# Stratification of Full Rank Polynomial Matrices

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(ongoing work)

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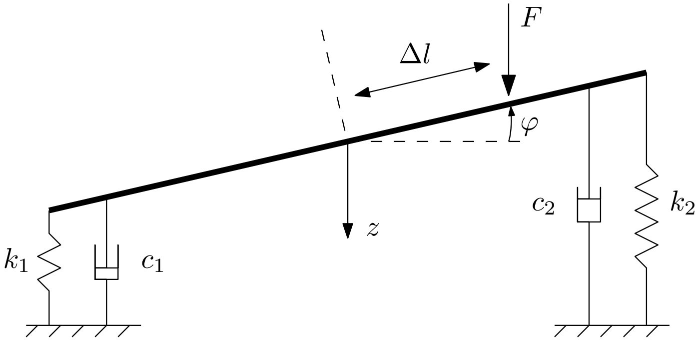
Umeå Universitet

Large Graphs and Networks
UCL Université catholique de Louvain

Swedish Foundation for Strategic Research




A uniform platform with mass *m* and length 2*l*, supported in both ends by springs

The control parameter of the system is the force *F* applied at distance Δ*l* from the center of the platform

Consider dynamical systems described by sets of algebraic-differential equations:

$$P_d x^{(d)}(t) + \cdots + P_1 x^{(1)}(t) + P_0 x(t) = f(t), \quad P_i \in \mathbb{C}^{m \times n}$$

Taking the Laplace transform yields the algebraic equation

$$P(s)\hat{x}(s) = \hat{f}(s) \quad \text{with} \quad P(s) := P_d s^d + \cdots + P_1 s + P_0$$

We study linearizations of $P(s)$ with full normal rank $r$ ($r = m$ or $r = n$)

*Goal:*

Derive stratification rules for *full normal rank polynomial matrices* $P(s)$

The *right linearization* of a $m \times n$ polynomial matrix $P(s)$ has the form

$$sB_r + A_r := s \begin{bmatrix} I_m & & & \\ & \ddots & & \\ & & I_m & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & & & P_0 \\ -I_m & \ddots & & P_1 \\ & \ddots & 0 & \vdots \\ & & -I_m & P_{d-1} \end{bmatrix}$$

If $P_d$ has full row rank $m$ this is "equivalent" to :

$$\mathbf{S}_{\mathrm{C}}(\lambda) = s \begin{bmatrix} I_{md} & 0 \end{bmatrix} + \begin{bmatrix} A & B \end{bmatrix}$$

associated with the matrix pair $(A, B)$

The *left linearization* of a $m \times n$ polynomial matrix $P(s)$ has the form

$$sB_l + A_l := s\begin{bmatrix} I_n & & & \\ & \ddots & & \\ & & I_n & \\ & & & P_d \end{bmatrix} + \begin{bmatrix} 0 & -I_n & & \\ & \ddots & \ddots & \\ & & 0 & -I_n \\ P_0 & P_1 & \ldots & P_{d-1} \end{bmatrix}$$

If $P_d$ has full column rank $n$ this is "equivalent" to :

$$\mathbf{S}_C(\lambda) = s\begin{bmatrix} I_{nd} \\ 0 \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}$$

associated with the matrix pair $(A, C)$

# Linearization

For monic scalar polynomials $p(s)$ we retrieve the first and second companion forms

$$sI_d + \begin{bmatrix} 0 & & & p_0 \\ -1 & \ddots & & p_1 \\ & \ddots & 0 & \vdots \\ & & -1 & p_{d-1} \end{bmatrix}$$

and

$$sI_d + \begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & 0 & -1 \\ p_0 & p_1 & \dots & p_{d-1} \end{bmatrix}$$

# Linearization

It is well known that the "structure" of the companion forms (i.e. its characteristic polynomial) is $p(s)$.

But is the "structure" of a general polynomial matrix $P(s)$ also that of its linearizations $sB_l + A_l$ and $sB_r + A_r$ ?

More importantly, can we study perturbations of $P(s)$ via (arbitrary) perturbations of $sB_l + A_l$ and $sB_r + A_r$ ?

We will see that we have to impose conditions for this !

Any matrix pencil $sB + A$ or system pencil $\mathbf{S}(s)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ($U$ and $V$ non-singular):

$$U^{-1}(\mathbf{S}(s))V = \mathrm{diag}(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, J(\mu_1), \ldots, J(\mu_t), N_{s_1}, \ldots, N_{s_k}, L^T_{\eta_1}, \ldots, L^T_{\eta_q})$$

# Kronecker canonical form

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Singular part:

- $L_{\epsilon_1}, \dots, L_{\epsilon_p}$ – Right singular blocks
- $L^T_{\eta_1}, \dots, L^T_{\eta_q}$ – Left singular blocks

$$L_k = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}$$

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$$J_k(\mu_i) = \begin{bmatrix} \mu_i - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_i - \lambda \end{bmatrix}$$

Singular part:

- $L_{\epsilon_1}, \ldots, L_{\epsilon_p}$ – Right singular blocks
- $L^T_{\eta_1}, \ldots, L^T_{\eta_q}$ – Left singular blocks

Regular part:

- $J(\mu_1), \ldots, J(\mu_t)$ – Each $J(\mu_i)$ is block-diagonal with *Jordan* blocks corresponding to the finite eigenvalue $\mu_i$
- $N_{s_1}, \ldots, N_{s_k}$ – Jordan blocks corresponding to the infinite eigenvalue

Any $P(s)$ can be transformed to *Smith canonical form* using unimodular transformations $M_l(s)$ and $M_r(s)$ :

$$M_l(s)P(s)M_r(s) = \left[\begin{array}{cccc|c} e_1(s) & 0 & \dots & 0 & \\ 0 & e_2(s) & \ddots & \vdots & O_{r,n-r} \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & e_r(s) & \\ \hline & O_{m-r,r} & & & O_{m-r,n-r} \end{array}\right]$$

where each $e_j(s)$ divides $e_{j+1}(s)$ for $j = 1, \dots, r-1$.

The invariant polynomials $e_j(s)$ define the elementary divisors $(s-\alpha)^{k_j}$ at a particular zero $\alpha$ of $e_j(s)$

The structural elements of $P(s)$ are :

Its invariant polynomials $e_j(s)$ and corresponding finite elementary divisors $(s-\alpha)^{k_j}$ at every zero $\alpha$

The elementary divisors at $\infty$
(defined as the elementary divisors of $\mu^d P(1/\mu)$ at $\mu = 0$)

The minimal indices $\epsilon_i$ of the left null space
(defined via $M_l(s)$)

The minimal indices $\eta_j$ of the right null space
(defined via $M_r(s)$)

**Theorem**
The linearization $sB_r + A_r$ shares the following structural elements with $P(s)$

Both have the same finite elementary divisors (and finite invariant factors)

Both have the same elementary divisors at $\infty$ (defined via $\mu^d P(1/\mu)$ and $\mu A_r + B_r$)

Both have the same right minimal indices (defined via the right null space)

Moreover, if $P(s)$ has normal rank $m$, then both have no left minimal indices (or left null space)

**Theorem (dual)**
The linearization $sB_l + A_l$ shares the following structural elements with $P(s)$

Both have the same finite elementary divisors (and finite invariant factors)

Both have the same elementary divisors at $\infty$ (defined via $\mu^d P(1/\mu)$ and $\mu A_r + B_r$)

Both have the same left minimal indices (defined via the left null space)

Moreover, if $P(s)$ has normal rank $n$, then both have no right minimal indices (or right null space)

But are these relations still holding when we allow arbitrary perturbations of the (structured pencils) $sB_r + A_r$ and $sB_l + A_l$ ?

Clearly not as the following simple example shows :

A companion matrix

$$sI_d + \begin{bmatrix} 0 & & & p_0 \\ -1 & \ddots & & p_1 \\ & \ddots & 0 & \vdots \\ & & -1 & p_{d-1} \end{bmatrix}$$

can not have multiple Jordan blocks at one eigenvalue $s = \alpha$ but a general matrix can of course.

# Possible structures for a polynomial matrix

**Theorem**

*An $m \times n$ polynomial matrix $P(s)$ of degree $d$ and normal rank $r = m$ has $m$ finite elementary divisors $(s-\lambda_i)^{k_{j,i}}, j = 1, \ldots, m$ for each zero $\lambda_i$, $m$ infinite elementary divisors $1/s^{k_{j,\infty}}$, and $n-m$ right minimal indices $\epsilon_j$, $j = 1, \ldots, n-m$ (some of these indices can be trivially zero) satisfying*

$$\sum_{j}^{m}\sum_{i} k_{j,i} + \sum_{j}^{m} k_{j,\infty} + \sum_{j}^{n-m} \epsilon_j = dm.$$

*All structures satisfying these constraints are possible for such a polynomial matrix.*

If $P_d$ has rank $m$, there are no infinite elementary divisors.

# Check this on the Kronecker form

Any matrix pencil $sB + A$ or system pencil $\mathbf{S}(s)$ can be transformed into *Kronecker canonical form* (KCF) using equivalence transformations ($U$ and $V$ non-singular):

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- Given a system pencil and its orbit: What other structures are found within its closure?

**Stratification**

The closure hierarchy of all possible orbits of Kronecker structures

We make use of:

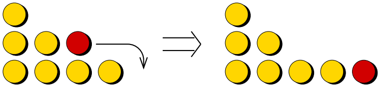
- Graphs to illustrate stratifications
- Dominance orderings for integer partitions in proofs and derivations

# Canonical structure indices

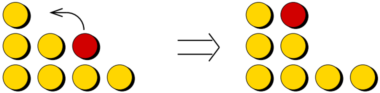
An *integer partition* $\kappa$ of an integer $K$ is defined as $\kappa = (\kappa_1, \kappa_2, \ldots)$ where $\kappa_1 \geq \kappa_2 \geq \cdots \geq 0$ and $K = \kappa_1 + \kappa_2 + \ldots$.

- $\mathcal{R} = (r_0, r_1, \ldots)$ where $r_i = \#L_k$ blocks with $k \geq i$
- $\mathcal{L} = (l_0, l_1, \ldots)$ where $l_i = \#L_k^T$ blocks with $k \geq i$
- $\mathcal{J}_{\mu_i} = (j_1, j_2, \ldots)$ where $j_i = \#J_k(\mu_i)$ blocks with $k \geq i$. $\mathcal{J}_{\mu_i}$ is known as the *Weyr characteristics* of the finite eigenvalue $\mu_i$
- $\mathcal{N} = (n_1, n_2, \ldots)$ where $n_i = \#N_k$ with $k \geq i$. $\mathcal{N}$ is known as the *Weyr characteristics* of the infinite eigenvalue

*Minimum rightward coin move*: rightward *one* column or downward *one* row (keep partition monotonic)



*Minimum leftward coin move*: leftward *one* column or upward *one* row (keep partition monotonic)



[Edelman, Elmroth & Kågström; 1999]

- By making use of linearizations, the cover relations for polynomial matrices are derived from general matrix pencils $sH + G$ and matrix pairs $(A, B)$:
  - The stratification rules for $P(s)$ with full normal rank from the rules for $sH + G$
  - The stratification rules for $P(s)$ with full normal rank and $rankP_d = m$ from the rules for $(A, B)$

# Orbit covering relations for full normal rank $P(s)$

## Theorem (Theorem 1)

*Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of $sB_r + A_r$ associated with a full normal rank polynomial matrix $P(s)$, where $\mu_i \in \overline{\mathbb{C}}$, one of the following if-and-only-if rules finds $s\widetilde{B}_r + \widetilde{A}_r$ such that:*

$\mathcal{O}(sB_r + A_r)$ **covers** $\mathcal{O}(s\widetilde{B}_r + \widetilde{A}_r)$

1. *Minimum* rightward *coin move in $\mathcal{R}$*
2. *If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially)*
3. *Minimum* leftward *coin move in any $\mathcal{J}_{\mu_i}$ as long as $j_1^{(i)}$ not exceed $m$*

$\mathcal{O}(sB_r + A_r)$ **is covered by** $\mathcal{O}(s\widetilde{B}_r + \widetilde{A}_r)$

1. *Minimum* leftward *coin move in $\mathcal{R}$*
2. *If the rightmost column in some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$*
3. *Minimum* rightward *coin move in any $\mathcal{J}_{\mu_i}$*

*Rules 1 and 2:* Coin moves that affect $r_0$ are not allowed

# Orbit covering relations for full normal rank $P(s)$

## Theorem (Theorem 1)

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$\mathcal{O}(sB_r + A_r)$ **covers** $\mathcal{O}(s\widetilde{B}_r + \widetilde{A}_r)$

1. *Minimum rightward coin move in $\mathcal{R}$*
2. *If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to a new rightmost column of some $\mathcal{J}_{\mu_i}$ (which may be empty initially)*
3. *Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$ as long as $j_1^{(i)}$ not exceed $m$*

$\mathcal{O}(sB_r + A_r)$ **is covered by** $\mathcal{O}(s\widetilde{B}_r + \widetilde{A}_r)$

1. *Minimum leftward coin move in $\mathcal{R}$*
2. *If the rightmost column in some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$*
3. *Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$*

*Rules 1 and 2:* Coin moves that affect $r_0$ are not allowed

# Bundle covering relations for full normal rank $P(s)$

## Theorem (Theorem 2)

*Given the structure integer partitions $\mathcal{R}$ and $\mathcal{J}_{\mu_i}$ of $sB_r + A_r$ associated with a full normal rank polynomial matrix $P(s)$, where $\mu_i \in \overline{\mathbb{C}}$, one of the following if-and-only-if rules finds $s\tilde{B}_r + \tilde{A}_r$ such that:*

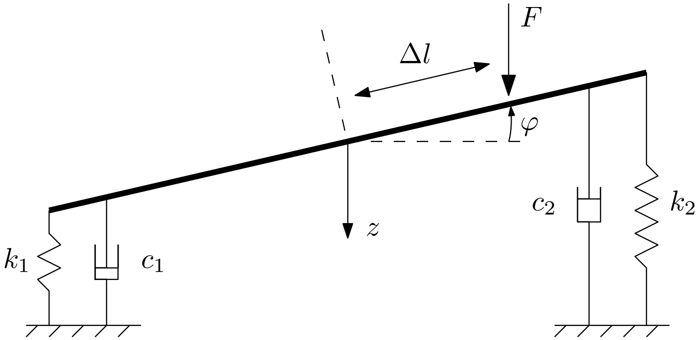
$\mathcal{B}(sB_r + A_r)$ **covers** $\mathcal{B}(s\tilde{B}_r + \tilde{A}_r)$

1. *Minimum rightward coin move in $\mathcal{R}$*
2. *If the rightmost column in $\mathcal{R}$ is one single coin, move that coin to the first column of $\mathcal{J}_{\mu_i}$ for a new eigenvalue $\mu_i$*
3. *Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$ as long as $j_1^{(i)}$ not exceed $m$*
4. *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins*

$\mathcal{B}(sB_r + A_r)$ **is covered by** $\mathcal{B}(s\tilde{B}_r + \tilde{A}_r)$

1. *Minimum leftward coin move in $\mathcal{R}$*
2. *If some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$*
3. *Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$*
4. *For any $\mathcal{J}_{\mu_i}$, divide the set of coins into two new sets so that their union is $\mathcal{J}_{\mu_i}$*

*Rules 1 and 2:* Coin moves that affect $r_0$ are not allowed




A uniform platform with mass $m$ and length $2l$, supported in both ends by springs

The control parameter of the system is the force $F$ applied at distance $\Delta l$ from the center of the platform

The equations of motion linearized near the equilibrium:

$$m\ddot{z} + (c_1 + c_2)\dot{z} + (k_1 + k_2)z + l(c_1 - c_2)\dot{\varphi} + l(k_1 - k_2)\varphi = F$$
$$J\ddot{\varphi} + l(c_1 - c_2)\dot{z} + l(k_1 - k_2)z + l^2(c_1 + c_2)\dot{\varphi} + l^2(k_1 + k_2)\varphi = -\Delta lF$$

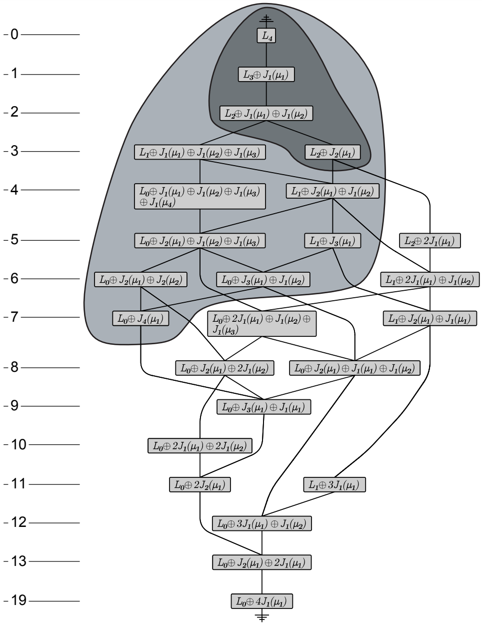
where $J = ml^2/3$ is the moment of inertia.

Equations of motion written on the form $M\ddot{x} + C\dot{x} + Kx = Eu$:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \ddot{x} + \begin{bmatrix} c_1 + c_2 & l(c_1 - c_2) \\ l(c_1 - c_2) & l^2(c_1 + c_2) \end{bmatrix} \dot{x} + \begin{bmatrix} k_1 + k_2 & l(k_1 - k_2) \\ l(k_1 - k_2) & l^2(k_1 + k_2) \end{bmatrix} x = \begin{bmatrix} 1 \\ -\Delta l \end{bmatrix} F$$

The right linearization:

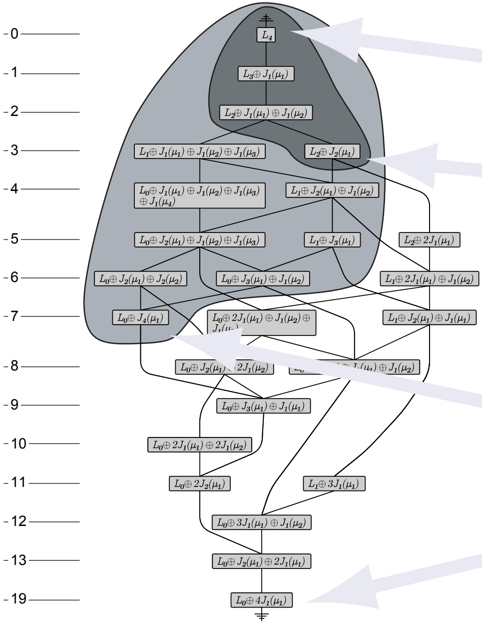
$$\left[\begin{array}{c|c} sI_4 + A & B \end{array}\right] = \left[\begin{array}{cc|c} sI_2 & M^{-1}K & M^{-1}E \\ -I_2 & sI_2 + M^{-1}C & 0 \end{array}\right]$$




The software tool StratiGraph is used for computing and visualizing the stratification

[Elmroth, P. Johansson & Kågström; 2001]

[P. Johansson; PhD Thesis 2006]

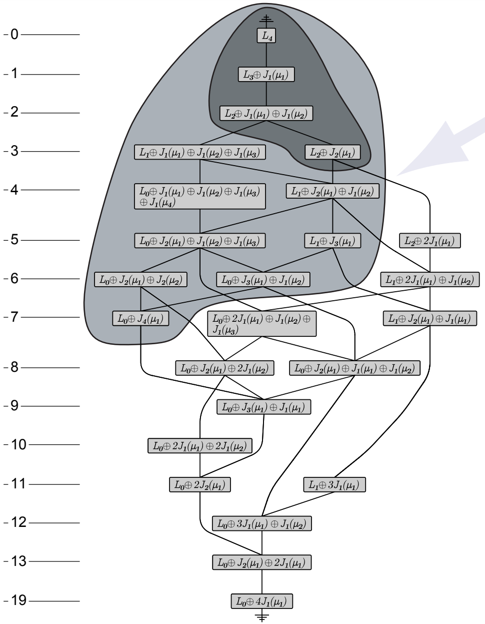



Most generic

Each node represents a bundle (or orbit) of a canonical structure

Least generic $4 \times 5$ $P(s)$

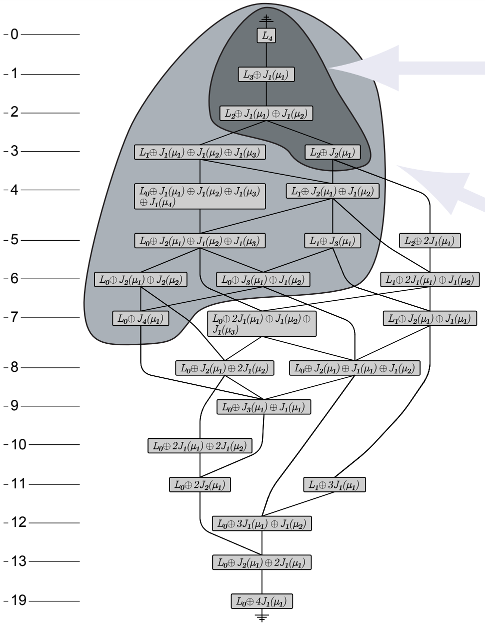
Least generic $4 \times 5$ $(A, B)$




Each edge represents a cover relation

It is always possible to go from any canonical structure (node) to another higher up in the graph by a small perturbation

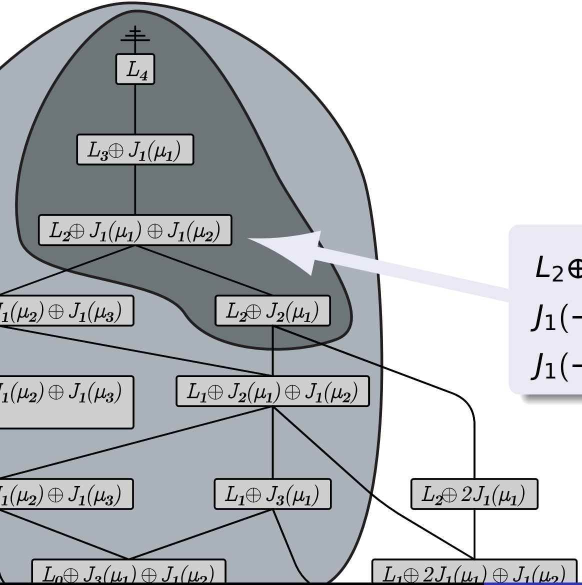
# Uniform platform – Illustrating the bundle stratification




Only possible bundles for the uniform platform

Only possible bundles for a full normal rank $4 \times 5$ polynomial matrix with $\det(P_d) \neq 0$

# Uniform platform – Illustrating the bundle stratification

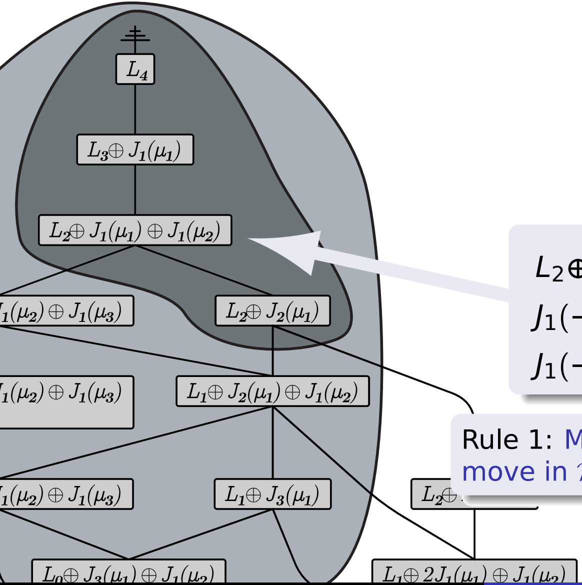



$L_2 \oplus$ $\quad$ $\mathcal{R}$:

$J_1(-0.02) \oplus$ $\quad$ $\mathcal{J}_{-0.02}$:

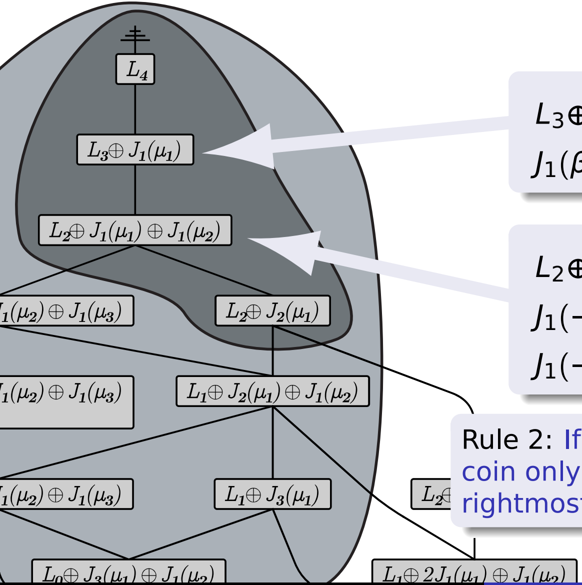
$J_1(-0.06)$ $\quad$ $\mathcal{J}_{-0.06}$:

# Uniform platform – Illustrating the bundle stratification




| | |
|---|---|
| $L_2 \oplus$ | $\mathcal{R}$: ○○○ |
| $J_1(-0.02) \oplus$ | $\mathcal{J}_{-0.02}$: ○ |
| $J_1(-0.06)$ | $\mathcal{J}_{-0.06}$: ○ |

Rule 1: Minimum leftward coin move in $\mathcal{R}$, without affecting $r_0$




$L_3 \oplus$ $\mathcal{R}$: 🟡🟡🟡🔴

$J_1(\beta)$ $\mathcal{J}_\beta$: 🟡

$\Uparrow$

$L_2 \oplus$ $\mathcal{R}$: 🟡🟡🟡

$J_1(-0.02) \oplus$ $\mathcal{J}_{-0.02}$: 🔴

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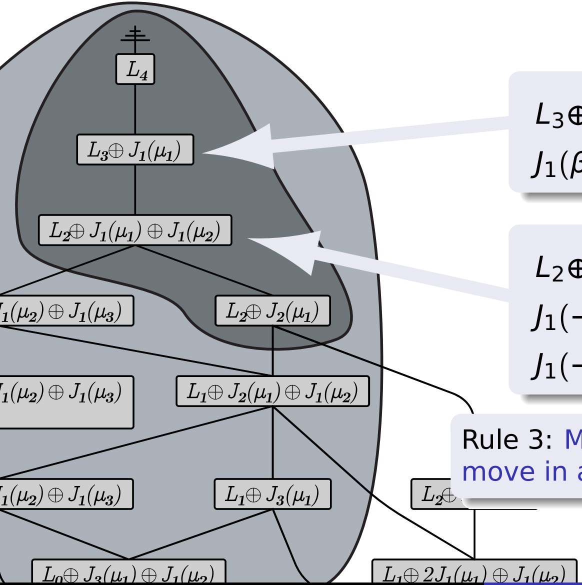
Rule 2: If some $\mathcal{J}_{\mu_i}$ consists of one coin only, move that coin to a new rightmost column in $\mathcal{R}$




$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$ $\mathcal{R}$:
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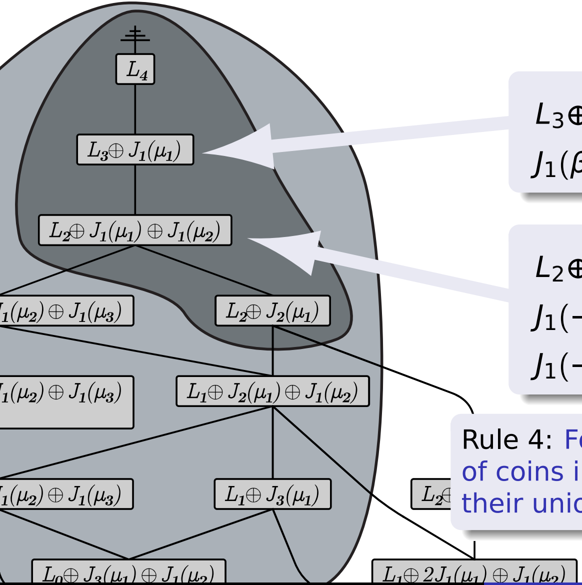
Rule 3: Minimum rightward coin move in any $\mathcal{J}_{\mu_i}$




$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$ $\mathcal{R}$: ● ● ●
$J_1(-0.02) \oplus$ $\mathcal{J}_{-0.02}$: ●
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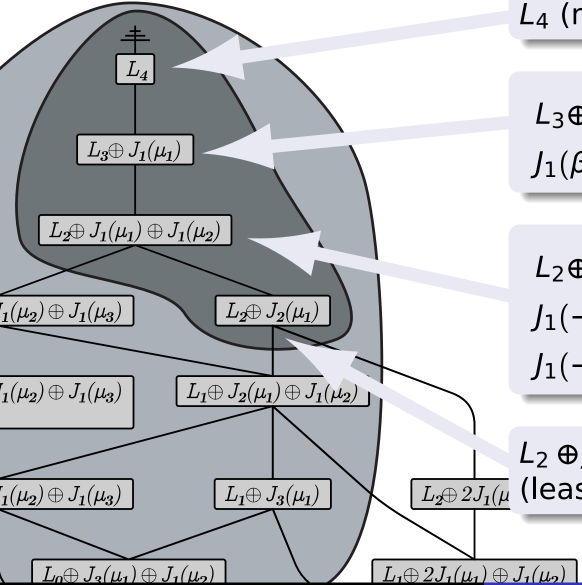
Rule 4: For any $\mathcal{J}_{\mu_i}$, divide the set of coins into two new sets so that their union is $\mathcal{J}_{\mu_i}$

# Uniform platform – Illustrating the bundle stratification




$L_4$ (most generic)

$L_3 \oplus$
$J_1(\beta)$

$L_2 \oplus$
$J_1(-0.02) \oplus$
$J_1(-0.06)$

$L_2 \oplus J_2(\alpha)$
(least generic)

Deformations of $P(s)$ can be treated via the linearization $sB_r + A_r$ provided

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It allows to understand deformations of the controllability structure of polynomial models.