Symmetry-Preserving Structure-Preserving Equivalences for the Quadratic Eigenvalue Problem

.. 2 open problems.

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A first *System*.

$$Q_1(\alpha,\beta) := \alpha^2 M_1 + \alpha\beta D_1 + \beta^2 K_1$$

System matrices $\{K_1, D_1, M_1\}$ are real and $(n \times n)$

No assumption that either K_1 or M_1 invertible but we do demand that $[K_1 \;\; D_1 \;\; M_1]$ full rank.

Homogeneous coordinates approach [1] allows eigenvalues to be considered to be $\lambda := \alpha/\beta$.

[1] Higham NJ & Tisseur F. 'More on pseudospectra for polynomial eigenvalue problems and applications in control theory'. *Linear Algebra & Appl<u>n</u>s*, 2001, 351-352(4), pp435-453

A second *System*.

$$Q_2(\alpha,\beta) := \alpha^2 M_2 + \alpha\beta D_2 + \beta^2 K_2$$

System matrices $\{K_2, D_2, M_2\}$ are real and $(n \times n)$

We outline the conditions that $\{K_1, D_1, M_1\}$ and $\{K_2, D_2, M_2\}$ are *isospectral* (share the same Jordan form).

Quantities of double dimension.

Underlined matrices, $\underline{A}$, are $(2n \times 2n)$

Underlined "vectors", $\underline{v}$, are $(2n \times 2)$

Define L.A.M.s

$$\underline{M_1} := \begin{bmatrix} 0 & K_1 \\ K_1 & D_1 \end{bmatrix}, \quad \underline{D_1} := \begin{bmatrix} K_1 & 0 \\ 0 & -M_1 \end{bmatrix}, \quad \underline{K_1} := \begin{bmatrix} -D_1 & -M_1 \\ -M_1 & 0 \end{bmatrix}$$

$$\underline{M_2} := \begin{bmatrix} 0 & K_2 \\ K_2 & D_2 \end{bmatrix}, \quad \underline{D_2} := \begin{bmatrix} K_2 & 0 \\ 0 & -M_2 \end{bmatrix}, \quad \underline{K_2} := \begin{bmatrix} -D_2 & -M_2 \\ -M_2 & 0 \end{bmatrix}$$

¥ Mackey, D. S.; Mackey, N.; Mehl, C; Mehrmann, V. 'Vector Spaces of Linearizations for Matrix Polynomials'
SIAM J. of Matrix Analysis & Appl<u>n</u>s, 2006, 28(4), pp971-1004

Isospectral Systems

Systems $\{K_1, D_1, M_1\}$ and $\{K_2, D_2, M_2\}$ are *isospectral* (share the Jordan form) if and only if there exist invertible matrices $\underline{T_L}^T, \underline{T_R}$ such that:

$$\underline{T_L}^T \underline{M_1} \underline{T_R} = \underline{M_2}$$

$$\underline{T_L}^T \underline{D_1} \underline{T_R} = \underline{D_2}$$

$$\underline{T_L}^T \underline{K_1} \underline{T_R} = \underline{K_2}$$

Parameterising using K_1, D_1, M_1

Matrices $\underline{T_L}^T, \underline{T_R}$ describing the SPE have the form

$$\underline{T_R} = \begin{bmatrix} \left(F_R - \frac{1}{2}G_R D_1\right) & \left(-G_R M_1\right) \\ \left(G_R K_1\right) & \left(F_R + \frac{1}{2}G_R D_1\right) \end{bmatrix}^{-1}$$

$$\underline{T_L} = \begin{bmatrix} \left(F_L - \frac{1}{2}G_L D_1{}^T\right) & \left(-G_L M_1{}^T\right) \\ \left(G_L K_1{}^T\right) & \left(F_L + \frac{1}{2}G_L D_1{}^T\right) \end{bmatrix}^{-1}$$

where $\{F_L, F_R, G_L, G_R\}$ are real $(n \times n)$ parameter matrices obeying:

$$F_R G_L{}^T + G_R F_L{}^T = 0$$

Parameterising using K_2, D_2, M_2

Matrices $\underline{T_L}^T, \underline{T_R}$ describing the SPE have the form

$$\underline{T_R} = \begin{bmatrix} \left(\tilde{F}_R - \tfrac{1}{2}\tilde{G}_R D_2\right) & \left(-\tilde{G}_R M_2\right) \\ \left(\tilde{G}_R K_2\right) & \left(\tilde{F}_R + \tfrac{1}{2}\tilde{G}_R D_2\right) \end{bmatrix}$$

$$\underline{T_L} = \begin{bmatrix} \left(\tilde{F}_L - \tfrac{1}{2}\tilde{G}_L D_2{}^T\right) & \left(-\tilde{G}_L M_2{}^T\right) \\ \left(\tilde{G}_L K_2{}^T\right) & \left(\tilde{F}_L + \tfrac{1}{2}\tilde{G}_L D_2{}^T\right) \end{bmatrix}$$

where $\{\tilde{F}_L, \tilde{F}_R, \tilde{G}_L, \tilde{G}_R\}$ are real ($n \times n$) parameter matrices obeying: $\tilde{F}_R \tilde{G}_L{}^T + \tilde{G}_R \tilde{F}_L{}^T = 0$

Open Problem: Explain why $\tilde{G}_L \equiv G_R$ and $\tilde{G}_R \equiv G_L$ *?*

All Symmetry Cases for the QEP !

Symmetry case:	Additional Constraints:
$K_1 = K_1^T, \quad D_1 = D_1^T, \quad M_1 = M_1^T.$	$F_L = F_R, \quad G_L = G_R$
$K_1 = K_1^T, \quad D_1 = -D_1^T, \quad M_1 = M_1^T.$	$F_L = F_R, \quad G_L = -G_R$
$K_1 = -K_1^T, \quad D_1 = D_1^T, \quad M_1 = -M_1^T.$	$F_L = F_R, \quad G_L = G_R$
$K_1 = -K_1^T, \quad D_1 = -D_1^T, \quad M_1 = -M_1^T$	$F_L = F_R, \quad G_L = -G_R$
$K_1 = M_1$	$G_L = 0, \quad G_R = 0$
$K_1 = -M_1, \quad D_1 = 0$	$G_L = 0, \quad G_R=0$
$K_1 = M_1^T, \quad D_1 = D_1^T$	$F_L = F_R, \quad G_L = G_R$
$K_1 = -M_1^T, \quad D_1 = -D_1^T$	$F_L = F_R, \quad G_L = -G_R$

The QEP in Dilated Form.

Theorem: The following two statements are equivalent:

(1) $$\left(\alpha^2 M_1 + \alpha\beta D_1 + \beta^2 K_1\right)v = 0$$

(2) For some scalars $\{\mu, \delta, \kappa\}$ not all zero

$$\left(\mu\begin{bmatrix} 0 & K_1 \\ K_1 & D_1 \end{bmatrix} + \delta\begin{bmatrix} K_1 & 0 \\ 0 & -M_1 \end{bmatrix} + \kappa\begin{bmatrix} -D_1 & -M_1 \\ -M_1 & 0 \end{bmatrix}\right)\begin{bmatrix} \alpha v \\ \beta v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\left(\mu\alpha^2 + \delta\alpha\beta + \kappa\beta^2\right) = 0$$

The QEP in Dilated Form.

It follows (after a little more work) that if you can find any scalar triple $\{\mu, \delta, \kappa\}$ such that

$$\left(\mu \underline{M_1} + \delta \underline{D_1} + \kappa \underline{K_1}\right)\underline{v} = 0$$

with rank($\underline{v}$)=2, then $\{\mu, \delta, \kappa\}$ implicitly defines a <u>pair</u> of eigenvalues of the system and $\underline{v}$ spans the subspace containing the corresponding two *long* eigenvectors.

The equation above is *the dilated form* of the QEP.

GEOMETRY evident in the QEP !

It is interesting that

$$\left(\begin{bmatrix} 0 & K_1 \\ K_1 & D_1 \end{bmatrix} \times \mu + \begin{bmatrix} K_1 & 0 \\ 0 & -M_1 \end{bmatrix} \times \delta + \begin{bmatrix} -D_1 & -M_1 \\ -M_1 & 0 \end{bmatrix} \times \kappa \right)$$

$$= \begin{bmatrix} (\delta K_1 - \kappa D_1) & (\mu K_1 - \kappa M_1) \\ (\mu K_1 - \kappa M_1) & (\mu D_1 - \delta M_1) \end{bmatrix}$$

$$= -\left(\begin{bmatrix} 0 & \kappa \\ \kappa & \delta \end{bmatrix} \otimes M_1 + \begin{bmatrix} \kappa & 0 \\ 0 & -\mu \end{bmatrix} \otimes D_1 + \begin{bmatrix} -\delta & -\mu \\ -\mu & 0 \end{bmatrix} \otimes K_1 \right)$$

There are strong flavours of *cross-product* (exterior product) about this!

Open Problem: What significance does this have?