}(c_{ij} + v_j^k)) \quad .$$



$$\begin{bmatrix}5\\0\\4\end{bmatrix}\quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3}))\\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3}))\\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2}))\end{aligned}$$

this game is fair

$$v = \frac{1}{2}(\max_{j:\, i \to j} v_j^k + \min_{j:\, i \to j} v_j^k) \enspace ,$$

v_i, $i \in$ boundary prescribed:

discrete variant of Laplacian infinity (Oberman), or Richman games (Tug of war).

The analogy explained: non linear Perron-Frobenius theory and log-glasses !

$F : \operatorname{int} \mathbb{R}_+^n \to \operatorname{int} \mathbb{R}_+^n$, order preserving and positively homogeneous (Krein-Rutman).

$$f(x) = \log F(\exp(x)), \qquad \mathbb{R}^n \to \mathbb{R}^n$$

$$F(u) = \mu u \iff f(v) = \lambda + v, \qquad \lambda = \log \mu, v_i = \log u_i \ .$$

f M,AH $\implies$ f is the one day operator of a stochastic game (Kolokoltsov), actually, the game can be deterministic

$$f(x) = \min_{y \in \mathbb{R}^n} f(y) + \max_i (x_i - y_i)$$

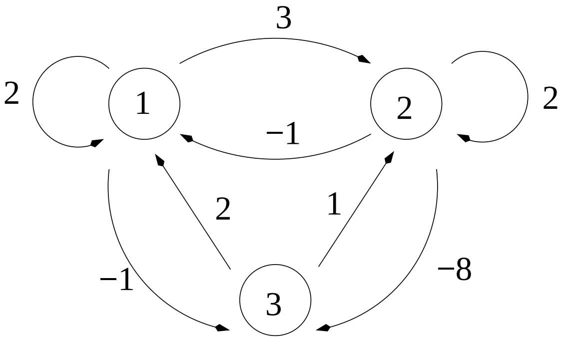
(Singer, Rubinov, Gunawardena, Sparrow).

Define the digraph $G(f)$, with an arc $i \mapsto j$ if $\lim_{t\to\infty} f_i(te_j) = +\infty$.

Theorem (SG, Gunawardena, TAMS'04)

If $G(f)$ is strongly connected, then $\exists v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $f(v) = \lambda + v$.

The Perron-Frobenius theorem, max-plus spectral theorem, Bather's theorem in stochastic control are special cases.



$$\begin{bmatrix}5\\0\\4\end{bmatrix} \quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3}) \\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3}) \\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2}) \end{aligned}$$

this game is fair

Many of the results of Perron-Frobenius/max-plus spectral theory can actually be proved for non-linear maps!

Eg Collatz-Wielandt formula (C closed pointed cone, finite dim):

$$\begin{aligned} \rho(F) &:= \max\{\mu \geq 0 \mid \exists u \in C \setminus \{0\},\ F(u) = \mu u\} \\ &= \max\{\mu \geq 0 \mid \exists u \in C \setminus \{0\},\ F(u) \geq \mu u\} \\ \mathrm{cw}(F) &:= \min\{\mu \geq 0 \mid \exists u \in \operatorname{int} C,\ F(u) \leq \mu u\} \\ &= \inf_{u \in \operatorname{int} C} \max_{1 \leq i \leq n} F_i(u)/u_i \end{aligned}$$

(Nussbaum, LAA 86)

For all $u \in \operatorname{int} \mathbb{R}_+^n$, there is one coordinate i such that

$$F_i^k(u) \geq \rho(F)^k u_i, \qquad \forall k$$

(SG, Gunawardena TAMS'04)

At the heart of the cyclicity/asymptotic results, is the fact that f M,AH is nonexpansive in the sup-norm and in Hilbert's seminorm

$$d(f(x), f(y)) \leq d(x, y) \qquad d(x, y) = \|x - y\|$$

$\|x\| = \max_i x_i$ or $\|x\|_H = \max_i x_i - \min_i x_i$.
A generalization of Denjoy-Wolff theorem (SG, Vigeral, 2010, arxiv): if f is non-expansive, then there exists an horofunction h such that

$$h(f(x)) \geq \rho(f) + f(x) \ ,$$

where $\rho(f) = \lim_k d(x, f^k(x))/k$.

Application to nonnegative tensors:

$$P_i(x) = \lambda x_i^d \ ,$$

$$P_i(x) = \sum_{j \in \mathbb{Z}_+^n} a_{ij_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n} \ , \qquad a_{ij} \geq 0$$

P_i homogeneous of degree d.

Friedland, SG, Han, LAA to appear, arxiv 0905.1626

Any bounded orbit converges to a periodic orbit of length

$$p \leq \max_{q+r+s=N} \frac{N!}{q!r!s!} = \frac{N!}{\lfloor \frac{N}{3} \rfloor ! \lfloor \frac{N+1}{3} \rfloor ! \lfloor \frac{N+2}{3} \rfloor !}$$

(more generally, in a polyhedral cone with N facets).

Akian, SG, Lemmens, Nussbaum, Math. Proc. Camb. Phil. Soc.,06, comes after a long series of works on periodic orbits of nonexpansive maps when the norm is polyhedral: Ackoglu and Krengel, Weller, Martus, Nussbaum, Sine, Scheutzow, Verdyun-Lunel,. . .

3. Deformation of the Perron root

Chain of spins (Ising)

$$Z = \sum_{\sigma_1,\dots,\sigma_n \in \Sigma^N} \exp(-\sum_{i=1}^{N} E(\sigma_i, \sigma_{i+1})/T), \qquad \sigma_{N+1} := \sigma_1$$

$$-E(\sigma,\sigma') = H\sigma + J\sigma\sigma',\ \sigma,\sigma' \in \{\pm 1\} \text{ (Ising)}$$

$$Z_N = \operatorname{tr} M_T^N, \qquad (M_T)_{\sigma\sigma'} = \exp(-E(\sigma,\sigma')/T)$$

$$F_N = N^{-1} T \log Z_N \sim T \log \rho(M_T) \quad \text{free energy per site,}$$

$T \to 0$, ground state

$$\epsilon := \exp(-1/T), \qquad (M_T)_{\sigma,\sigma'} = \epsilon^{E(\sigma,\sigma')}$$

Similar to perturbation problems, but now, the “Puiseux series” have real exponents (Dirichlet series).

Kingman 61:

$$\log \circ \rho \circ \exp \qquad \text{convex [entrywise exp]}$$

Let $A, B \geq 0$, and $C = A^{(s)} \circ B^{(t)}$, with $s + t = 1, s, t \geq 0$ [entrywise product and exponent] then

$$\rho(C) \leq \rho(A)^s \rho(B)^t \quad .$$

Indeed, $\log \rho(C) = \lim_m \log \|C^m\|/m$ is a pointwise limit of convex functions of $(\log C_{ij})$, for any monotone norm. □

So

$$\rho(A \circ B) \leq \rho(A^{(p)})^{1/p} \rho(B^{(q)})^{1/q} \qquad 1/p + 1/q = 1$$

Friedland (86) observed that

$$\rho(B^{(q)})^{1/q} \to \max_{i_1,\ldots,i_m} (B_{i_1 i_2} \cdots B_{i_{m-1} i_m})^{1/m} =: \rho_\infty(B)$$

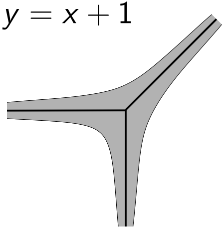
and so for all $A \in \mathbb{C}^{n\times n}$,

$$\rho(A) \leq \rho(\text{pattern}(A))\rho_\infty(|A|) \leq n\rho_\infty(|A|)$$

also

$$\rho(A) \geq \rho_\infty(A) \qquad \text{if } A_{ij} \geq 0 \ .$$

Explanation, special case of amoeba (Gelfand, Kapranov, Zelevinsky). $V \subset (\mathbb{C}^*)^n$,
$A(V) = \{(\log|z_1|, \ldots, \log|z_n|) \mid x \in V$.



Cf. Passare, Rüllgaard; Purbhoo

4. Location of roots of polynomials

Given

$$f(z) = a_0 + a_1 z + \cdots + a_k z^k + \cdots + a_n z^n, \qquad a_i \in \mathbb{C}$$

Let $\zeta_1, \ldots, \zeta_n$ be the solutions of $f(z) = 0$, ordered by $|\zeta_1| \geq \cdots \geq |\zeta_n|$. Bound $|\zeta_i|$? E.g.,

Cauchy (1829)

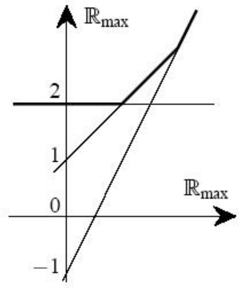
$$|\zeta_1| \leq 1 + \max_{0 \leq k \leq n-1} \frac{|a_k|}{|a_n|} \ .$$

Fujiwara (1916)

$$|\zeta_1| \leq 2 \max_{0 \leq k \leq n-1} \sqrt[n-k]{\frac{|a_k|}{|a_n|}} \ .$$

Tropical polynomial functions. . .

are convex piecewise-linear with nonnegative integer slopes



$$p(x) = \text{“}(-1)x^2 + 1x + 2\text{”} = \max(-1 + 2x, 1 + x, 2)$$

"Fondamental theorem of algebra"

A tropical polynomial function

$$p(x) = \text{``}\sum_{0\leq k\leq n} b_k x^k\text{''} = \max_{0\leq k\leq n} b_k + kx \enspace .$$

can be factored uniquely (Cuninghame-Green & Meijer, 80) as

$$p(x) = \text{``}b_n \prod_{1\leq k\leq n} (x+\alpha_k)\text{''}$$
$$= b_n + \sum_{1\leq k\leq n} \max(x,\alpha_k) \enspace .$$

The points $\alpha_1,\dots,\alpha_n$ are the tropical roots: the maximum is attained twice.

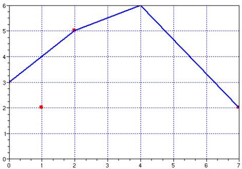
The Newton polygon Δ is the concave hull of the points (k, b_k), $k = 0, \dots, n$.

Proposition

Two formal (tropical) polynomials yield the same polynomial function iff their Newton polygons coincide

Indeed, the function $x \mapsto \max_{0 \le k \le n} b_k + kx$ is the Legendre-Fenchel transform of $k \mapsto -b_k$.

The tropical roots $\alpha_1, \dots, \alpha_k$ are the opposite of the slopes of Δ. They can be computed in $O(n)$ time.



$$p(x) = \max(2 + 7x, 6 + 4x, 5 + 2x, 2 + x, 3)$$
$$= 2 + 2\max(-1, x) + 2\max(-1/2, x) + \max(4/3, x)$$

Associate to $f = a_0 + \cdots + a_n z^n$, $a_i \in \mathbb{C}$, the tropical polynomial

$$p(x) = \max_{0 \le k \le n} \log |a_k| + kx \ .$$

The maximal tropical root is

$$\alpha_1 = \max_{1 \le k \le n-1} \frac{\log |a_k| - \log |a_n|}{n-k}$$

Fujiwara's bound reads

$$|\zeta_1| \le 2 \max_{0 \le k \le n-1} \sqrt[n-k]{\frac{|a_k|}{|a_n|}} \ .$$

Associate to $f = a_0 + \cdots + a_n z^n$, $a_i \in \mathbb{C}$, the tropical polynomial

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Fujiwara's bound reads

$$|\zeta_1| \leq 2 \exp(\alpha_1) \enspace .$$

Geom. interp. of Fujiwara's bound : the open polyhedron

$$\log|z| > \log 2 + \max_{0\leq k\leq n-1} (\log|a_k| - \log|a_n|)/(n-k)$$

in the variables $\log|z|$, $\log|a_k|$, $0 \leq k \leq n$, is included in the complement of the amoeba of $a_0 + \cdots + a_n z^n = 0$ (tought of as an hypersurface of $(\mathbb{C}^*)^{n+2}$ in the variables $a_0, \ldots, a_n, z$).

The components of the complement of an amoeba are convex.

Theorem (Ostrowski, 1940, see also Hadamard, Polya)

$$\frac{1}{C_n^k}\exp(\alpha_1+\cdots+\alpha_k) \leq |\zeta_1\cdots\zeta_k| \leq cst_k \exp(\alpha_1+\cdots+\alpha_k)$$

Corollary

$$cst''_{n,k}\exp(\alpha_k) \leq |\zeta_k| \leq cst'_{n,k}\exp(\alpha_k)$$

Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann (¹);

PAR M. J. HADAMARD.

1. La décomposition d'une fonction entière $F(x)$ en facteurs primaires, d'après la méthode de M. Weierstrass,

$$F(x) = e^{G(x)} \prod_{p=1}^{\infty} \left(1 - \frac{x}{\xi_p}\right) e^{Q_p(x)} \tag{1}$$

a conduit à la notion du genre de la fonction F.

sion géométrique ayant pour premier terme $\frac{1}{|a_{m_1}|}$ et pour dernier $\frac{1}{|a_{m_2}|}$. La raison de cette progression, à savoir $\left|\sqrt[m_2-m_1]{\frac{a_{m_1}}{a_{m_2}}}\right|$, sera plus grande que la précédente, car l'inégalité

$$\left|\sqrt[m_2-m_0]{\frac{a_{m_2}}{a_{m_0}}}\right| < \left|\sqrt[m_1-m_0]{\frac{a_{m_1}}{a_{m_0}}}\right|$$

ou

$$\left|\frac{a_{m_0}}{a_{m_1}}\right|^{m_1-m_0}\left|\frac{a_{m_2}}{a_{m_2}}\right|^{m_1-m_0} > \left|\frac{a_{m_0}}{a_{m_1}}\right|^{m_2-m_0}$$

peut s'écrire

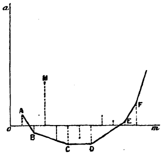
$$\left|\sqrt[m_2-m_1]{\frac{a_{m_1}}{a_{m_2}}}\right| > \left|\sqrt[m_1-m_0]{\frac{a_{m_0}}{a_{m_1}}}\right|.$$

On considérera ensuite la valeur m_3 de m pour laquelle $\left|\sqrt[m-m_2]{\frac{a_m}{a_{m_2}}}\right|$ sera le plus grand, et l'on continuera ainsi indéfiniment.

4. Au reste, ces opérations peuvent se ramener à la construction bien connue du polygone de Newton.

Pour cela on considérera m comme l'abscisse d'un point M (*fig.* 1) dont l'ordonnée sera fournie par la valeur correspondante de $\mathrm{L}\left|\frac{1}{a_m}\right|$.

Fig. 1.



Nous aurons ainsi une suite indéfinie de points représentant les différents coefficients de notre série.

Prenons alors une demi-droite, tout d'abord parallèle à la partie

Theorem (Ostrowski, 1940, see also Hadamard, Polya)

$$\frac{1}{C_n^k}\exp(\alpha_1+\cdots+\alpha_k) \leq |\zeta_1\cdots\zeta_k| \leq cst_k \exp(\alpha_1+\cdots+\alpha_k)$$

Corollary

$$cst''_{n,k}\exp(\alpha_k) \leq |\zeta_k| \leq cst'_{n,k}\exp(\alpha_k)$$

Ostrowski: $\text{cst}_k \leq 2k+1$ (hidden in his memoir on the Graeffe's method (1940), the "numerical newton polygon" is Δ). Early (simpler) instance of Viro's patchworking.

Hadamard: $\text{cst}_k \leq k+1$ (1891)

Polya: $\text{cst}_k < e\sqrt{k+1}$ (reproduced in Ostrowski).

Specht: $|\zeta_1 \cdots \zeta_k| \leq (k+1)\exp(k\alpha_1)$ (1938, weaker!), followup by Mignotte and Moussa.

Application to scaling of matrix pencils

$$P(\lambda) = \lambda^2 10^{-18} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 & 10 \\ 16 & 45 \end{pmatrix} + 10^{-18} \begin{pmatrix} 12 & 15 \\ 34 & 28 \end{pmatrix}$$

- Apply the QZ algorithm to the companion form of $P(\lambda)$ Matlab (7.3.0) [similar in Scilab]
- We get: $-Inf, -7.731e-19, Inf, 3.588e-19$
- Scaling of Fan, Lin and Van Dooren (2004): $-Inf, Inf, -3.250e-19, 3.588e-19$
- tropical scaling, cf. F. Tisseur talk this morning $-7.250E-18 \pm 9.744E-18i$, $-2.102E+17 \pm 7.387E+17i$ the correct answer (agrees with `Pari`).

Application to scaling of matrix pencils

$$P(\lambda) = A_0 + A_1\lambda + \cdots + A_d\lambda^d, \qquad A_k \in \mathbb{C}^{n\times n}$$

Considering the tropical polynomial

$$p(x) = \max_{0\leq m\leq d}(\log \|A_m\| + mx)$$

with tropical roots α_i,

If each of the matrices A_k is well conditioned, we expect precisely n roots of order $\exp(\alpha_i)$, for all $1 \leq i \leq d$.

Substitute $\lambda = \exp(\alpha_i)\mu$, rescale

$$\begin{aligned}\tilde{P}(\mu) &= \exp(-p(\alpha_i))P(\lambda) \\ &= \tilde{A}_0 + \tilde{A}_1\mu + \cdots + \tilde{A}_d\mu^d \\ \tilde{A}_k &= \exp(k\alpha_i - p(\alpha_i))A_k\end{aligned}$$

For at least two indices r, s (belonging to the edge of the Netwon polygon corresponding to α_i)

$$\|\tilde{A}_r\| = \|\tilde{A}_s\| = 1$$

and

$$\|\tilde{A}_k\| \leq 1, \; \textit{for } k \neq r, s$$

Idea: perform such a scaling for each α_i, QZ is expected to compute accurately the group of eigenvalues of order $\exp(\alpha_i)$.

Ex, for

$$P(\lambda) = \lambda^2 10^{-18} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 & 10 \\ 16 & 45 \end{pmatrix} + 10^{-18} \begin{pmatrix} 12 & 15 \\ 34 & 28 \end{pmatrix}$$

two tropical eigenvalues, approx $-18 \log 10$ and $18 \log 10$. We called QZ once for each tropical eigenvalue, that's how we got the four complex eigenvalues:

$-7.250E - 18 \pm 9.744E - 18i$,

$-2.102E + 17 \pm 7.387E + 17i$

In the quadratic case, Fan, Lin and Van Dooren (2004) proposed a scaling with a unique call to QZ which coincides with our only when the two tropical roots coincide. When these are far away from each other, a single scaling cannot work!

Tropical splitting of eigenvalues

Definition (eigenvalue variation)

Let $\lambda_1, \dots \lambda_n$ and $\mu_1 \dots \mu_n$ denote two sequences of complex numbers. The variation between λ and μ is defined by

$$v(\lambda, \mu) := \min_{\pi \in S_n} \{\max_i |\mu_{\pi(i)} - \lambda_i|\} \ ,$$

where S_n is the set of permutations of $\{1, 2, \dots, n\}$.
If $A, B \in \mathbb{C}^{n \times n}$, the eigenvalue variation of A and B is defined by $v(A, B) := v(\operatorname{spec} A, \operatorname{spec} B)$.

Theorem (quadratic case, SG, Sharify POSTA 09)

Let $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0, \quad A_i \in \mathbb{C}^{n\times n}$, $\alpha^+ > \alpha^-$ *tropical roots,* $\delta := \exp(\alpha^+ - \alpha^-)$*. Let* $\zeta_1, \dots, \zeta_n$ *denote the eigenvalues of the pencil* $\lambda A_2 + A_1$*, and* $\zeta_{n+1} = \cdots = \zeta_{2n} = 0$*. Then,*

$$v(\operatorname{spec} P, \zeta) \leq \frac{C\alpha^+}{\delta^{1/2n}}$$

where

$$C := 4\times 2^{-1/2n}\Big(2+2\operatorname{cond} A_2+\frac{\operatorname{cond} A_2}{\delta}\Big)^{1-1/2n}\Big(\operatorname{cond} A_2\Big)^{1/2n}$$

$$\alpha^+(\operatorname{cond} A_1)^{-1} \leq |\zeta_i| \leq \alpha^+ \operatorname{cond} A_2, \qquad 1 \leq i \leq n$$

So, there are precisely n eigenvalues of the order of the maximal tropical root if

- δ (measuring the separation between the two tropical roots) is sufficiently large, and
- the matrices A_2, A_1 are well conditioned,

Under the dual assumption (A_0, A_1 well conditioned), there are precisely n eigenvalues of the order of the minimal tropical root.

- Proof relies on Bathia, Elsner, and Krause (1990):

$$\nu(\operatorname{spec} A, \operatorname{spec} B) \leq 4 \times 2^{-1/n}(\|A\| + \|B\|)^{1-1/n}\|A - B\|^{1/n}$$

Experimental results

To estimate the accuracy of computing an eigenpair, we consider the normwise backward error (Tisseur 1999)

$$\eta(\tilde{x}, \tilde{\lambda}) = \min\{\epsilon : (P(\tilde{\lambda}) + \Delta P(\tilde{\lambda}))\tilde{x} = 0, \|\Delta A_l\|_2 \leq \epsilon \|E_l\|_2\}$$

$$\eta(\tilde{x}, \tilde{\lambda}) = \frac{\|r\|_2}{\tilde{\alpha}\|\tilde{x}\|_2}$$

where $r = P(\tilde{\lambda})\tilde{x}$, $\tilde{\alpha} = \sum |\tilde{\lambda}|^l \|E_l\|_2$ and the matrices E_l represent tolerances

Backward error for quadratic pencils

- η: no scaling, η_s Fan, Lin, and Van Dooren (2004), η_t tropical
- Backward error for the 5 smallest eigenvalues of 100 randomly generated quadratic pencils,
  - $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0 \quad A_i \in \mathbb{C}^{10\times 10}$
  - $\|A_2\|_2 \approx 5.54 \times 10^{-5}$, $\|A_1\|_2 \approx 4.73 \times 10^{3}$, $\|A_0\|_2 \approx 6.01 \times 10^{-3}$

$\lvert\lambda\rvert$	$\eta(\zeta,\lambda)$	$\eta_s(\zeta,\lambda)$	$\eta_t(\zeta,\lambda)$
2.98E-07	1.01E-06	5.66E-09	6.99E-16
5.18E-07	1.37E-07	8.48E-10	2.72E-16
7.38E-07	5.81E-08	4.59E-10	2.31E-16
9.53E-07	3.79E-08	3.47E-10	2.08E-16
1.24E-06	3.26E-08	3.00E-10	1.98E-16

Backward error for a matrix pencil with an arbitrary degree

Backward error for 20 randomly generated matrix pencils

- $P(\lambda) = \lambda^5 A_5 + \lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0 \quad A_i \in \mathbb{C}^{20\times 20}$
- $\|A_5\|_2 \approx 10^5, \|A_4\|_2 \approx 10^{-4}, \|A_3\|_2 \approx 10^{-1}, \|A_2\|_2 \approx 10^2, \|A_1\|_2 \approx 10^2, \|A_0\|_2 \approx 10^{-3}$

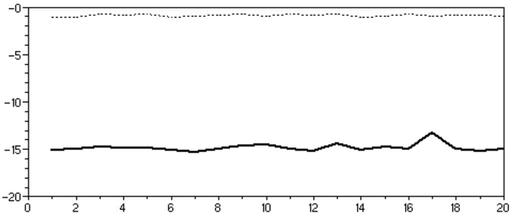


Figure: Backward error before and after scaling for the smallest eigenvalue

Backward error for a matrix pencil with an arbitrary degree

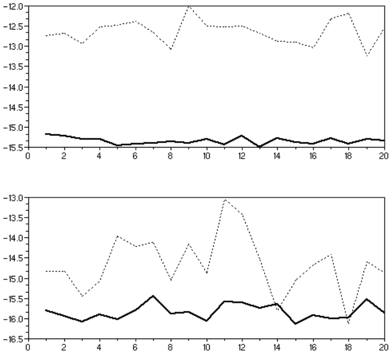


Figure: Backward error before and after scaling for the “central” 50^{th} eigenvalue and the maximum one from top to down

- The tropical roots could be used as a warning, if they are too separated, the “naive” computations are likely to be inaccurate.
- What precedes is suboptimal: calls $O(d)$ times QZ (number of times equal to the different orders of tropical eigenvalues), so the execution time can be slowed down by a factor d.
- If the matrices A_i are badly conditioned, we cannot estimate the eigenvalues based only on the norms $\|A_i\|$ ... but then we can use a finer estimation, the tropical eigenvalues...

5. Lidskiĭ, Višik, Ljusternik perturbation theory

Theorem (Lidskiĭ 65; also Višik, Ljusternik 60)

Let $a \in \mathbb{C}^{n\times n}$ be nilpotent, with m_i Jordan blocks of size ℓ_i. For a generic perturbation $b \in \mathbb{C}^{n\times n}$, the matrix

$$a + \epsilon b$$

has precisely $m_i \ell_i$ eigenvalues of order ϵ^{1/ℓ_i} as $\epsilon \to 0$.

See survey Moro, Burke, Overton, SIMAX 97 or Baumgärtel's book

$$
a = \left[\begin{array}{ccc|ccc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
$$

6 eigenvalues $\sim \omega\epsilon^{1/3}$, $\omega^3 = \lambda$, λ eigenvalue of

$$
\begin{bmatrix} b_{31} & b_{34} \\ b_{61} & b_{64} \end{bmatrix}
$$

$$a = \left[\begin{array}{ccc|ccc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

2 eigenvalues $\sim \omega\epsilon^{1/2}$, $\omega^2 = \lambda$,

$$\lambda = b_{87} - \begin{bmatrix} b_{81} & b_{84} \end{bmatrix} \begin{bmatrix} b_{31} & b_{34} \\ b_{61} & b_{64} \end{bmatrix}^{-1} \begin{bmatrix} b_{37} \\ b_{67} \end{bmatrix}$$

$$
a = \left[\begin{array}{ccc|ccc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot}
\end{array}\right]
$$

1 eigenvalue $\sim \lambda\epsilon$,

$$
\lambda = b_{99} - \begin{bmatrix} b_{91} & b_{94} & b_{97} \end{bmatrix} \begin{bmatrix} b_{31} & b_{34} & b_{37} \\ b_{61} & b_{64} & b_{67} \\ b_{81} & b_{84} & b_{87} \end{bmatrix}^{-1} \begin{bmatrix} b_{39} \\ b_{69} \\ b_{89} \end{bmatrix}
$$

Lidskiĭ's approach does not give the correct orders in degenerate cases. . .

If the matrix

$$\begin{bmatrix} b_{31} & b_{34} \\ b_{61} & b_{64} \end{bmatrix}$$

has a zero-eigenvalue, then, $a + \epsilon b$ has less than 6 eigenvalues of order $\epsilon^{1/3}$.

Moreover, the Schur complement

$$b_{87} - \begin{bmatrix} b_{81} & b_{84} \end{bmatrix} \begin{bmatrix} b_{31} & b_{34} \\ b_{61} & b_{64} \end{bmatrix}^{-1} \begin{bmatrix} b_{37} \\ b_{67} \end{bmatrix}$$

is not defined, and there may be no eigenvalue of order $\epsilon^{1/2}$

Finding the correct order of magnitude of all eigenvalues (Puiseux series) $\iff$ characterizing (combinatorially) the Newton polygon of the curve

$$\{(\lambda, \epsilon) \mid \det(a + \epsilon b - \lambda I) = 0\}$$

open problem.

This talk: tropical algebra yields the correct order of magnitudes, in degenerate cases for Lidskiı (new degenerate cases appear but of a higher order).

The (algebraic) tropical eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are the roots of

$$\text{“}\operatorname{per}(A + xI)\text{”}$$

where

$$\text{“}\operatorname{per}(M)\text{”} := \text{“}\sum_{\sigma\in S_n} \prod_{i\in[n]} M_{i\sigma(i)}\text{”}$$

All geom. eigenvalues λ (“$Au = \lambda u$”) are algebraic eigenvalues, but the converse does not hold. $\rho_{|max}(A)$ is the max algebraic eigenvalue.

The (algebraic) tropical eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are the roots of

$$\text{“}\mathrm{per}(A + xI)\text{”}$$

where

$$\text{“}\mathrm{per}(M)\text{”} := \max_{\sigma\in S_n} \sum_{i\in[n]} M_{i\sigma(i)}$$

- All geom. eigenvalues λ (“$Au = \lambda u$”) are algebraic eigenvalues, but the converse does not hold. $\rho_{|max}(A)$ is the max algebraic eigenvalue.
- Trop. eigs. can be computed in $O(n)$ calls to an optimal assignment solver (Butkovič and Burkard) (not known whether the formal characteristic polynomial can be computed in polynomial time).

Majorization inequalities

We associate to $a+\epsilon b$, $a,b\in\mathbb{C}^{n\times n}$ the matrix $A=v(a+\epsilon b)\in\mathbb{R}_{\max}^{n\times n}$, i.e.,

$$A_{ij}=\begin{cases}0 & \text{if } a_{ij}\neq 0\\ -1 & \text{if } a_{ij}=0,\, b_{ij}\neq 0\\ -\infty & \text{otherwise}\end{cases}$$

Let $\gamma_1\geq\cdots\geq\gamma_n$ trop. eigs., and let $\mathcal{L}_1(\epsilon),\ldots,\mathcal{L}_n(\epsilon)$ denote the eigenvalues of $a+\epsilon b$, $v(\mathcal{L}_1)\geq\cdots\geq v(\mathcal{L}_n)$.

Theorem (Akian, Bapat, SG, combining CRAS 2004 and arXiv:0402090)

$$v(\mathcal{L}_1)+\cdots+v(\mathcal{L}_k)\leq\gamma_1+\cdots+\gamma_k$$

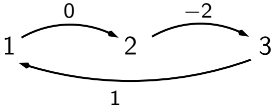
and $=$ for generic values of b.

Retrospectively, the bounds on the modulus of polynomial roots appear as “log-majorization” inequalities.

Not only the valuations of the eigenvalues, but their leading coefficients can be obtained: Akian, Bapat, SG CRAS 2004…

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} , \qquad A = \begin{bmatrix} 1 & 0 & 4 \\ \infty & 1 & -2 \\ 1 & 2 & \infty \end{bmatrix} .$$

We have $\gamma_1 = -1/3$, corresponding to the critical circuit:



Eigenvalues:

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

The idea

Replace

$$\det(a + \epsilon b - \lambda I)$$

by

$$\operatorname{diag}(\epsilon^{-U})(a + \epsilon b - \epsilon^{\gamma}\mu)\operatorname{diag}(\epsilon^{-V}) \to \text{nonsingular limit}(\mu)$$

How to find the scaling U, V, μ?

Dual variables

$A(\gamma) =$ "$A + \gamma I$". The dual of the linear programming formulation of the optimal assignment problem reads:

$$\text{"per } A(\gamma)\text{"} = \min \sum_i U_i + \sum_j V_j ; \quad A(\gamma)_{ij} \leq U_i + V_j \ .$$

Let U, V be "Hungarian" (optimal dual) variables. By complementary slackness, a permutation σ is optimal iff it is supported by $G^s := \{(i,j) \mid A_{ij}(\gamma) = U_i + V_j\}$.

$$(a+\epsilon b)_{ij} \sim c_{ij}\epsilon^{-v(A_{ij})}$$

$c_{ij} = a_{ij}$ if $v(A_{ij}) = 0$, $c_{ij} = b_{ij}$ if $v(A_{ij}) = -1$, $c_{ij} = 0$ otherwise.

$G^0 = \{(i,j) \in G^s \mid \text{“}(A+\gamma I)_{ij}\text{”} = A_{ij}\}$,
$G^1 = \{(i,i) \in G^s \mid \text{“}(A+\gamma I)_{ii}\text{”} = \gamma\}$

$(c^G)_{ij} := c_{ij}$ if $(i,j) \in G$, 0 otherwise.

Idea: $A(\gamma)_{ij} \leq U_i + V_j$ implies, as $\epsilon \to 0$,

$$\mathrm{diag}(\epsilon^{-U})(a+\epsilon b - \epsilon^{\gamma}\mu)\,\mathrm{diag}(\epsilon^{-V}) \to c^{G^0} - \mu I^{G^1} \quad .$$

Theorem (Akian, Bapat, SG CRAS 2004)

If the pencil $c^{G^0} - \mu I^{G^1}$ determined from the optimal dual variables for the tropical eigenvalue γ has m non-zero eigenvalues $\lambda_1, \dots, \lambda_m$, then $a + b\epsilon$ has m eigenvalues $\sim \lambda_i \epsilon^\gamma$, and all the other ones are either $o(\epsilon^\gamma)$ or $\omega(\epsilon^\gamma)$.

Generically, $m =$ tropical multiplicity of γ, so we get the eigenvalues.

$\det(c^{G^0} - \mu I^{G^1})$ indep. of the choice of optimal dual variables (only optimal permutations matter).

Murota 90, alternative algorithmic approach: "combinatorial relaxation".

This extends Lidskiĭ's theorem: when a is nilpotent in Jordan form, $-1/\ell_i$ are precisely the tropical eigenvalues, and the boxes in

$$\left[\begin{array}{ccc|ccc|cc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot} \\
\hline
\boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \cdot & \boxed{\cdot} & \cdot & \boxed{\cdot}
\end{array}\right]$$

correspond to the union of the saturation digraphs for the different tropical eigenvalues.

This theorem also extends: Ma, Edelman 98; Najman 99

This explains why some attempts to extend Lidskiĭ failed:

the perturbed eigenvalues are controlled by pencils. . .

even if the original problem is a standard eigenvalue problem (not all eigenvalues of pencils can be expressed as eigenvalues of Schur complements).

Example

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22}\epsilon & b_{23}\epsilon \\ b_{31} & b_{32}\epsilon & b_{33}\epsilon \end{bmatrix}$$

$$A(x) = \begin{bmatrix} 0 \oplus x & 0 & 0 \\ 0 & -1 \oplus X & -1 \\ 0 & -1 & -1 \oplus X \end{bmatrix}$$

$$P_A(x) = (x \oplus 0)^2 (x \oplus -1) \quad .$$

Tropical roots: $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = -1$.

If $\gamma = 0$, then $U = V = (0, 0, 0)$

$$A(0) = \begin{bmatrix} 0_{01} & 0_0 & 0_0 \\ 0_0 & 0_1 & 1 \\ 0_0 & 1 & 0_1 \end{bmatrix} ,$$

(0 and 1 subscripts correspond to G^0 and G^1).

$$\det \begin{bmatrix} b_{11} - \lambda & b_{12} & b_{13} \\ b_{21} & -\lambda & 0 \\ b_{31} & 0 & -\lambda \end{bmatrix} = \lambda(-\lambda^2 + \lambda b_{11} + b_{12} b_{21} + b_{31} b_{31})$$

The theorem predicts that this equation has, for generic values of the parameters b_{ij}, two non-zero roots, λ_1, λ_2, which yields two eigenvalues of $a + \epsilon b$, $\sim \lambda_m \epsilon^0 = \lambda_m$, for $m = 1, 2$.

Tropical eigenvalue $\gamma = -1$. $U = (0, -1, -1)$, $V = (1, 0, 0)$,

$$A(1) = \begin{bmatrix} 0 & 0_0 & 0_0 \\ 0_0 & -1_{01} & -1_0 \\ 0_0 & -1_0 & -1_{01} \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & b_{22} - \lambda & b_{23} \\ b_{31} & b_{31} & b_{33} - \lambda \end{bmatrix} = 0 \ .$$

This yields $\lambda(b_{12}b_{21} + b_{13}b_{31}) + b_{12}b_{23}b_{31} + b_{13}b_{32}b_{21} - b_{21}b_{12}b_{33} - b_{31}b_{13}b_{22} = 0$. The theorem predicts that this equation has generically a unique nonzero root, λ_1, and that $a + \epsilon b$ has a third eigenvalue $\sim \lambda_1 \epsilon$.

Current work with Akian and Sharify. When the A_i are ill conditioned, the scaling based on $\|A_i\|$ does not help to compute the eigenvalues of:

$$P(\lambda) = A_0 + A_1\lambda + \cdots + A_d\lambda^d, \qquad A_k \in \mathbb{C}^{n\times n}$$

Reduce to companion form $A - \lambda B$, take now $\log|\cdot|$ as the valuation, and perform the scaling

$$\mathrm{diag}(\exp(-U))(A - \exp(\gamma)\mu B)\,\mathrm{diag}(\exp(-V))$$

Reduces experimentally the backward error, prove it!

Conclusion.

Simpler results in the non-archimedian case (Puiseux series).

Much remains to do in the case of log-glasses:

- finer location of the spectrum
- **numerical applications**: can make accurate (and certified?) computations with few digits.

Thank you!