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Stable and Efficient Spectral Divide and Conquer Algorithms for the Symmetric Eigenvalue Decomposition and the SVD

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Joint work with Nick Higham

Perspectives on Parallel Numerical Linear Algebra

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Symmetric eigendecomposition and SVD

- **Symmetric eigenvalue decomposition** (symeig): for $A = A^T \in \mathbb{R}^{n\times n}$,

$$A = V \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \cdot V^T \ \left(\equiv V\Lambda V^T\right),$$

where $V^T V = I_n$. $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of A.

- **SVD**: for any $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$,

$$A = U \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_n) \cdot V^T \ \left(\equiv U\Sigma V^T\right),$$

where $U^T U = I_m$, $V^T V = I_n$. $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values of A.

Standard algorithm for symmetric eigendecomposition

1. Reduce matrix to tridiagonal form

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \xrightarrow{H_L, H_R} \begin{bmatrix} * & * & & & \\ * & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix} \xrightarrow{H_L, H_R} \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & * & * & * \\ & & * & * & * \\ & & * & * & * \end{bmatrix} \xrightarrow{H_L, H_R} \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & * & * & \\ & & * & * & * \\ & & & * & * \end{bmatrix} \equiv T$$

   Accumulate orthogonal factors: $A = V_A T V_A^H$ where $V_A = (\prod H_L)^H$
2. Compute eigendecomposition $T = V_B \Lambda V_B^H$ (QR, divide-and conquer, MRRR, ...)
3. Eigendecomposition: $A = (V_A V_B)\Lambda(V_B^H V_A^H) = V \Lambda V^H$

Standard algorithm for symmetric eigendecomposition

1. Reduce matrix to tridiagonal form
   $\Rightarrow$ Expensive in communication cost

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \overset{H_L, H_R}{\rightarrow} \begin{bmatrix} * & * & & & \\ * & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix} \overset{H_L, H_R}{\rightarrow} \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & * & * & * \\ & & * & * & * \\ & & * & * & * \end{bmatrix} \overset{H_L, H_R}{\rightarrow} \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & * & * & \\ & & * & * & * \\ & & & * & * \end{bmatrix} \equiv T$$

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We develop completely different algorithms based on

spectral divide-and-conquer

Spectral divide-and-conquer algorithm for symeig

$$A = \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

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Spectral divide-and-conquer algorithm for symeig

$$A = \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \rightarrow V_1^* A V_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} V_{21} & V_{22} \end{bmatrix}^* V_1^* A V_1 \begin{bmatrix} V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} * & * \\ * & * & & & & & & \\ & & * & * & & & & \\ & & * & * & & & & \\ & & & & * & * & & \\ & & & & * & * & & \\ & & & & & & * & * \\ & & & & & & * & * \end{bmatrix}$$

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$$A = \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \rightarrow V_1^* A V_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

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$$\rightarrow \begin{bmatrix} V_{31} & & & \\ & V_{32} & & \\ & & V_{33} & \\ & & & V_{34} \end{bmatrix}^* \begin{bmatrix} V_{21} & V_{22} \end{bmatrix}^* V_1^* A V_1 \begin{bmatrix} V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} V_{31} & & & \\ & V_{32} & & \\ & & V_{33} & \\ & & & V_{34} \end{bmatrix} = \mathrm{diag}(\lambda_i)$$

Spectral divide-and-conquer algorithm for symeig

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– not divide-and-conquer for symmetric tridiagonal eigenproblem!

Executing spectral divide-and-conquer

To obtain

$$[V_+ \; V_-]^* A [V_+ \; V_-] = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix},$$

- V_+, V_- each spans an invariant subspace of A corresponding to a subset of eigenvalues.

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- V_+, V_- each spans an invariant subspace of A corresponding to a subset of eigenvalues.
- A natural idea: let V_+ and V_- span the positive and negative eigenspaces.

  $\rightarrow [V_+\ V_-] \begin{bmatrix} I & \\ & -I \end{bmatrix} [V_+\ V_-]^T = \text{sign}(A)$, the matrix sign function of A. Since $A = A^T$, $\text{sign}(A)$ is equivalent to U, the unitary polar factor in the polar decomposition.

Polar decomposition

For any $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$,

$$A = UH,$$

where $U^T U = I_n$, H: symmetric positive semidefinite

- U is called the unitary polar factor of A.
- Connection with SVD: $A = U_* \Sigma V_*^T = (U_* V_*^T) \cdot (V_* \Sigma V_*^T) = UH$.
- Can be used as first step for computing the SVD.

–[Higham and Papadimitriou (1993)]

polar/eigen decompositions of symmetric matrices

The polar/eigen decompositions of symmetric A:

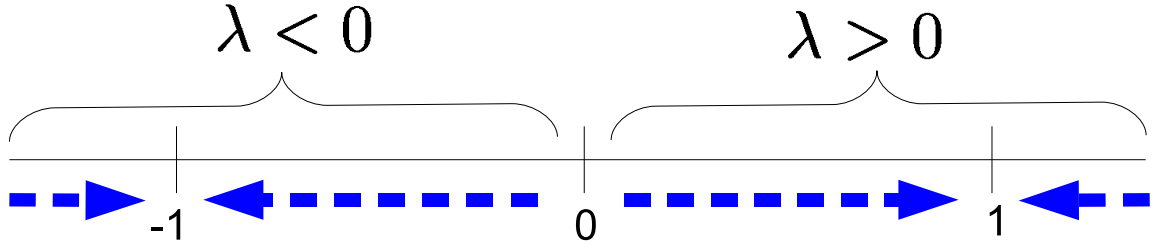
$$A = UH$$

$$= [V_+ \; V_-] \begin{bmatrix} \Lambda_+ & \\ & \Lambda_- \end{bmatrix} [V_+ \; V_-]^T$$

where $\mathrm{diag}(\Lambda_-) < 0$, are related by $U = [V_+ \; V_-] \begin{bmatrix} I & \\ & -I \end{bmatrix} [V_+ \; V_-]^T$:

$$UH = \left([V_+ \; V_-] \begin{bmatrix} I & \\ & -I \end{bmatrix} [V_+ \; V_-]^T \right) \cdot \left([V_+ \; V_-] \begin{bmatrix} \Lambda_+ & \\ & |\Lambda_-| \end{bmatrix} [V_+ \; V_-]^T \right)$$

- $U = f(A)$, where $f(\lambda)$ maps $\lambda > 0$ to 1, $\lambda < 0$ to -1



QR-based iteration for polar decomposition

DWH (Dynamically Weighted Halley): –[N., Bai, Gygi, SIMAX 2010]

$$X_{k+1} = X_k(a_k I + X_k^T X_k)(b_k I + c_k X_k^T X_k)^{-1}$$
$$= \frac{b_k}{c_k} X_k + \left(a_k - \frac{b_k}{c_k}\right) X_k (I + c_k X_k^T X_k)^{-1}, \quad X_0 = A/\|A\|_2$$

– $\lim_{k\to\infty} X_k = U$ in $A = UH$

– a_k, b_k, c_k chosen s.t. $\frac{x_k(a_k+x_k^2)}{b_k+c_k x_k^2}$ is the best $(3,2)$-type rational approximation to sign(x) on $[\sigma_{\min}(X_k), \sigma_{\max}(X_k)]$

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- Lemma:

$$Q_1 Q_2^T = X(I + X^T X)^{-1}, \text{ where } \begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \mathrm{R}$$

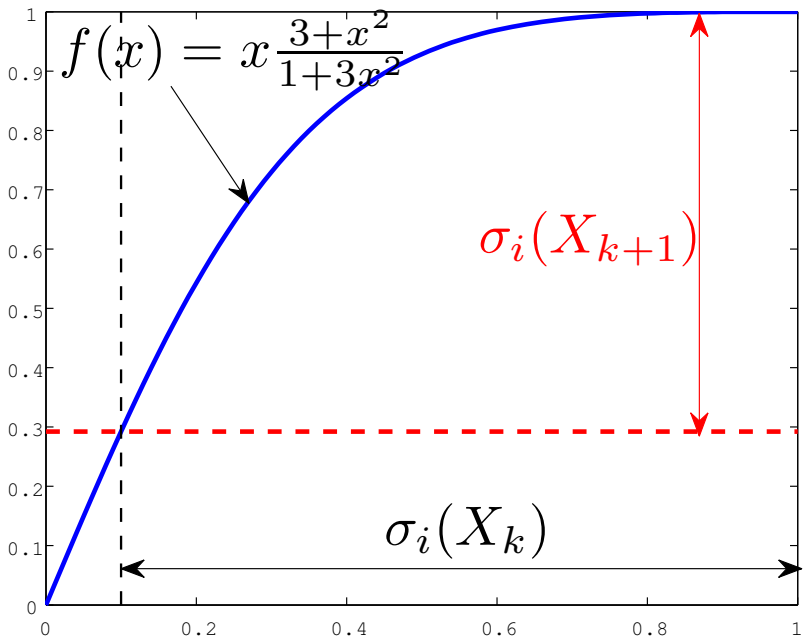
- QR-based DWH (QDWH)

$$\begin{cases} \begin{bmatrix} \sqrt{c_k} X_k \\ I \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R, \\ X_{k+1} = \frac{b_k}{c_k} X_k + \frac{1}{\sqrt{c_k}} \left(a_k - \frac{b_k}{c_k}\right) Q_1 Q_2^T \end{cases}$$

Halley vs. Dynamically weighted Halley

$$X_{k+1} = \begin{cases} X_k(a_k I + X_k^T X_k)(b_k I + c_k X_k^T X_k)^{-1} & \text{(Dynamical Halley)} \\ X_k(3I + X_k^T X_k)(I + 3X_k^T X_k)^{-1} & \text{(Halley)} \end{cases}$$

Halley



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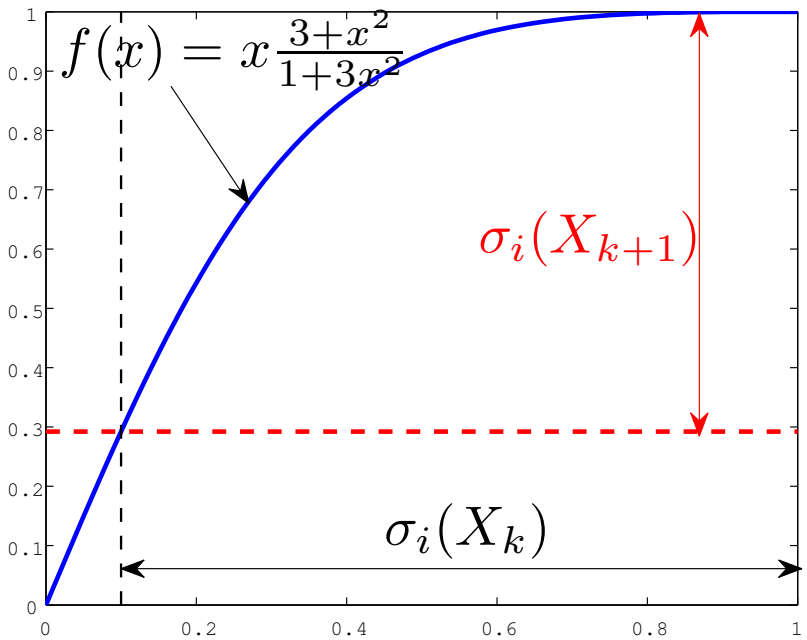
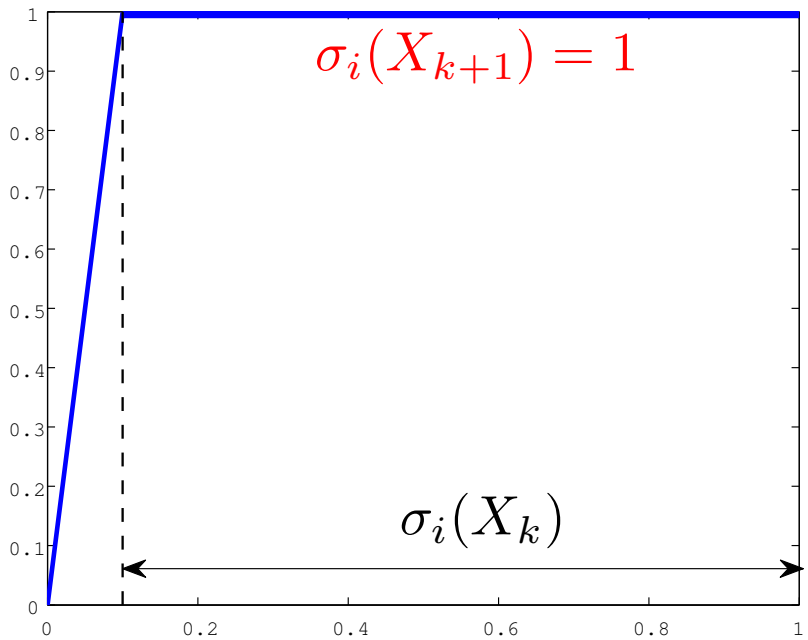


”Optimal” function



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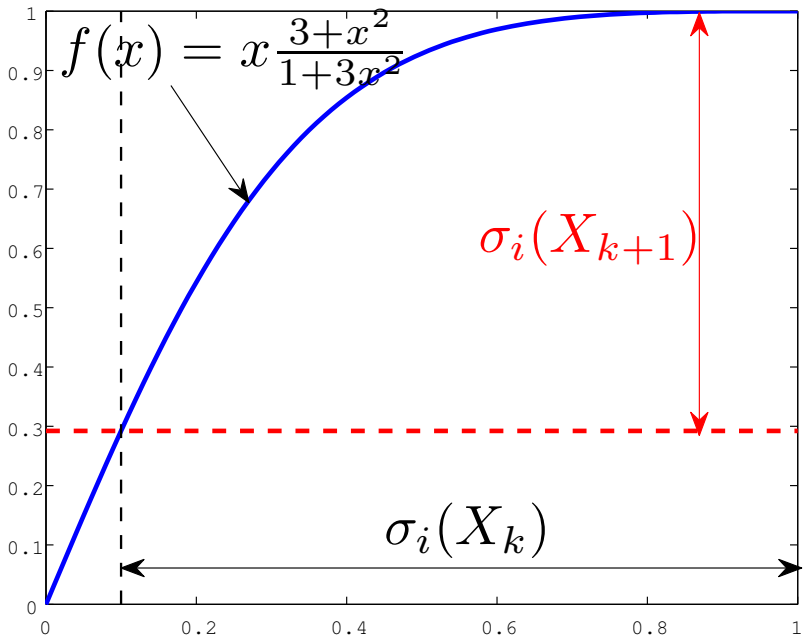
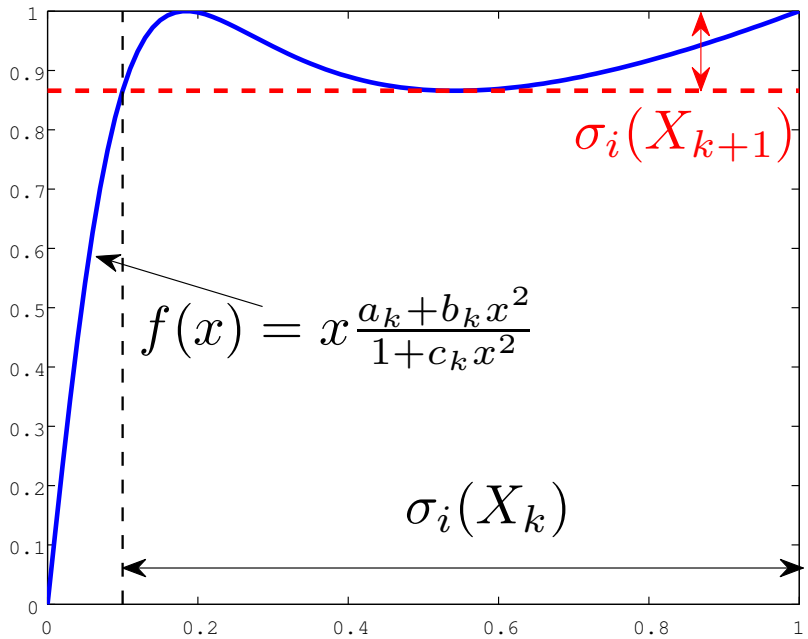
Halley



Dynamical weighting



- Dynamical weighting uses the best type-(3,2) rational approximation to sign(x), brings $\sigma_i(X_{k+1})$ much closer to 1, thereby speeding up convergence

Algorithm for symmetric eigendecomposition

Algorithm **QDWH-eig**

1. compute polar decomposition $A - \sigma I = UH$
2. compute unitary $[V_+ \; V_-]$ s.t. $(U + I)/2 = V_+ V_+^T$
3. compute $[V_+ \; V_-]^T A [V_+ \; V_-] = \begin{bmatrix} A_+ & E^T \\ E & A_- \end{bmatrix}$
4. repeat steps 1–3 with $A := A_+, A_-$

- Ideal choice of σ: median of eig$(A) \Rightarrow$ dim$(A_+) \simeq$ dim(A_-)
- Dominant cost is in step 1, computing the polar decomposition
  - QDWH needs six or fewer iterations (QR + matrix multiply)

Backward stability

Theorem 1

If subspace iteration is successful and the polar decomposition $A \simeq \widehat{U}\widehat{H}$ *is computed backward stably s.t.*

$$A = \widehat{U}\widehat{H} + \epsilon_1 \|A\|_2, \quad \widehat{U}^T\widehat{U} - I = \epsilon_2, \tag{1}$$

then $A - \widehat{V}\widehat{\Lambda}\widehat{V}^T = \epsilon\|A\|_2$*, that is, QDWH-eig is backward stable.*

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Theorem 2

(1) *is satisfied if QR factorizations are computed with pivoting in QDWH (in practice even without pivoting).*

–[Nakatsukasa and Higham, SIMAX 2012]

Previous spectral d-c: Implicit Repeated Squaring (IRS)

Algorithm **IRS** —Ballard, Demmel, Dumitriu [2010,LAWN 237]

1. $A_0 = A, B_0 = I$
2. for $j = 0, 1, \ldots,$

$$\begin{bmatrix} B_j \\ -A_j \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_j \\ 0 \end{bmatrix},$$

$$A_{j+1} = Q_{12}^T A_j,$$

$$B_{j+1} = Q_{22}^T B_j$$

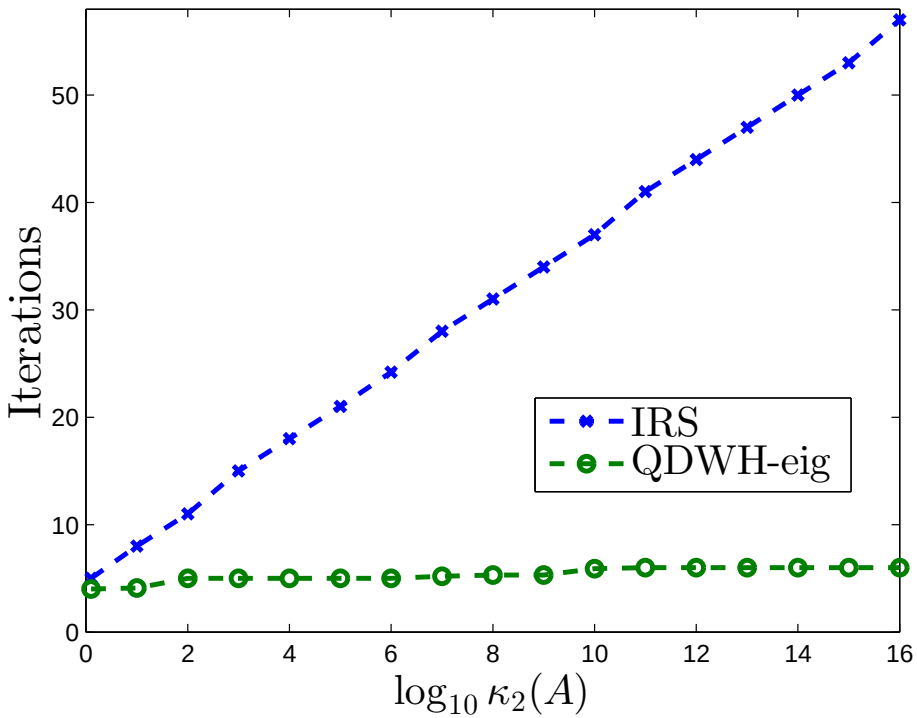
3. rank-revealing QR decomposition $(A_j + B_j)^{-1} A_j = QR$
4. form $Q^T A Q = \begin{bmatrix} A_{11} & E^T \\ E & A_{22} \end{bmatrix}$, confirm $\|E\|_F \lesssim \epsilon \|A\|_F$, repeat 1-4 with $A := A_{11}, A_{22}$

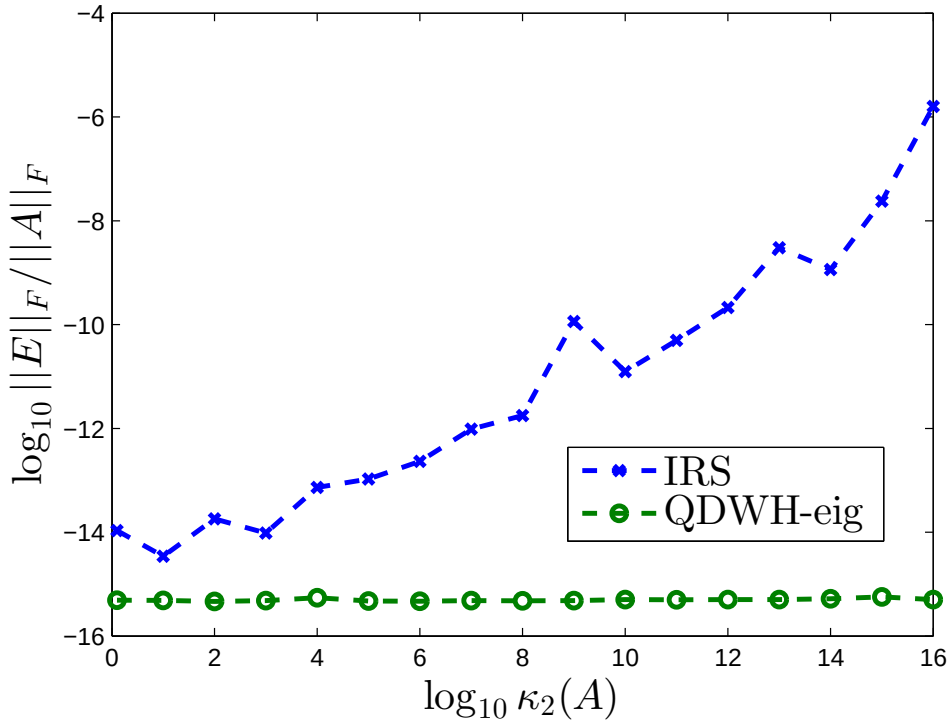
- Motivated by design of **communication-minimizing** algorithms. (QR, matmul, Cholesky are "good" operations)
- $(A_j + B_j)^{-1} A_j = (I + A^{-2^j})^{-1}$: maps $|\lambda| < 1$ to 0, $|\lambda| > 1$ to 1.
- SVD can be computed via eigendecomposition of $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$.

Spectral d-c algorithms: QDWH vs. IRS

100-by-100 MATLAB randsvd matrices,

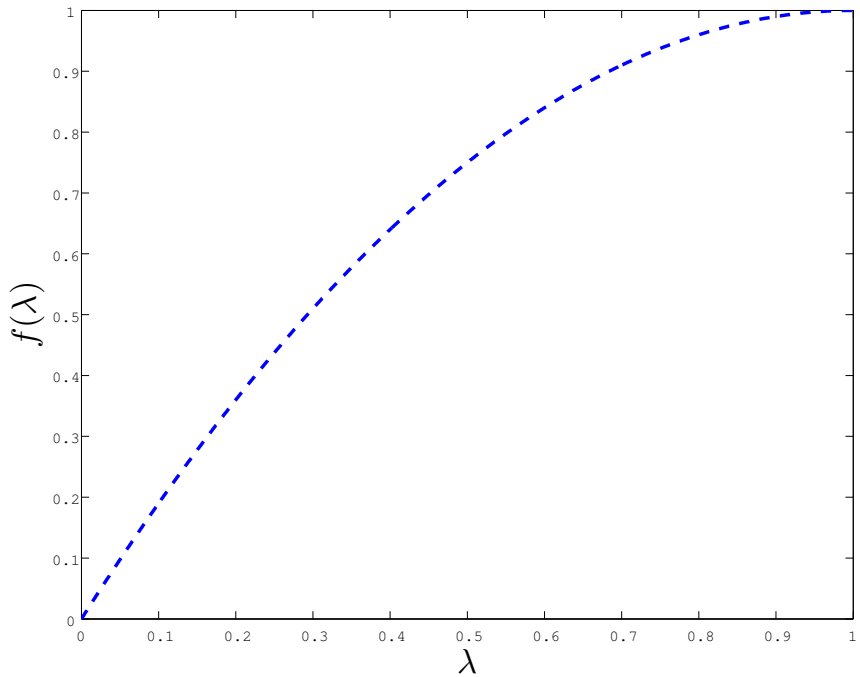
spectral d-c splitting at $\lambda = 0$



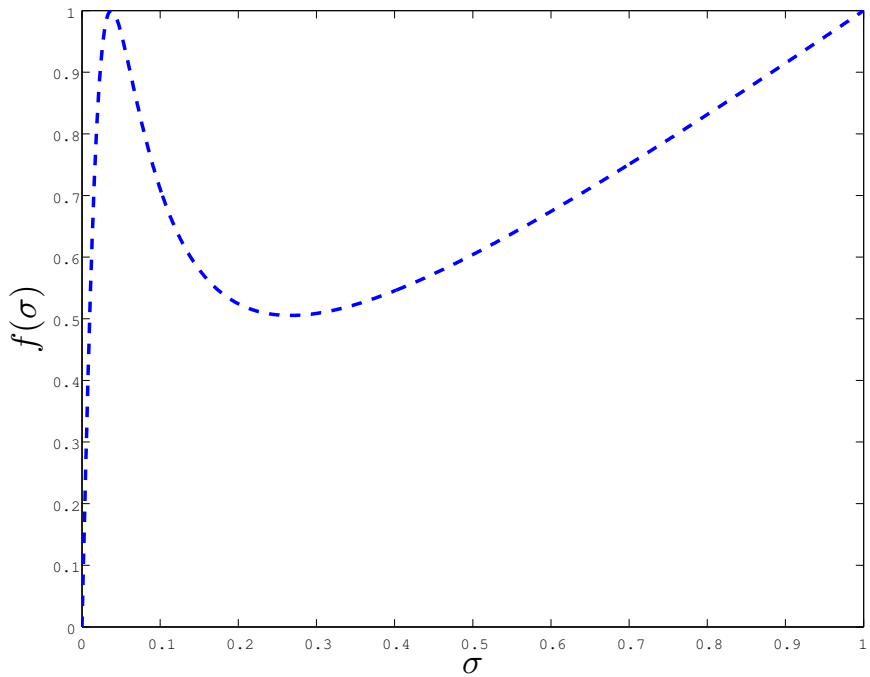


IRS vs. QDWH convergence

IRS: static parameters



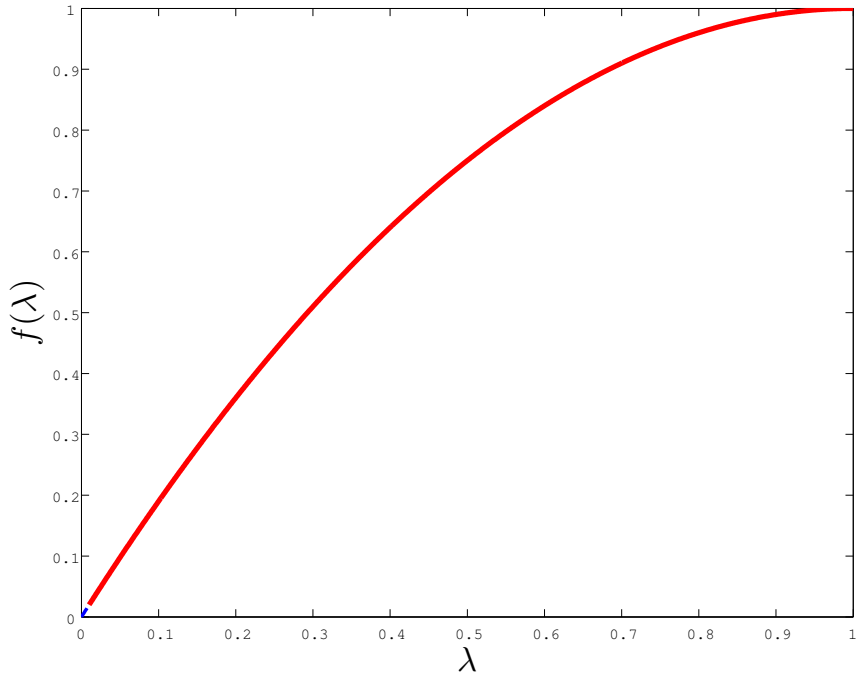
QDWH: dynamical parameters



IRS vs. QDWH convergence

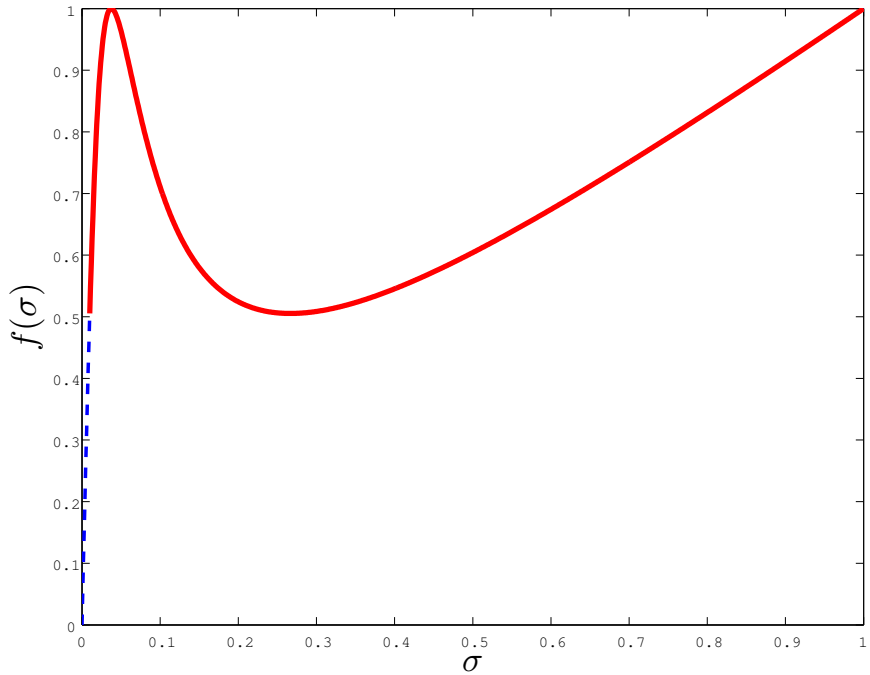
IRS: static parameters

it=0



QDWH: dynamical parameters

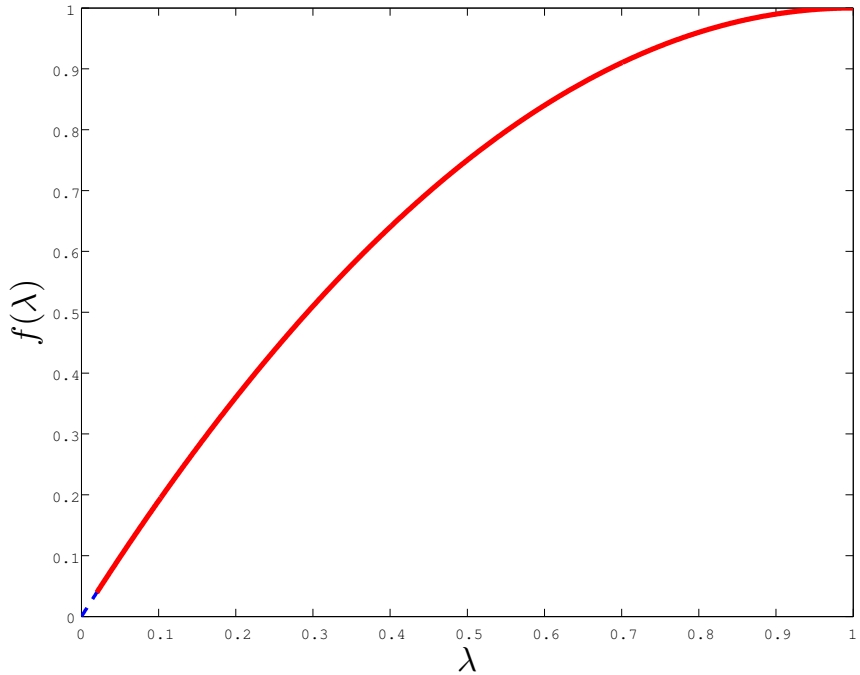
it=0



IRS vs. QDWH convergence

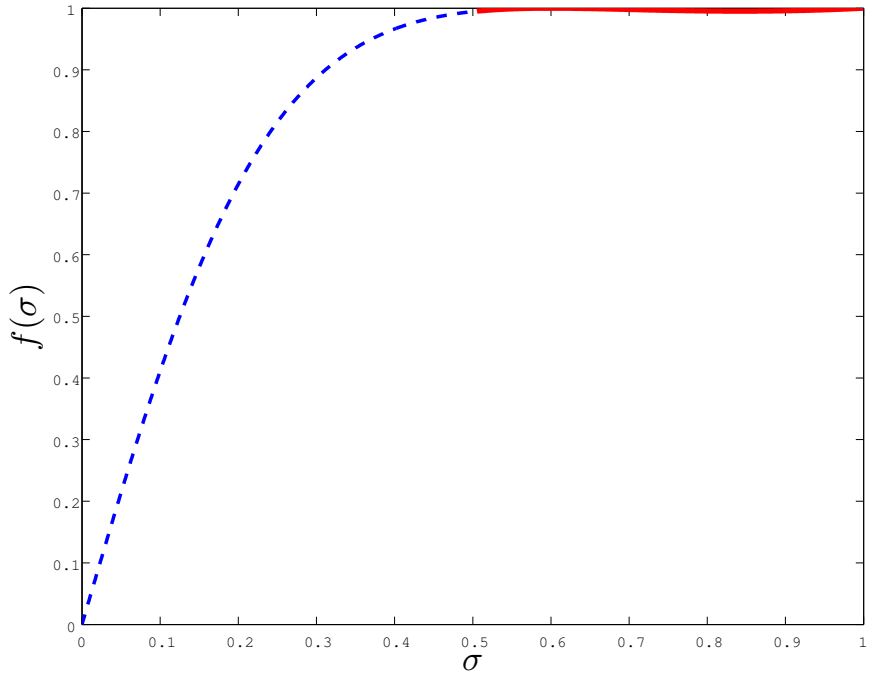
IRS: static parameters

it=1



QDWH: dynamical parameters

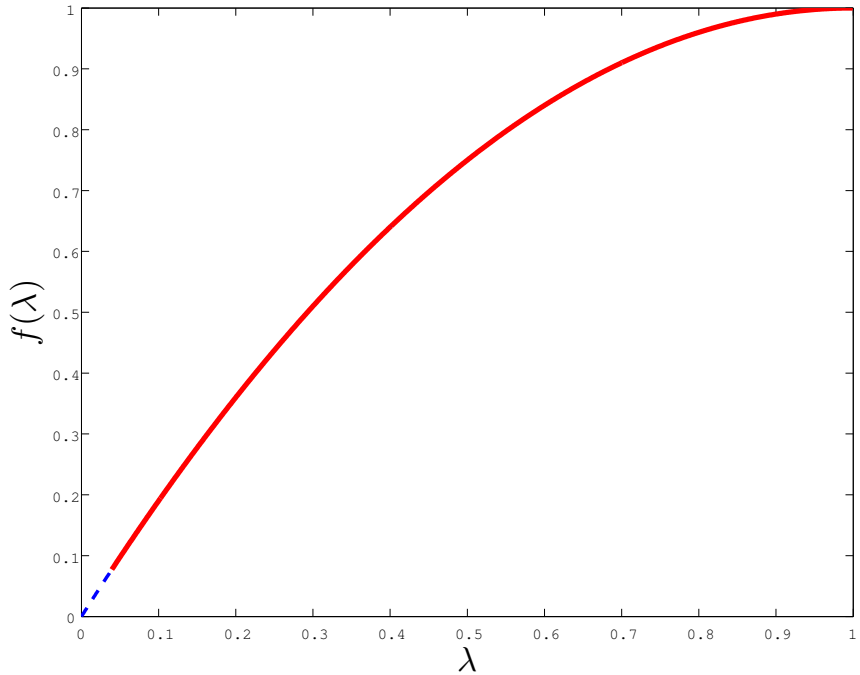
it=1



IRS vs. QDWH convergence

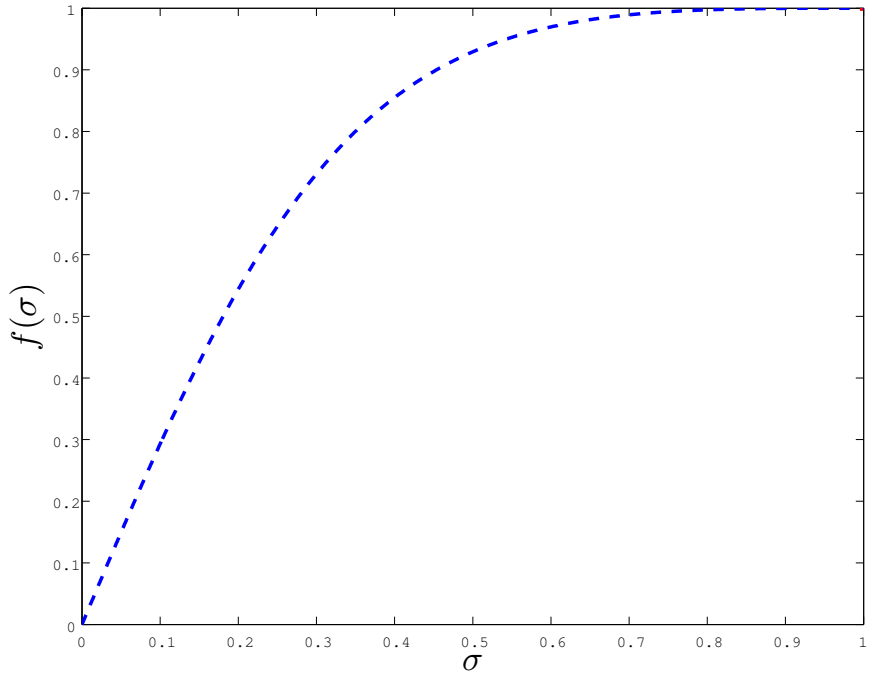
IRS: static parameters

it=2



QDWH: dynamical parameters

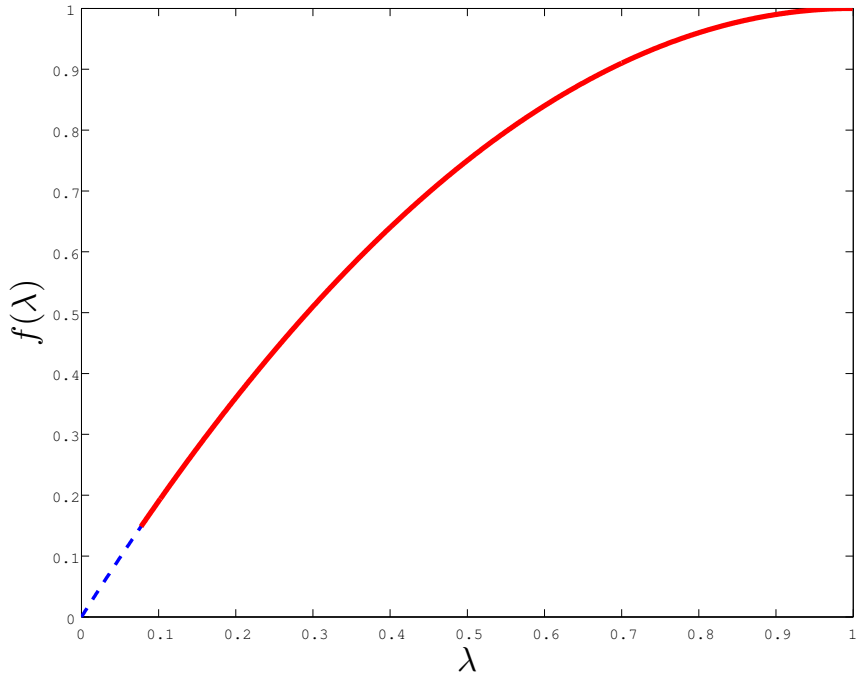
it=2



IRS vs. QDWH convergence

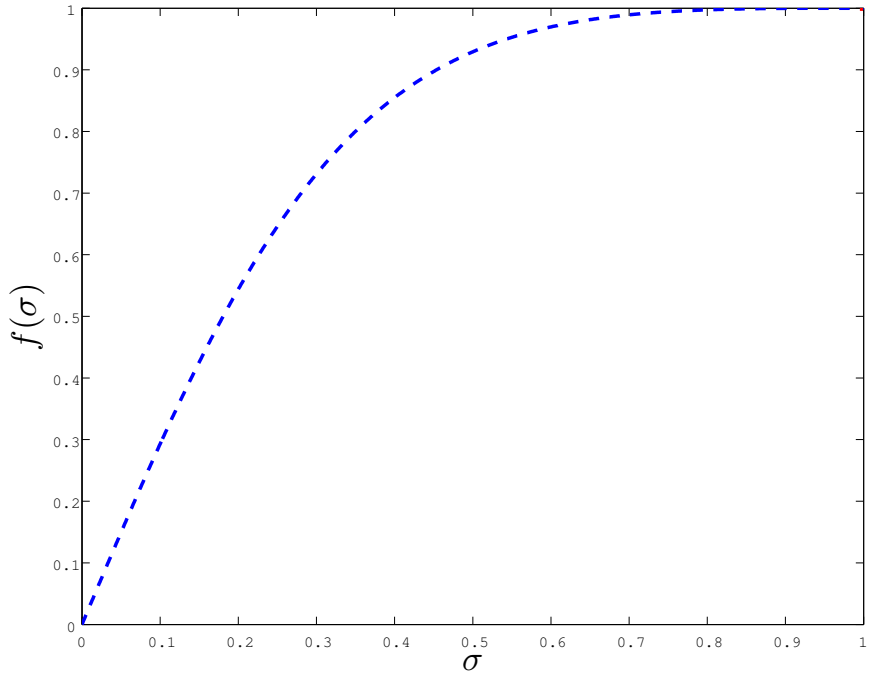
IRS: static parameters

it=3



QDWH: dynamical parameters

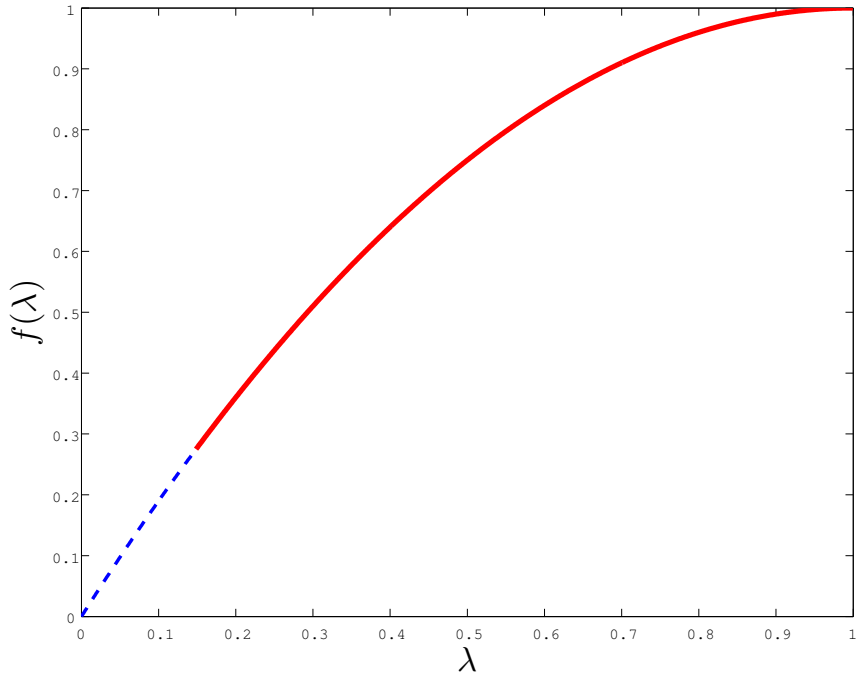
it=2



IRS vs. QDWH convergence

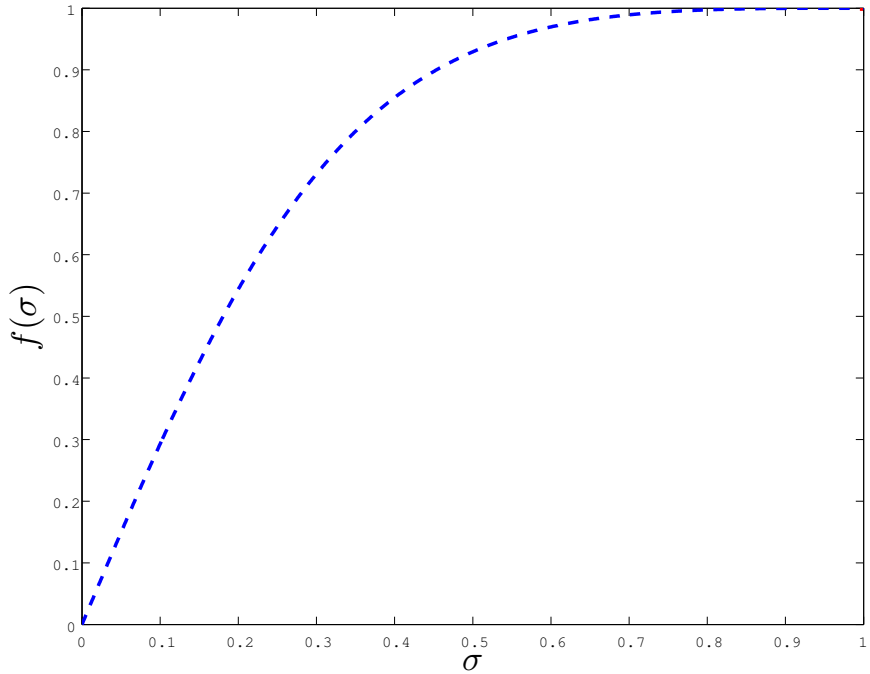
IRS: static parameters

it=4



QDWH: dynamical parameters

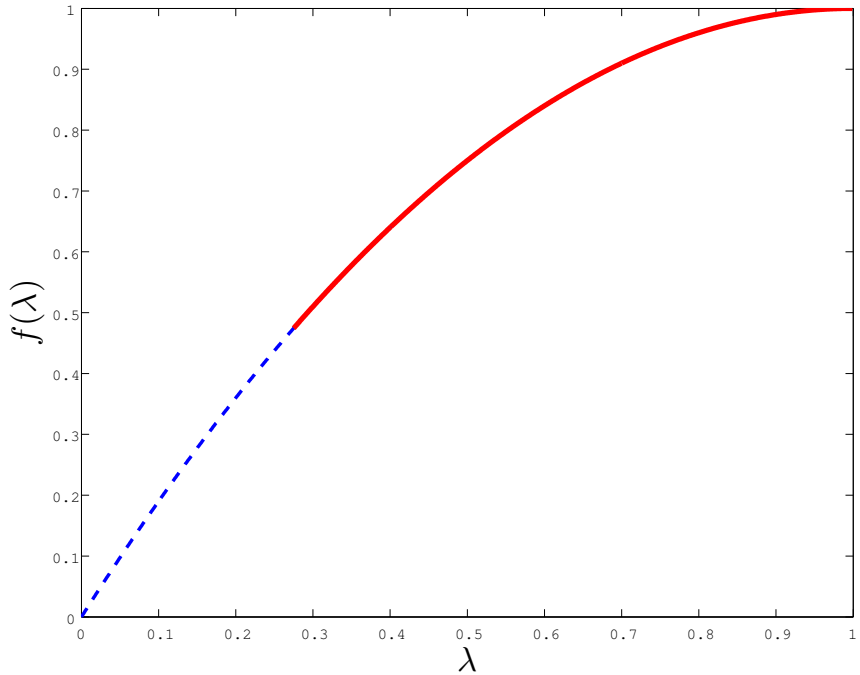
it=2



IRS vs. QDWH convergence

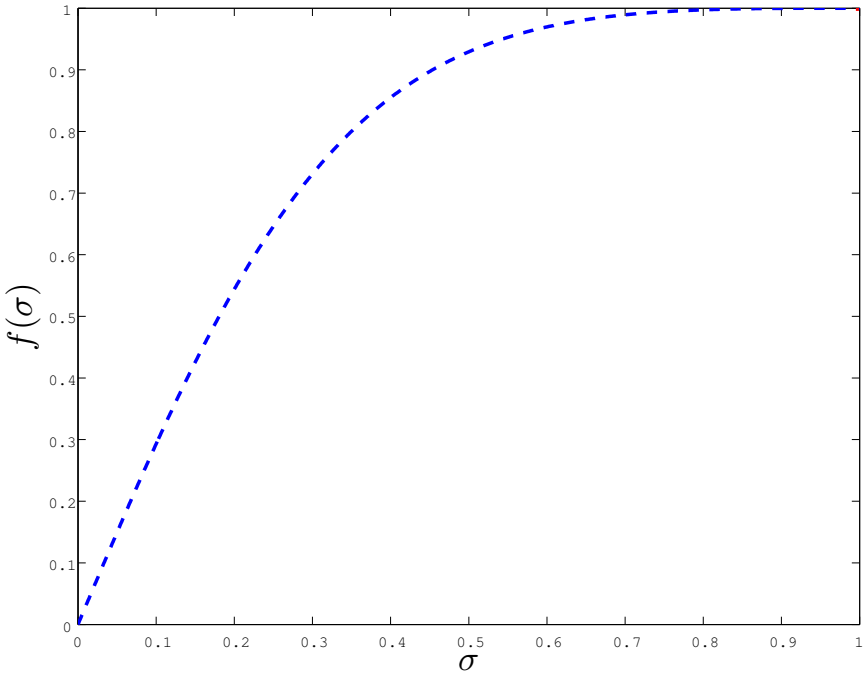
IRS: static parameters

it=5



QDWH: dynamical parameters

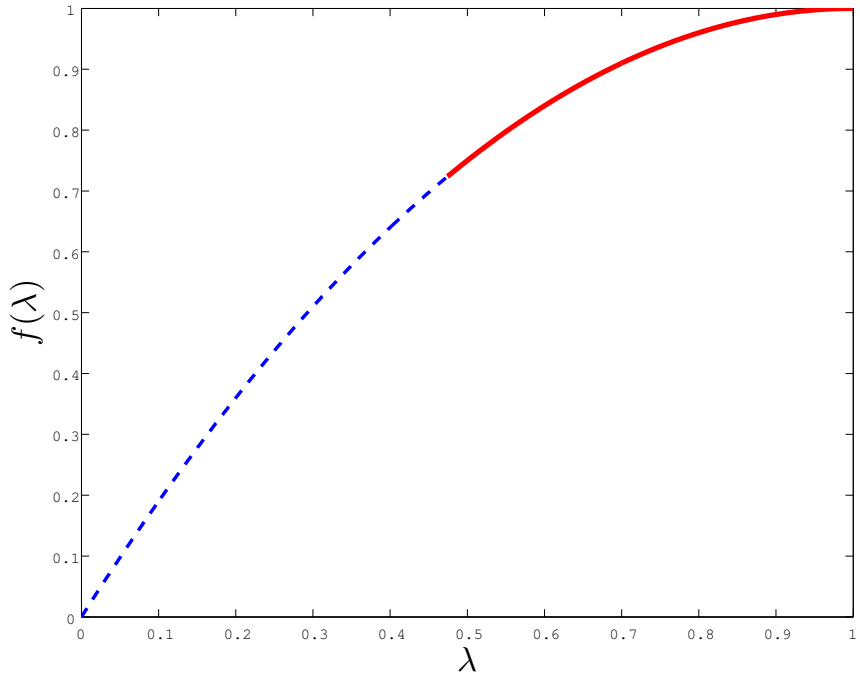
it=2



IRS vs. QDWH convergence

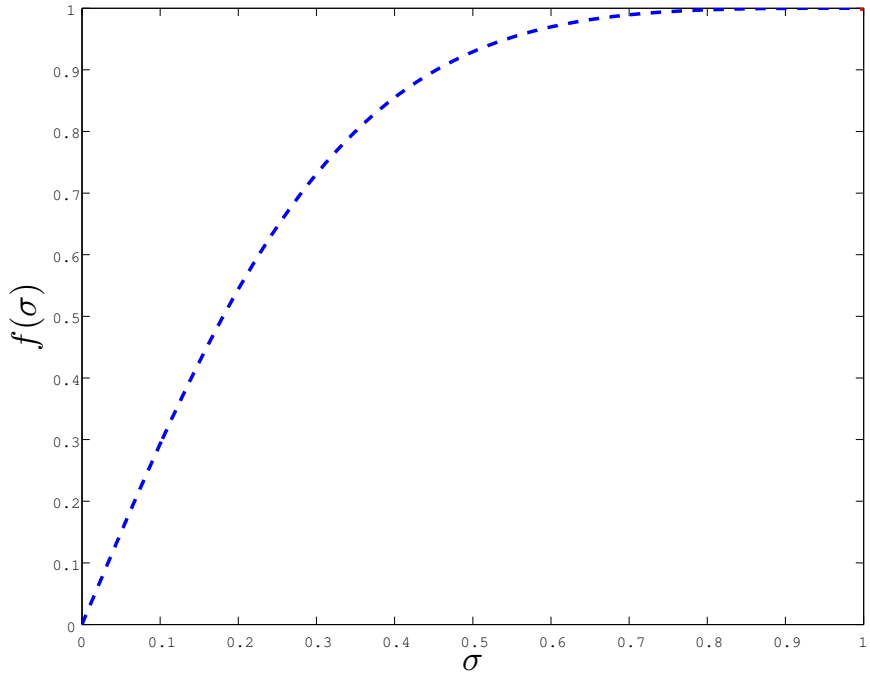
IRS: static parameters

it=6



QDWH: dynamical parameters

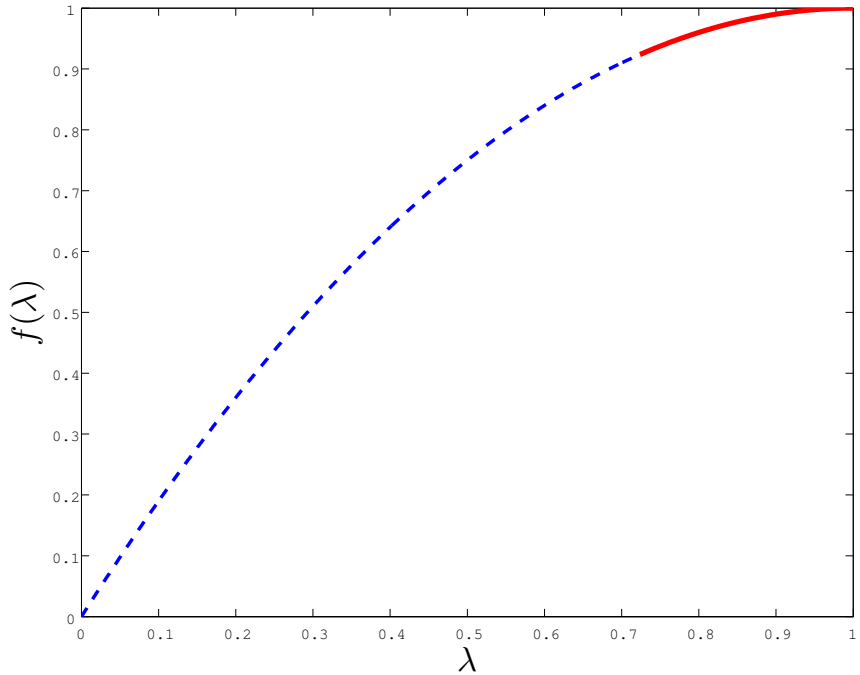
it=2



IRS vs. QDWH convergence

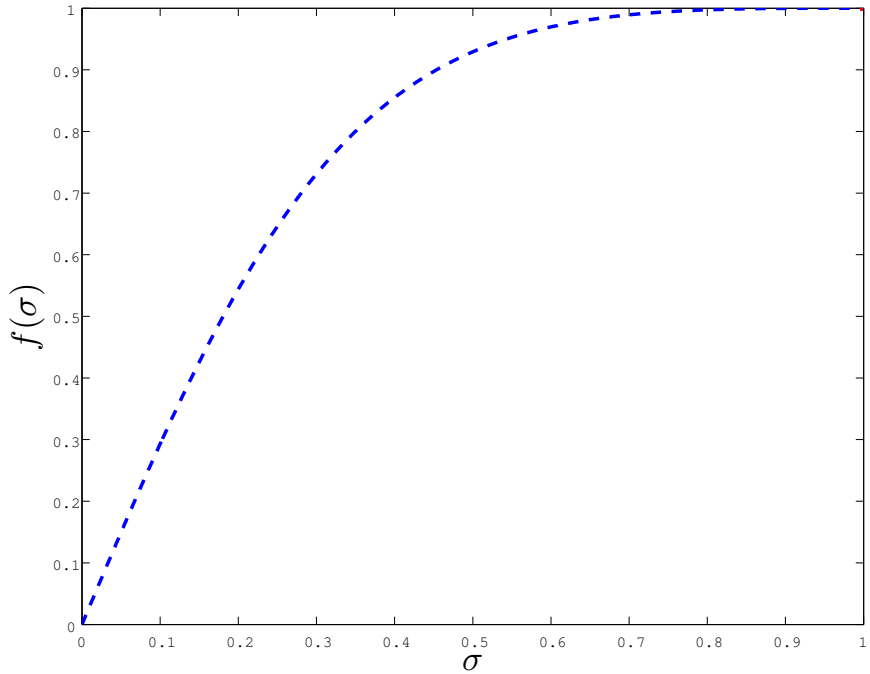
IRS: static parameters

it=7



QDWH: dynamical parameters

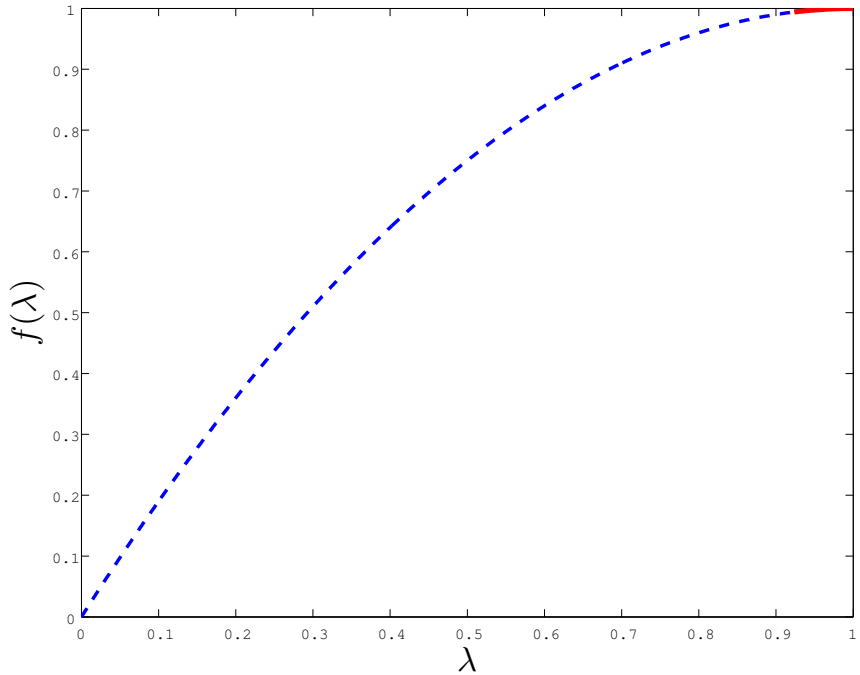
it=2



IRS vs. QDWH convergence

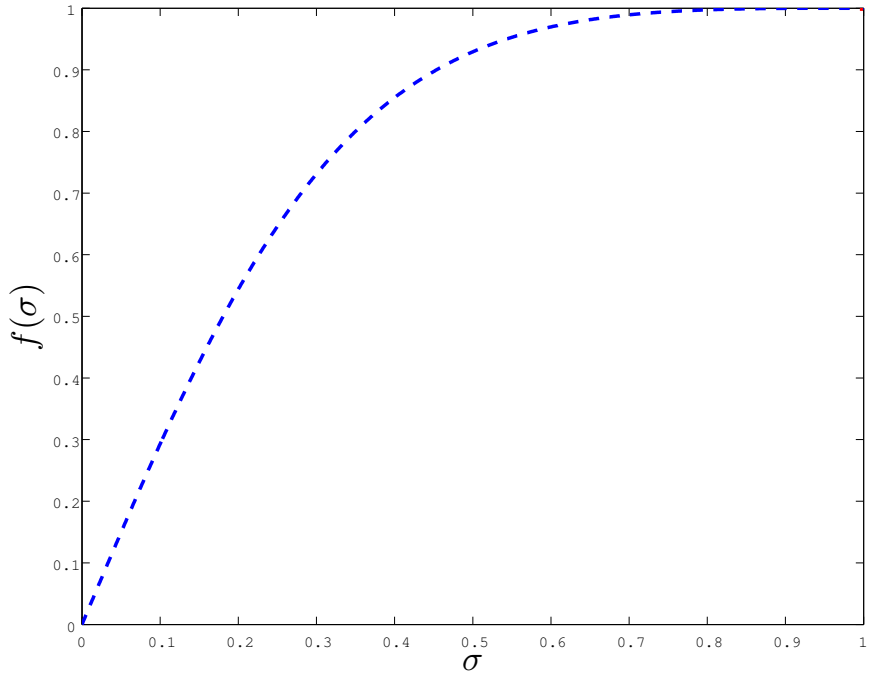
IRS: static parameters

it=8



QDWH: dynamical parameters

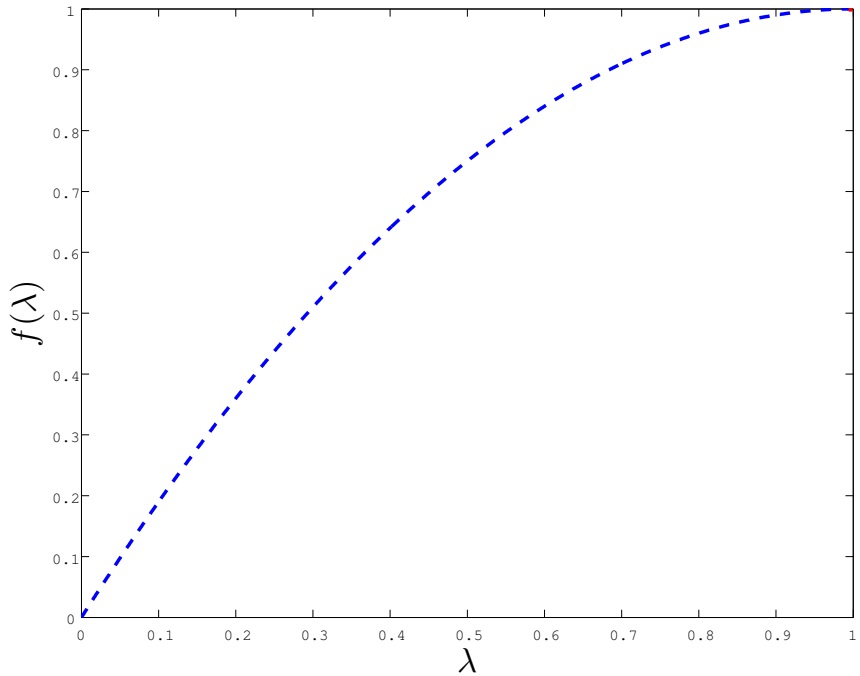
it=2



IRS vs. QDWH convergence

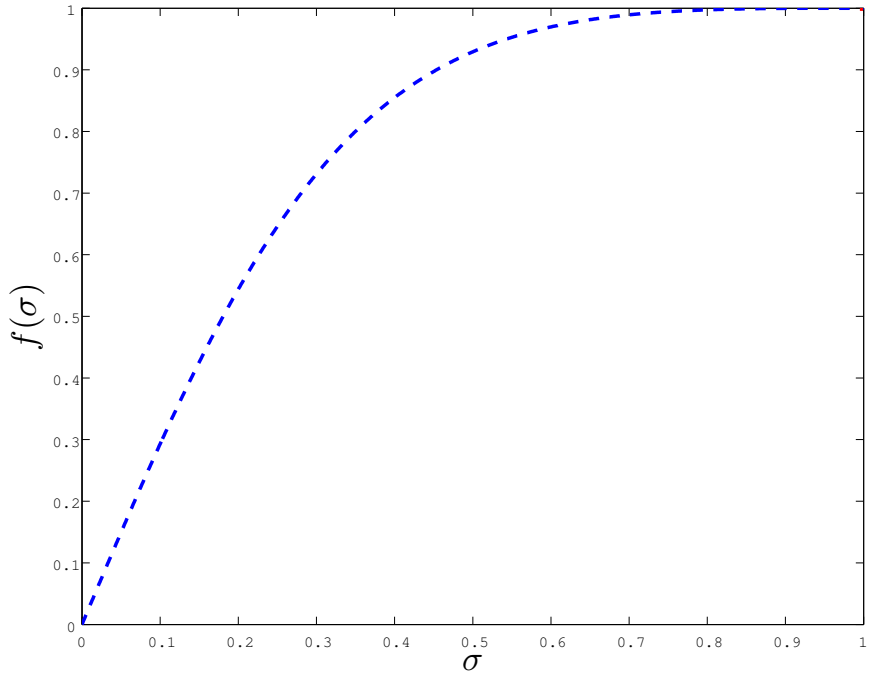
IRS: static parameters

it=9



QDWH: dynamical parameters

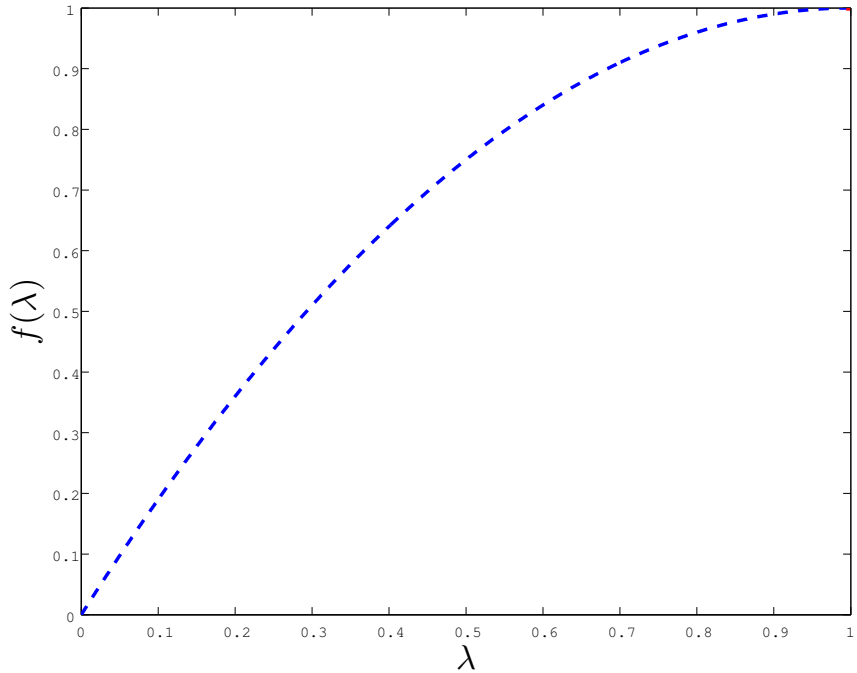
it=2



IRS vs. QDWH convergence

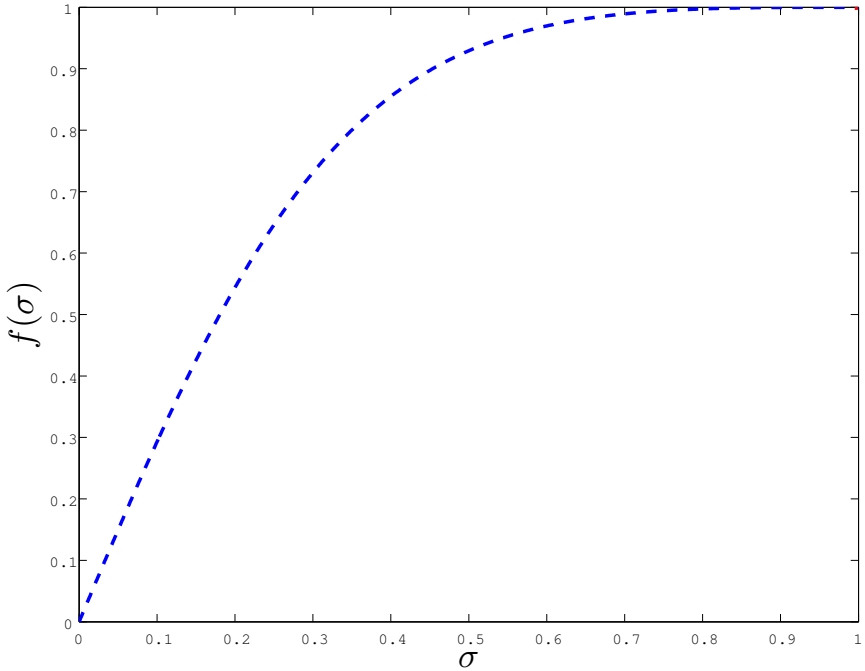
IRS: static parameters

it=9



QDWH: dynamical parameters

it=2



dynamical parameters speed up convergence significantly

In fact, no more than 6 iterations are needed for QDWH for any A with $\kappa_2(A) < 10^{16}$

Polar decomposition + symmetric eigenproblem = SVD

Algorithm **SVD**

1. compute polar decomposition

$$A = U_P H \tag{2}$$

2. compute symmetric eigendecomposition

$$H = V^T \Sigma V \tag{3}$$

3. get SVD $A = (U_P V^T)\Sigma V = U \Sigma V^T$

Polar decomposition + symmetric eigenproblem = SVD

Algorithm **SVD**

1. compute polar decomposition

$$A = U_P H \tag{2}$$

2. compute symmetric eigendecomposition

$$H = V^T \Sigma V \tag{3}$$

3. get SVD $A = (U_P V^T)\Sigma V = U \Sigma V^T$

- The computed SVD is backward stable if both (2) and (3) are

–[Higham and Papadimitriou (1993)]

$\Rightarrow$ **Backward stability again rests on that of polar decomposition**

- For $n \times n$ matrices, SVD is less expensive than **2** runs of QDWH-eig

note: SVD via eig $\left(\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\right)$ costs $\simeq \times 8$

Comparison

- symmetric eigenproblem

	standard	ZZY	IRS	QDWH-eig
Arithmetic cost	$9n^3$	$\simeq 370n^3$	$\approx 720n^3$	$27n^3$
Backward stable?	$\surd$	unknown	$\times$	$\surd$
Minimize communication?	$\times$	$\surd$	$\surd$	$\surd$

- SVD

	standard	IRS	QDWH-SVD
Arithmetic cost	$26n^3$	$\simeq 5700n^3$	$34n^3 \sim 52n^3$
Backward stable?	$\surd$	$\times$	$\surd$
Minimize communication?	$\times$	$\surd$	$\surd$

nb: IRS is applicable to generalized non-symmetric eigenproblems

Experiments: spectral d-c algorithms for symeig

$$A = A^T \in \mathbb{R}^{100\times 100},$$

$\lambda(A) = \{1, r, r^2, \ldots, r^{n-1}\}$ with $r = -\kappa^{-1/(n-1)}$, 100 runs

$\kappa_2(A)$		10^2		10^8		10^{15}	
		min	max	min	max	min	max
iter	QDWH-eig	4	5	5	6	6	6
	ZZY	12	12	32	32	55	55
	IRS	12	12	32	32	54	55
back error	QDWH-eig	4.2e-16	5.1e-16	4.3e-16	5.5e-16	5.0e-16	6.3e-16
	ZZY	1.3e-15	1.6e-15	1.9e-15	2.5e-15	2.2e-15	3.1e-15
	IRS	1.9e-15	4.2e-13	3.6e-13	3.4e-11	5.4e-10	1.1e-6

Numerical experiments: symeig accuracy

NAG MATLAB toolbox experiments with n-by-n matrices $A^T = A$, uniformly distributed eigenvalues

Backward error $\|A - \widehat{V}\widehat{\Lambda}\widehat{V}^T\|_F / \|A\|_F$

	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-eig	2.1e-15	2.4e-15	2.6e-15	2.8e-15	2.9e-15
Divide-Conquer	5.3e-15	7.3e-15	8.7e-15	9.8e-15	1.1e-14
MATLAB	1.4e-14	1.8e-14	2.3e-14	2.7e-14	3.0e-14
MR^3	4.2e-15	5.8e-15	7.1e-15	8.1e-15	8.9e-15
QR	1.5e-14	2.8e-14	2.5e-14	2.9e-14	3.2e-14

Distance from orthogonality $\|\widehat{V}^T\widehat{V} - I\|_F / \sqrt{n}$.

	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-eig	7.7e-16	8.0e-16	8.2e-16	8.4e-16	8.5e-16
Divide-Conquer	4.6e-15	6.3e-15	7.6e-15	8.7e-15	9.6e-15
MATLAB	1.1e-14	1.5e-14	1.9e-14	2.1e-14	2.3e-14
MR^3	2.6e-15	3.6e-15	4.2e-15	4.8e-15	5.3e-15
QR	1.1e-14	2.4e-14	1.9e-14	2.2e-14	2.5e-14

Numerical experiments: symeig runtime

NAG MATLAB toolbox experiments with n-by-n matrices $A^T = A$, uniform eigenvalues

	Runtime(s)				
	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-eig	11.0	64.1	188	526	959
Divide-Conquer	2.0	17.8	60.8	141	275
MATLAB	2.4	18.8	64.9	151	290
MR^3	4.0	37.0	120.7	262	504
QR	11.8	105.4	354.7	834	1631

Numerical experiments: SVD accuracy

NAG MATLAB toolbox experiments with n-by-n MATLAB randsvd matrices with uniform singular values, $\kappa_2(A) = 1.5$

Backward error $\|A - \widehat{U}\widehat{\Sigma}\widehat{V}^T\|_F / \|A\|_F$

	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-SVD	2.0e-15	2.3e-15	2.5e-15	2.7e-15	2.8e-15
Divide-Conquer	5.6e-15	7.8e-15	9.2e-15	1.0e-14	1.2e-14
MATLAB	5.8e-15	7.8e-15	9.2e-15	1.1e-14	1.2e-14
QR	1.6e-14	2.2e-14	2.7e-14	3.1e-14	3.5e-14

Distance from orthogonality $\max\{\|\widehat{U}^T\widehat{U} - I\|_F / \sqrt{n}, \|\widehat{V}^T\widehat{V} - I\|_F / \sqrt{n}\}$.

	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-SVD	7.7e-16	8.0e-16	8.2e-16	8.4e-16	8.5e-16
Divide-Conquer	5.2e-15	7.2e-15	8.4e-15	9.5e-15	1.1e-14
MATLAB	8.1e-15	7.1e-15	8.4e-15	9.8e-15	1.1e-14
QR	1.2e-14	1.7e-14	2.1e-14	2.5e-14	2.8e-14

Numerical experiments: SVD runtime

NAG MATLAB toolbox experiments with n-by-n MATLAB randsvd matrices with uniform singular values, $\kappa_2(A) = 1.5$

	Runtime(s)				
	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QDWH-SVD	15.4	90.3	269	581	1079
Divide-Conquer	5.9	43.5	140	316	605
MATLAB	4.0	32.0	105	243	470
QR	20.5	151.8	458	1071	2010

Summary and future work

- Proposed spectral divide-and-conquer algorithms for symeig, SVD
  - Arithmetic cost within a factor 3 of standard algorithms.
  - Backward stability is proved and empirically better than standard algorithms.
  - Uses only matrix multiplication, QR and Cholesky, hence minimizes communication.

for details, see –[Nakatsukasa and Higham, MIMS EPrint May 2012]
(codes also available at MATLAB Central Exchange 36830)

Summary and future work

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Work in progress

- Higher-order iteration using Zolotarev's rational approximation
  $\rightarrow$ further enhanced parallelizability

–[with R. W. Freund]

- Performance benchmarking with a parallel implementation

–[with J. Poulson, G. Quintana, R. van de Geijn]

- Extension to generalized eigenvalue problems

–[with N. J. Higham, F. Tisseur]

Spectral d-c for generalized eigenproblem

Goal: given (A, B), compute nonsingular Q, Z such that

$$(QAZ, QBZ) = \left(\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & \\ & B_{22} \end{bmatrix} \right).$$

Spectral d-c for generalized eigenproblem: outline

Goal: given (A, B), compute nonsingular Q, Z such that

$$QAZ = \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}, \quad QBZ = \begin{bmatrix} B_{11} & \\ & B_{22} \end{bmatrix}. \tag{4}$$

Idea: map eigenvalues to ± 1; $Re(\lambda) > 0$ to 1, $Re(\lambda) < 0$ to -1.

Suppose (A, B) diagonalizable: $A = X \begin{bmatrix} \Lambda_+ & \\ & \Lambda_- \end{bmatrix} Y, \quad B = XY.$

1. Compute (NB: left eigenmatrix is $W \neq X$)

$$\widetilde{A} = W \begin{bmatrix} I & \\ & -I \end{bmatrix} Y, \qquad \widetilde{B} = WY.$$

2. Form

$$(\widetilde{A} + \widetilde{B})^* = Y^* \begin{bmatrix} 2I & \\ & 0 \end{bmatrix} W^* = [Q_{1+}\ Q_{2+}] \begin{bmatrix} R_+ \\ 0 \end{bmatrix},$$

$$-(\widetilde{A} - \widetilde{B})^* = Y^* \begin{bmatrix} 0 & \\ & 2I \end{bmatrix} W^* = [Q_{1-}\ Q_{2-}] \begin{bmatrix} R_- \\ 0 \end{bmatrix}.$$

3. Then letting $Z = [Q_{2+}, Q_{2-}]$, $\exists$ nonsingular $Q = [Q_a, Q_b]$ such that (4) holds (Q_a, Q_b computed via QR)

Mapping eigenvalues to ± 1

Given

$$A = X \begin{bmatrix} \Lambda_+ & \\ & \Lambda_- \end{bmatrix} Y, \quad B = XY,$$

the goal is to compute

$$\widetilde{A} = W \begin{bmatrix} I & \\ & -I \end{bmatrix} Y, \quad \widetilde{B} = WY.$$

We do this via computing

$$A_k = W_k \begin{bmatrix} f_k(\Lambda_{+,k-1}) & \\ & f_k(\Lambda_{-,k-1}) \end{bmatrix} Y, \quad B_k = W_k Y,$$

where $f_k(\cdots f_2(f_1(x)) \to \text{sign}(x)\ (= \pm 1)$ for $x \in \Lambda(A, B)$.

- $f(x)$ are the same functions as in QDWH, has the form

$$\begin{aligned} f(x) = \frac{x(a + bx^2)}{1 + cx^2} &= \frac{b}{c}x + \left(a - \frac{b}{c}\right)\frac{x}{1 + cx^2} \\ &= \frac{b}{c}x + \frac{1}{2i}\left(a - \frac{b}{c}\right)\left(\frac{x}{\sqrt{c}x + i} - \frac{x}{\sqrt{c}x - i}\right) \end{aligned}$$

Mapping eigenvalues of (A, B) by rational functions

Let

$$\begin{bmatrix} \alpha A + \beta B \\ -(\gamma A + \delta B) \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} Q_{12}^T & Q_{22}^T \end{bmatrix} \begin{bmatrix} \alpha A + \beta B \\ -(\gamma A + \delta B) \end{bmatrix} = Q_{12}^T(\alpha A + \beta B) - Q_{22}^T(\gamma A + \delta B) = 0,$$

that is,

$$(\alpha A + \beta B)(\gamma A + \delta B)^{-1} = Q_{12}^{-T} Q_{22}^T.$$

This means

$$A^{(1)} = Q_{22}^T A, \quad B^{(1)} = Q_{12}^T B,$$

satisfies (recall $A = X\Lambda Y, B = XY$)

$$A^{(1)} = W^{(1)} g(\Lambda) Y, \quad B^{(1)} = W^{(1)} Y, \quad \text{where} \quad g(x) = x\frac{\alpha x + \beta}{\gamma x + \delta}.$$

Forming $W_2(\Lambda_1+\Lambda_2, I)Y$ from $W_1(\Lambda_1, I)Y, W_2(\Lambda_2, I)Y$

Goal: obtain $\widetilde{A} = Wf(\Lambda)Y,\ \widetilde{B} = WY$, where $A = X\Lambda Y,\ B = XY$,

$$f(x) = \frac{b}{c}x + \frac{1}{2i}\left(a - \frac{b}{c}\right)\left(\frac{x}{\sqrt{c}x+i} - \frac{x}{\sqrt{c}x-i}\right) := \frac{b}{c}x + g_1(x) + g_2(x).$$

We have $A^{(i)} = W^{(i)}g_i(\Lambda)Y,\ B^{(i)} = W^{(i)}Y = W^{(i,B)}B,\ i = 1, 2$. Hence if

$$\begin{bmatrix} B^{(1)} \\ -B^{(2)} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix},$$

then $Q_{22}^* W^{(2)} = Q_{12}^* W^{(1)} := W$, so $Q_{22}^* B^{(2)} = Q_{12}^* B^{(1)} = WY = \widetilde{B}$ and

$$\begin{aligned} \widetilde{A} :&= \frac{b}{c}Q_{12}^* W^{(1,B)}A + Q_{12}^* A^{(1)} + Q_{22}^* A^{(2)} \\ &= W(\frac{b}{c}\Lambda + g_1(\Lambda) + g_2(\Lambda))Y = Wf(\Lambda)Y \end{aligned}$$

give the required $(\widetilde{A}, \widetilde{B})$.

Properties: Spectral d-c for GEP

– When (A, B) are symmetric but indefinite, Spectral d-c takes full advantage of symmetry by congruence:

$$Z^* A Z = \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}, \quad Z^* B Z = \begin{bmatrix} B_{11} & \\ & B_{22} \end{bmatrix}.$$

- Conventional approach uses the QZ algorithm:
  - Destroys symmetry
  - Yields upper-triangular (QAZ, QAB), not (block) diagonal
- No need to check for definiteness of (A, B)

– Spectral d-c uses (heavy use of)QR and matmul, minimizes communication

– Seems to work even for singular (A, B)

MATLAB experiments with 1000×1000 matrices

Showing backward error $\max\left(\frac{\|\widehat{P}A\widehat{Q}-\widehat{\Lambda}\|_F}{\|\widehat{\Lambda}\|_F}, \frac{\|\widehat{P}B\widehat{Q}-I\|_F}{\sqrt{n}}\right)$.

- A, B symmetric, $B > 0$, $\kappa_2(B) = 10^{10}$.

	backward error
QDWH-GEP	2.7e-13
Matlab eig	6.9e-11

- A, B symmetric indefinite, $B = X^T \mathrm{diag}(\pm 1)X$, $\kappa_2(B) = 10^{10}$.

	backward error
QDWH-GEP	2.7e-13
Matlab eig	5.0e-11

- A, B nonsymmetric, 998 real eigenvalues

	backward error
QDWH-GEP	3.4e-13
Matlab eig	1.8e-10

References

- G. Ballard, J. Demmel, I. Dumitriu, Minimizing communication for eigenproblems and the singular value decomposition. LAWN237.
- Y. Nakatsukasa, Z. Bai, and F. Gygi. Optimizing Halley's iteration for computing the matrix polar decomposition. SIAM J. Matrix Anal. Appl., 2010.
- Y. Nakatsukasa and N. J. Higham. Backward stability of iterations for computing the polar decomposition. SIAM J. Matrix Anal. Appl., 2012.
- Y. Nakatsukasa and N. J. Higham. Efficient, communication-minimizing algorithms for the symmetric eigenvalue decomposition and the singular value decomposition. MIMS EPrint 2012.52.

Standard algorithm for SVD

1. Reduce matrix to bidiagonal form

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \overset{H_L}{\rightarrow} \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{bmatrix} \overset{H_R}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{bmatrix}$$

$$\overset{H_L}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \overset{H_R}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \\ & & * & * \end{bmatrix} \overset{H_L}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \\ & & & * \end{bmatrix} \equiv B$$

–[Golub and Kahan (1965)]

Accumulate orthogonal factors: $A = U_A B V_A^H$ where $U_A = (\prod H_L)^H, V_A = \prod H_R$

2. Compute SVD $B = U_B \Sigma V_B^H$ (divide-conquer, QR, dqds + twisted factorization)
3. SVD: $A = (U_A U_B)\Sigma(V_B^H V_A^H) = U \Sigma V^H$

Standard algorithm for SVD

1. Reduce matrix to bidiagonal form
   $\Rightarrow$ Expensive in communication cost

$$A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \overset{H_L}{\rightarrow} \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{bmatrix} \overset{H_R}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & * \\ & * & * & * \\ & * & * & * \end{bmatrix}$$

$$\overset{H_L}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \overset{H_R}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \\ & & * & * \end{bmatrix} \overset{H_L}{\rightarrow} \begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \\ & & & * \end{bmatrix} \equiv B$$

–[Golub and Kahan (1965)]

   Accumulate orthogonal factors: $A = U_A B V_A^H$ where $U_A = (\prod H_L)^H, V_A = \prod H_R$
2. Compute SVD $B = U_B \Sigma V_B^H$ (divide-conquer, QR, dqds + twisted factorization)
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We develop completely different algorithms based on

spectral divide-and-conquer

Backward stability proof of QDWH-eig

Goal: show $\|E\|_2 = \|\epsilon A\|_2$ where $\widehat{V}^T A \widehat{V} = \begin{bmatrix} A_+ & E^T \\ E & A_- \end{bmatrix}$

Assumptions:

$$A = \widehat{U}\widehat{H} + \epsilon\|A\|_2, \quad \widehat{U}^T\widehat{U} - I = \epsilon.$$

$$\widehat{V}^T \begin{bmatrix} I & \\ & -I \end{bmatrix} \widehat{V} = \widehat{U} + \epsilon$$

- By the assumptions $A = \widehat{V}\begin{bmatrix} I & \\ & -I \end{bmatrix}\widehat{V}^T\widehat{H} + \epsilon\|A\|_2$, so

$$\begin{aligned} 0 &= A - A^T \\ &= \left(\widehat{V}\begin{bmatrix} I & \\ & -I \end{bmatrix}\widehat{V}^T\widehat{H} - \widehat{H}^T\widehat{V}\begin{bmatrix} I & \\ & -I \end{bmatrix}\widehat{V}^T\right) + \epsilon\|A\|_2 \end{aligned}$$

- Therefore

$$\epsilon\|A\|_2 = \widehat{V}^T\widehat{H}\widehat{V} - \begin{bmatrix} I & \\ & -I \end{bmatrix}\widehat{V}^T\widehat{H}\widehat{V}\begin{bmatrix} I & \\ & -I \end{bmatrix} = 2\begin{bmatrix} 0 & E^T \\ -E & 0 \end{bmatrix}$$

Subspace iteration to obtain V_+

$$U = [V_+ \; V_-] \begin{bmatrix} I & \\ & -I \end{bmatrix} [V_+ \; V_-]^T$$

- U is Hermitian with eigenvalues ± 1
- $\frac{1}{2}(U + I) = [V_+ \; V_-] \begin{bmatrix} I & \\ & 0 \end{bmatrix} [V_+ \; V_-]^T = V_+ V_+^T$ is the projection onto the positive eigenspace of A

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Obtain V_+ from $C \equiv \frac{1}{2}(U + I) = V_+ V_+^T$:

- Simultaneous iteration with initial guess $Q_1 = C(:, 1 : \sqrt{\|Q\|_F^2 + 1})$
  $C = Q_1 R$, $CQ_1 = Q_2 R$, ...
  - Theoretical convergence factor is $\simeq 1/\epsilon$
  - Stop when $\widehat{C\widehat{V}_1} = \widehat{V}_1 + \epsilon$ and $\widehat{C\widehat{V}_2} = \epsilon$: in practice, at most 2 iterations are necessary
  - Use a random initial guess if fails (never happened in experiments)

Newton–Schulz postprocessing to improve orthogonality

In practice, the computed $\widehat{U}, \widehat{V}$ in $A \simeq \widehat{U}\Sigma\widehat{V}^*$ have distance from orthogonality $\|\widehat{U}^*\widehat{U} - I\|_F, \|\widehat{V}^*\widehat{V} - I\|_F > 0$.

A (empirically) powerful way to improve the orthogonality is to run a step of the Newton–Schulz iteration $U := \frac{1}{2}U(3I - U^*U)$.

$\max\{\|\widehat{U}^*\widehat{U} - I\|_F/\sqrt{n}, \|\widehat{V}^*\widehat{V} - I\|_F/\sqrt{n}\}$

	$n = 2000$	$n = 4000$	$n = 6000$	$n = 8000$	$n = 10000$
QR	1.2e-14	1.7e-14	2.1e-14	2.5e-14	2.8e-14
QDWH before N–S	2.8e-15	3.1e-15	3.4e-15	3.6e-15	3.8e-15
QDWH after N–S	7.7e-16	8.0e-16	8.2e-16	8.4e-16	8.5e-16

Faster iterations using Cholesky factorization

$$X_{k+1} = X_k(a_k I + b_k X_k^H X_k)(I + c_k X_k^H X_k)^{-1}, \quad X_0 = A/\alpha \tag{5}$$

- The QR-kernel

$$\begin{bmatrix} \sqrt{c_k} X_k \\ I \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R, \quad X_{k+1} = \frac{b_k}{c_k} X_k + \frac{1}{\sqrt{c_k}} \left(a_k - \frac{b_k}{c_k} \right) Q_1 Q_2^H$$

Note $\kappa_2 \left(\begin{bmatrix} \sqrt{c_k} X_k \\ I \end{bmatrix} \right) \leq \sqrt{c_k} + 1$, since $\|X_k\|_2 \leq 1$

- well-conditioned if c_k is not too large (e.g., $c_k \leq 100$), then **always** true for $k \geq 3$
- then, safely use CholeskyQR to compute Q_1, Q_2:

$$Z = I + c_k X_k^H X_k, \quad W = chol(Z),$$
$$Q_1 = \sqrt{c_k} X_k W^{-1}, \quad Q_2 = W^{-1}$$

communication-optimal + fewer arithmetic

Backward stability of iterations for polar decomposition

–[Nakatsukasa and Higham, SIMAX 2012]

An iteration $X_{k+1} = f_k(X_k), \quad X_0 = A, \lim_{k\to\infty} X_k = U$ for computing the polar decomposition is backward stable ($A - \widehat{U}\widehat{H} = \epsilon\|A\|$) if

1. Each iteration is mixed backward–forward stable:
   $\widehat{X}_{k+1} = f_k(\widetilde{X}_k) + \epsilon\|\widehat{X}_{k+1}\|, \quad \widetilde{X}_k = X_k + \epsilon\|X_k\|$

2. The mapping function $f_k(x)$ does not significantly decrease the relative size of σ_i for $i = 1 : n$

$$d\frac{f_k(\sigma_i)}{\|f_k(\widetilde{X})\|_2} \geq \frac{\sigma_i}{\|\widetilde{X}\|_2}$$

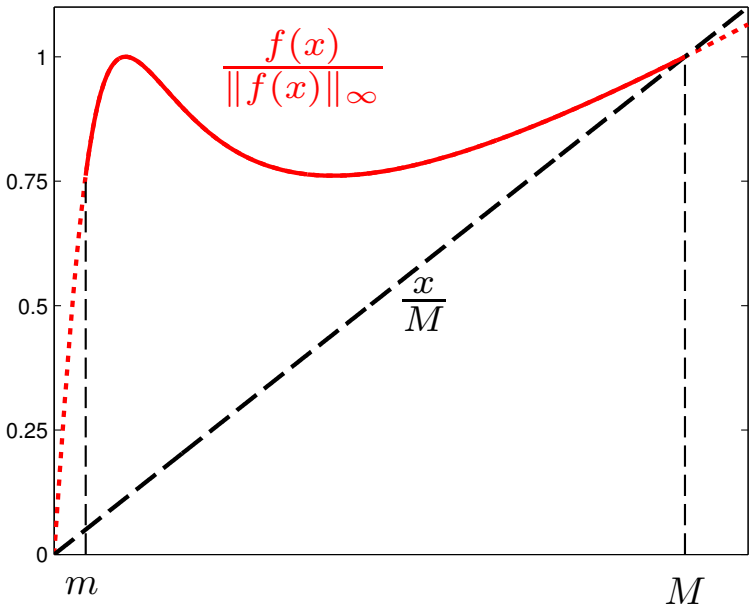
for $d = O(1)$.

(we prove $A - \widehat{U}\widehat{H} = d\epsilon\|A\|$, $H - \widehat{H} = d\epsilon\|H\|$)

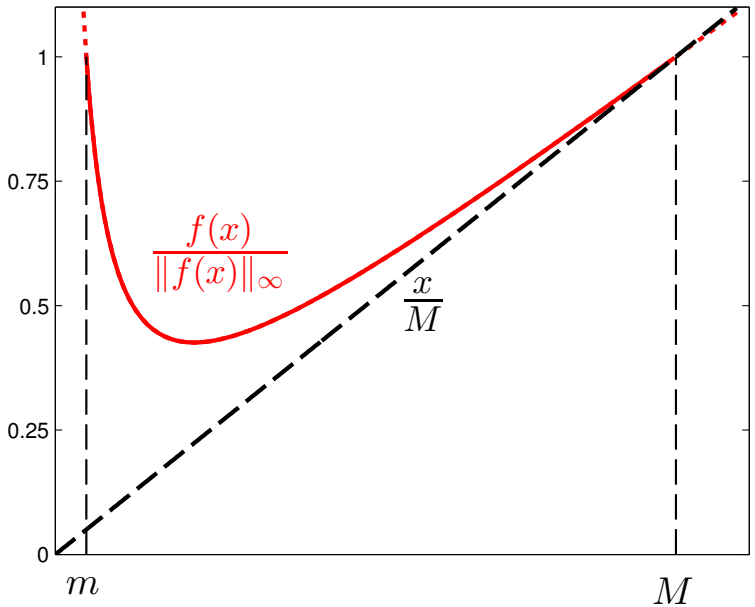
– for details, please go to Nick Higham's talk (MS 40, Wed 12:15)

Mapping function: stable (top) and unstable (bottom)

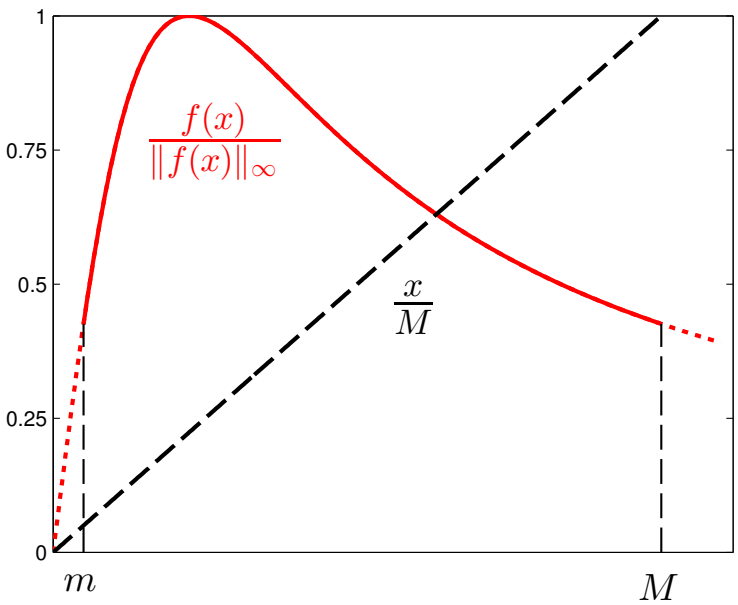
QDWH: $d = 1$



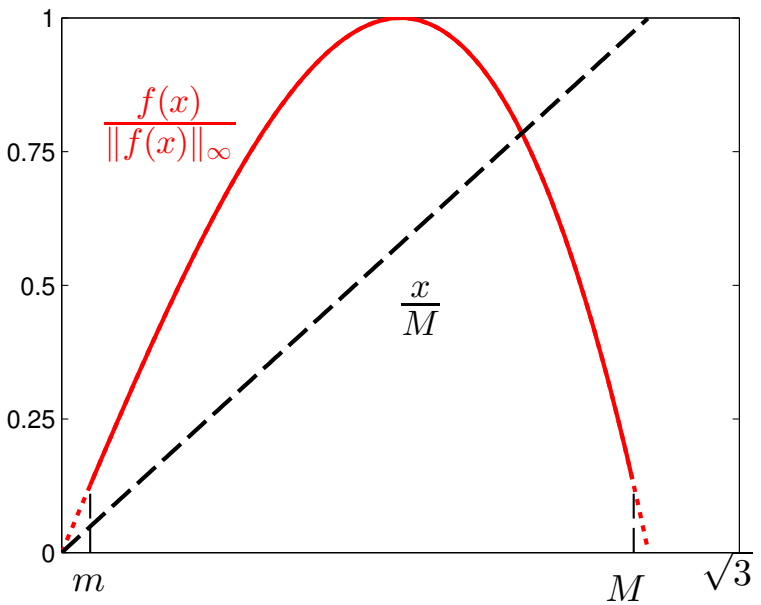
scaled Newton: $d = 1$



inverse Newton: $d \geq (\kappa_2(A))^{1/2}$



Newton–Schulz



Scaled Newton iteration

$$X_{k+1} = \frac{1}{2}(\zeta_k X_k + (\zeta_k X_k)^{-*}), \quad X_0 = A$$

– $\lim_{k\to\infty} X_k = U$ of $A = UH$ quadratically with appropriate scalings ζ_k

– Higham(86) : $\zeta_k = \left(\frac{\|X_k^{-1}\|_1 \|X_k^{-1}\|_\infty}{\|X_k\|_1 \|X_k\|_\infty} \right)^{1/4}$

Byers-Xu (08) : $\zeta_{k+1} = \frac{2}{\sqrt{(\zeta_k + 1/\zeta_k)}}$, $\zeta_0 = 1/\sqrt{ab}$, $a \geq \|A\|_2, b \leq \sigma_{\min}(A)$

– Both scalings work well in practice

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- **However, X_k^{-1} needed explicitly**
  - Efforts in designing inverse-free algorithms
    –[Bai, Demmel and Gu (97), Crudge (98), Byers and Xu (01)]
    1. Instability of computing inverses
    2. High communication cost (pivoting is expensive)

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Known inverse-free algorithms faced instability or slow convergence

Halley iterations

Halley iteration:

$$X_{k+1} = X_k(3I + X_k^* X_k)(I + 3X_k^* X_k)^{-1}, \quad X_0 = A.$$

- $X_k \to U$ as $k \to \infty$, convergence is global and asymptotically cubic
- Can be implemented inverse-free via QR
- Applicable to rectangular/singular matrices
- Slow convergence for ill-conditioned A

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(New): Dynamically Weighted Halley iteration (DWH)

$$X_{k+1} = X_k(a_k I + b_k X_k^* X_k)(I + c_k X_k^* X_k)^{-1}, \quad X_0 = A/\alpha, \quad \alpha \simeq \|A\|_2$$

- Choose a_k, b_k, c_k dynamically to get "suboptimal" convergence speed

Choosing weighting parameters

$$X_{k+1} = X_k(a_k I + b_k X_k^* X_k)(I + c_k X_k^* X_k)^{-1}$$

Suppose $[\sigma_{\min}(X_k), \sigma_{\max}(X_k)] \subseteq [\ell_k, 1]$, then

- $\sigma_i(X_{k+1}) = f_k(\sigma_i(X_k))$, where $f_k(x) = x\dfrac{a_k + b_k x^2}{1 + c_k x^2}$
- $\sigma_i(X_{k+1}) \in [\min_{\ell_k \le x \le 1} f_k(x), \max_{\ell_k \le x \le 1} f_k(x)]$

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- $\sigma_i(X_{k+1}) \in [\min_{\ell_k \le x \le 1} f_k(x), \max_{\ell_k \le x \le 1} f_k(x)]$

Idea: Find (a_k, b_k, c_k) such that

$$\max_{a_k, b_k, c_k > 0} \left\{ \min_{\ell_k \le x \le 1} f_k(x) \right\},$$

under $\quad 0 < f_k(x) \le 1 \quad$ for $\quad \ell_k \le x \le 1$

Solving rational optimization problem

$$\max_{a,b,c}\left\{\min_{\ell\le x\le 1} f(x)\right\} \quad \text{under} \quad f(x)=x\frac{a+bx^2}{1+cx^2}, \quad 0<f(x)\le 1 \quad \text{on} \quad [\ell,1]$$

- Analytic solution (a,b,c):

$$a=h(\ell), \quad b=(a-1)^2/4, \quad c=a+b-1,$$

where $h(\ell)=\sqrt{1+d}+\frac{1}{2}\sqrt{8-4d+\frac{8(2-\ell^2)}{\ell^2\sqrt{1+d}}}$, $d=\sqrt[3]{\frac{4(1-\ell^2)}{\ell^4}}$

– tedious, but doable

- Alternatively, convert the problem to semi-definite programming, then use solver such as SeDuMi

– [Nie (09)]

DWH Algorithm

$$X_{k+1} = X_k(a_k I + b_k X_k^* X_k)(I + c_k X_k^* X_k)^{-1}, \quad X_0 = A/\alpha$$

- Estimate α, ℓ_0 such that
  1. $\alpha \simeq \|A\|_2$,
  2. $\ell_0 \lesssim \sigma_{\min}(A/\alpha)$

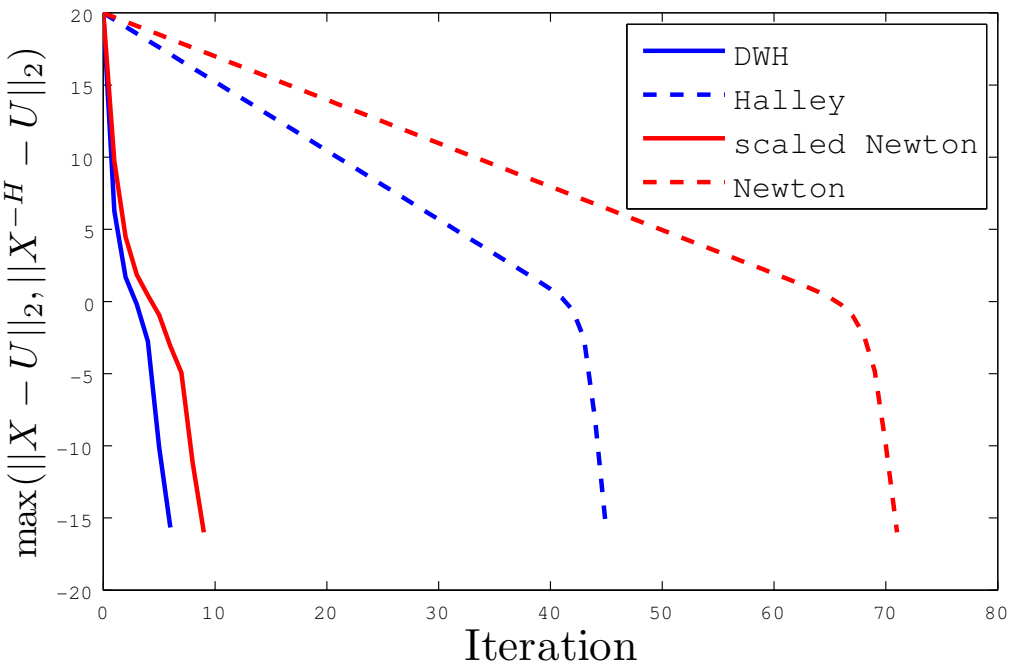
- $a_k = h(\ell_k), \quad b_k = (a_k - 1)^2/4, \quad c_k = a_k + b_k - 1,$

  where $h(\ell) = \sqrt{1+d} + \frac{1}{2}\sqrt{8 - 4d + \frac{8(2-\ell^2)}{\ell^2\sqrt{1+d}}}, \quad d = \frac{2^{2/3}(1-\ell^2)^{1/3}}{\ell^{4/3}}$

- Update: $\ell_{k+1} = \ell_k \dfrac{a_k + b_k \ell_k^2}{1 + c_k \ell_k^2}$

Efficiency of dynamical weighting

Cond. number		2	10	10^2	10^5	10^{10}	10^{15}	10^{20}
iter	DWH	3	4	4	5	5	6	6
	Halley	4	5	7	14	24	35	45
	scaled Newton	5	6	7	7	8	9	9



QR-based implementation of DWH

- Lemma:

$$Q_1 Q_2^* = A(I + A^*A)^{-1}, \text{ where } \begin{bmatrix} A \\ I \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \mathrm{R}$$

- DWH

$$\begin{aligned} X_{k+1} &= X_k(a_k I + X_k^* X_k)(b_k I + c_k X_k^* X_k)^{-1} \\ &= \frac{b_k}{c_k} X_k + \left(a_k - \frac{b_k}{c_k}\right) X_k (I + c_k X_k^* X_k)^{-1}, \end{aligned}$$

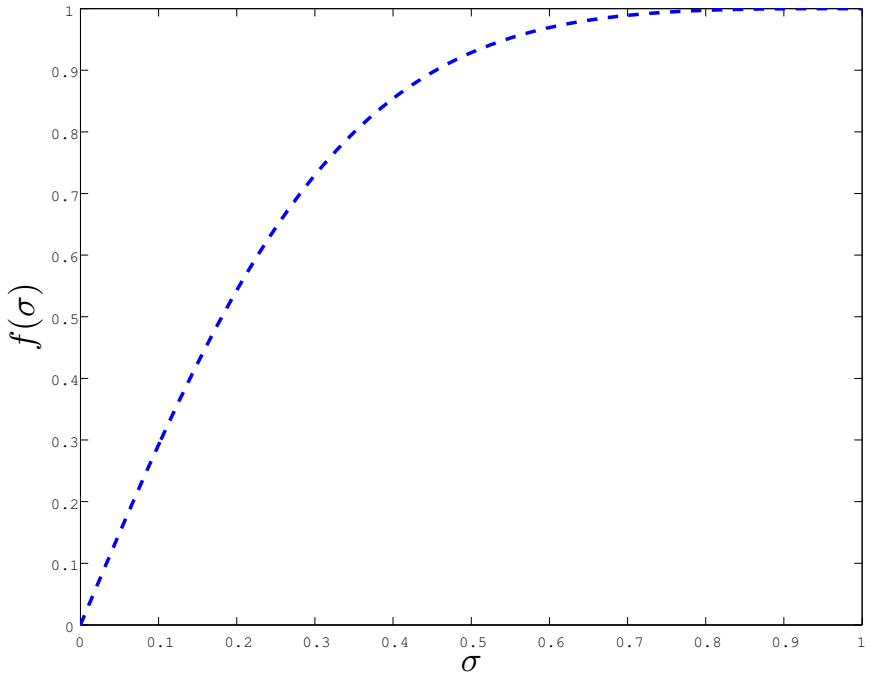
- QR-based DWH

$$\begin{cases} \begin{bmatrix} \sqrt{c_k} X_k \\ I \end{bmatrix} &= \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R, \\ X_{k+1} &= \frac{b_k}{c_k} X_k + \frac{1}{\sqrt{c_k}} \left(a_k - \frac{b_k}{c_k}\right) Q_1 Q_2^* \end{cases}$$

Halley vs. Dynamical weighted Halley

Halley: static parameters

it=0



DWH: dynamical parameters

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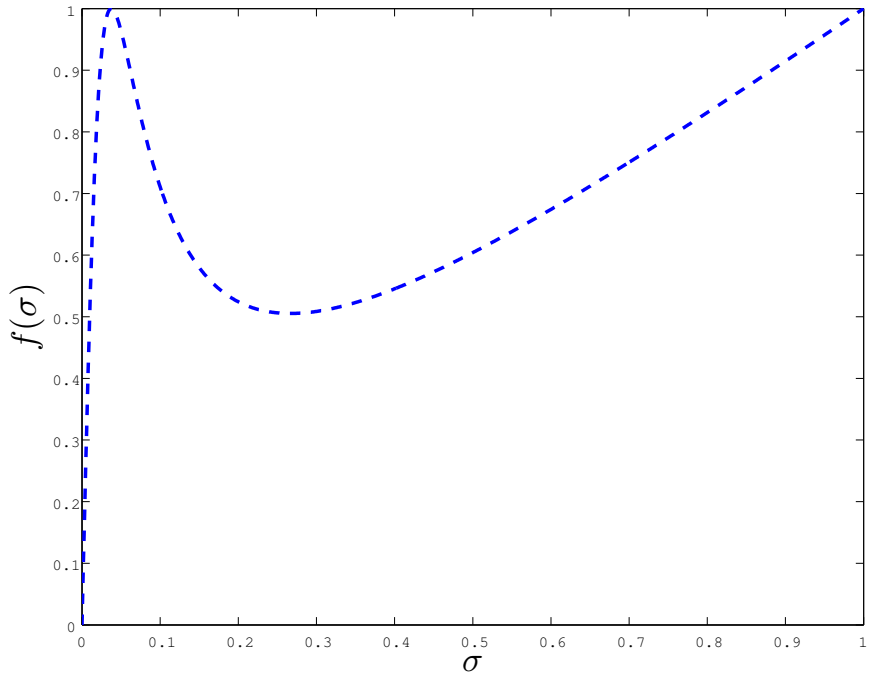

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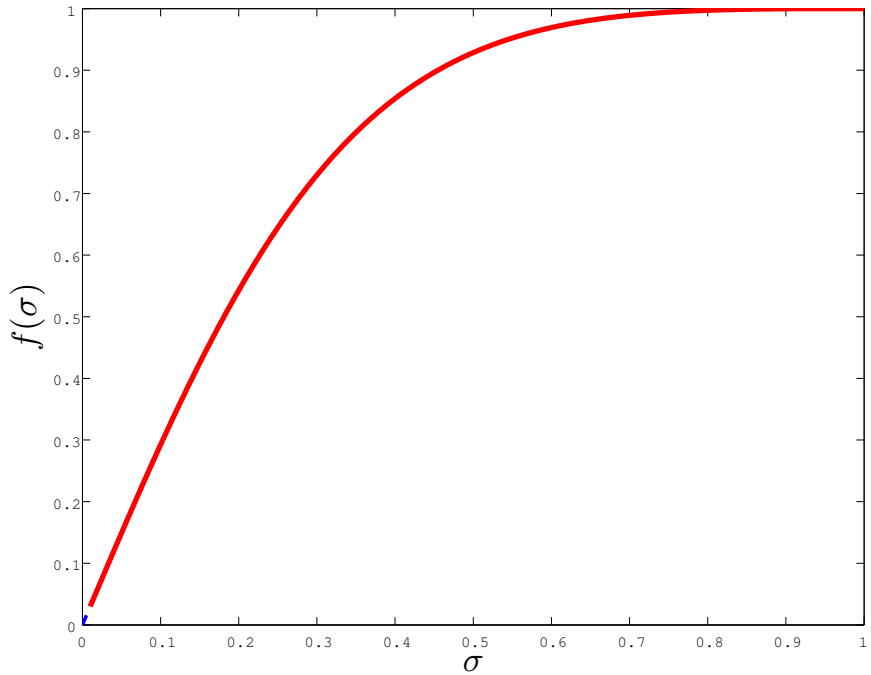

DWH: dynamical parameters

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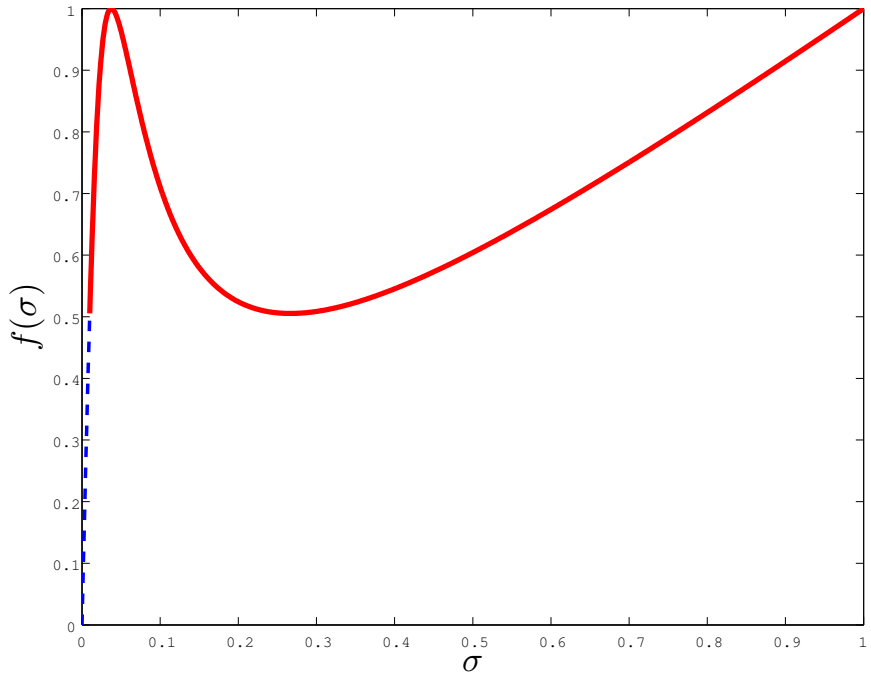
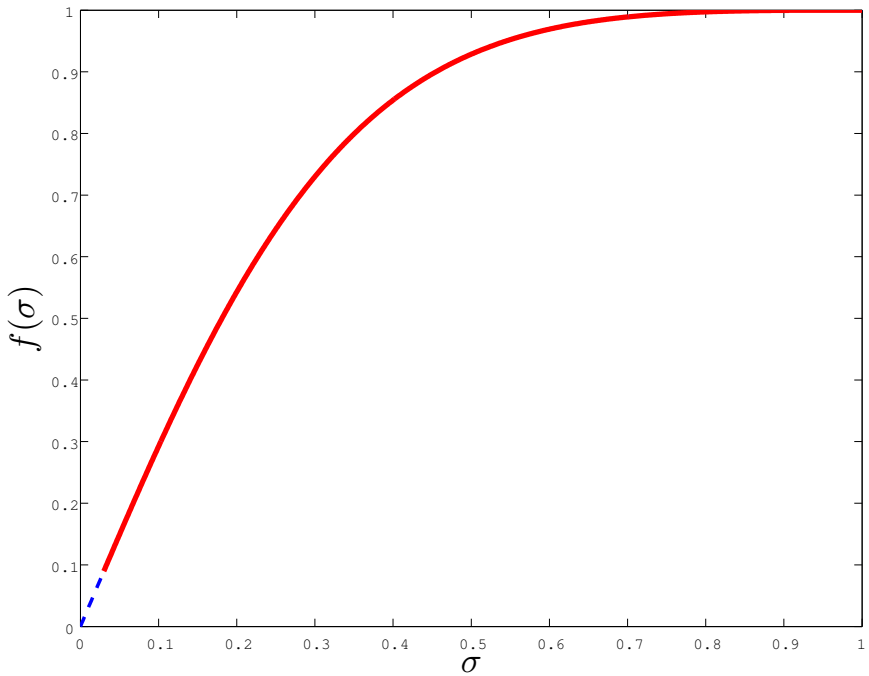


Halley vs. Dynamical weighted Halley

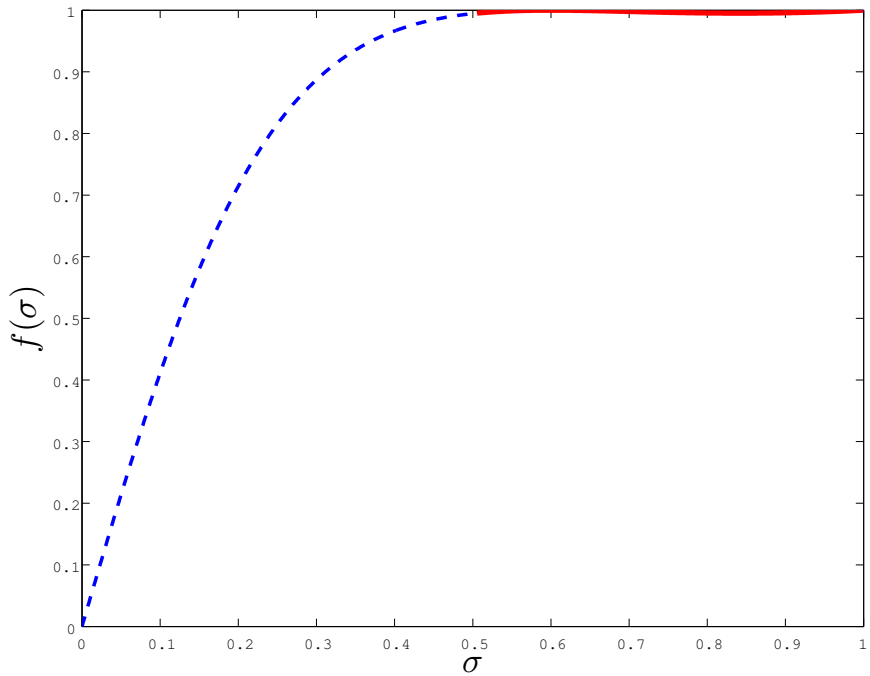
Halley: static parameters

it=1



DWH: dynamical parameters

it=1



Halley vs. Dynamical weighted Halley

Halley: static parameters

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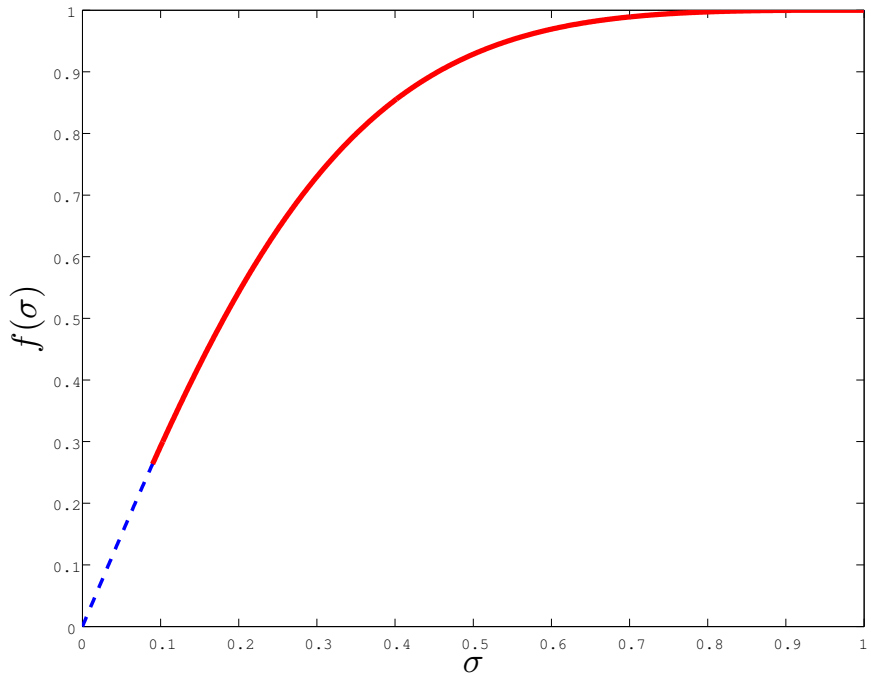
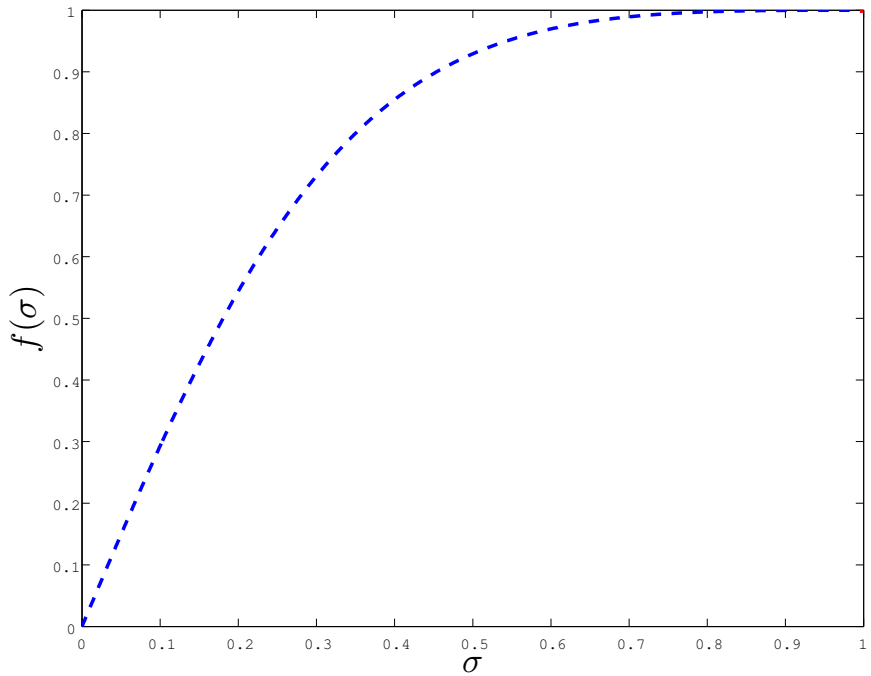


DWH: dynamical parameters

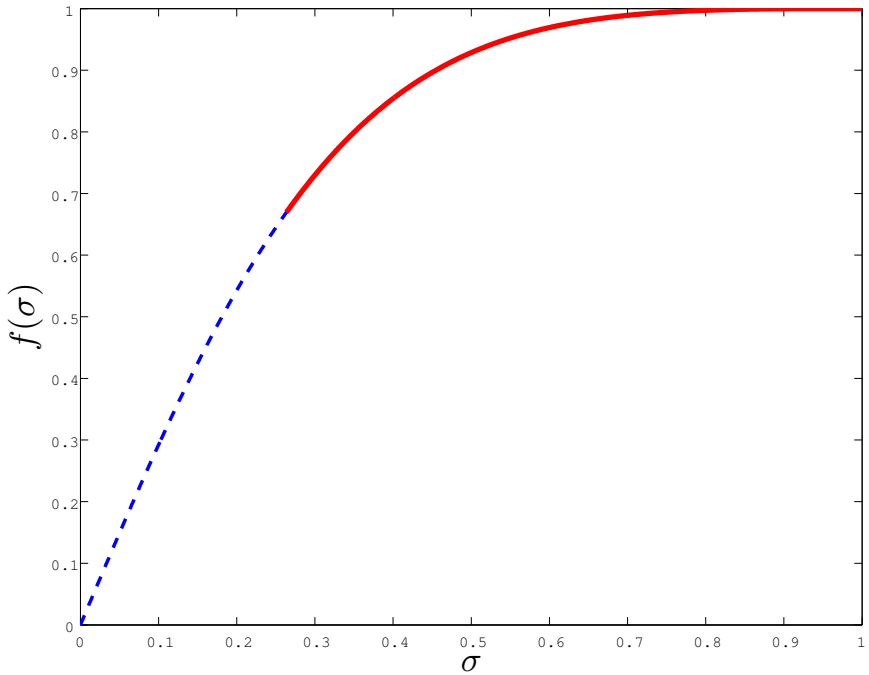
it=2



Halley vs. Dynamical weighted Halley

Halley: static parameters

it=3



DWH: dynamical parameters

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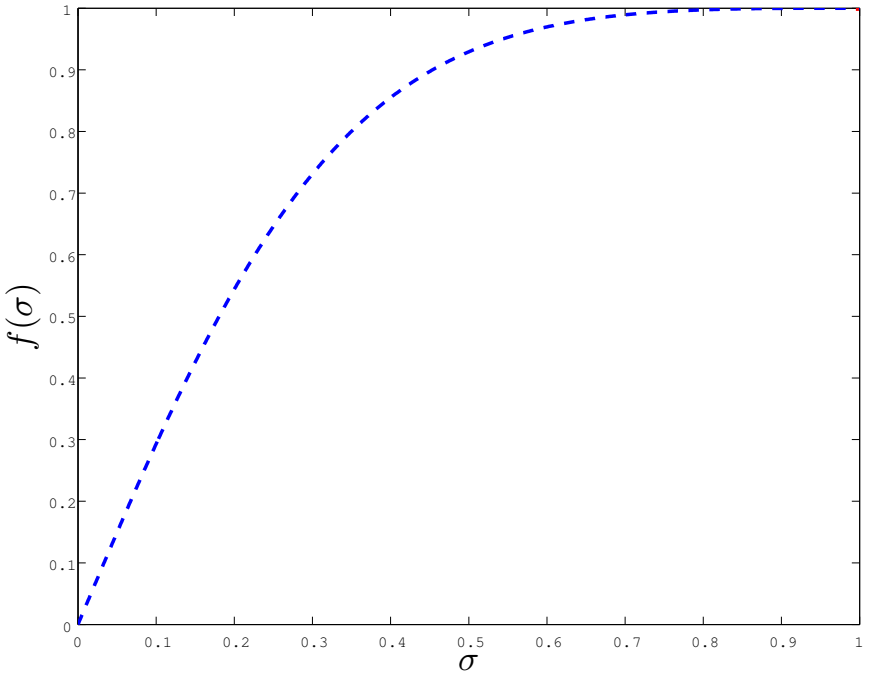
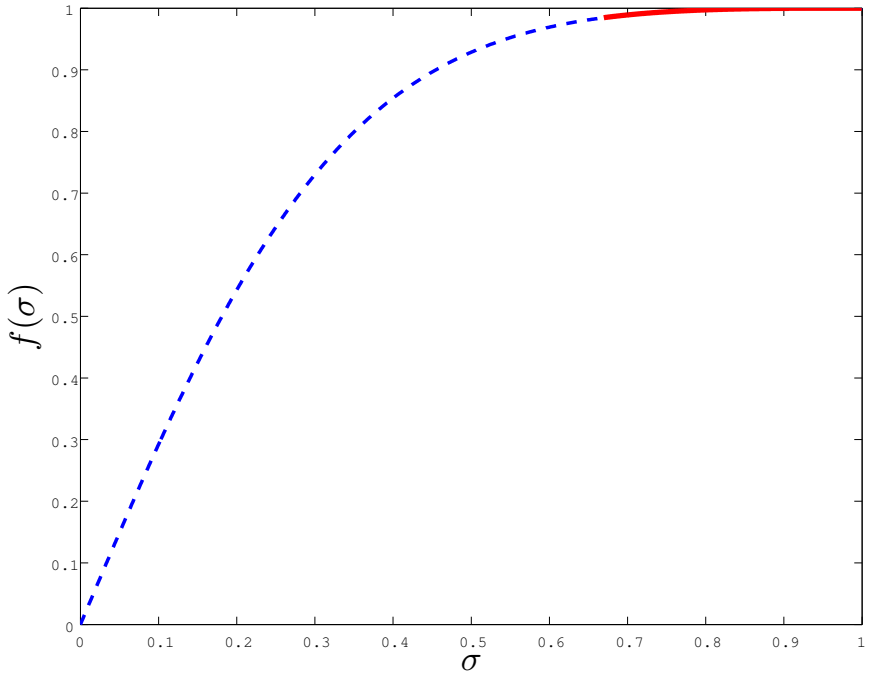


Halley vs. Dynamical weighted Halley

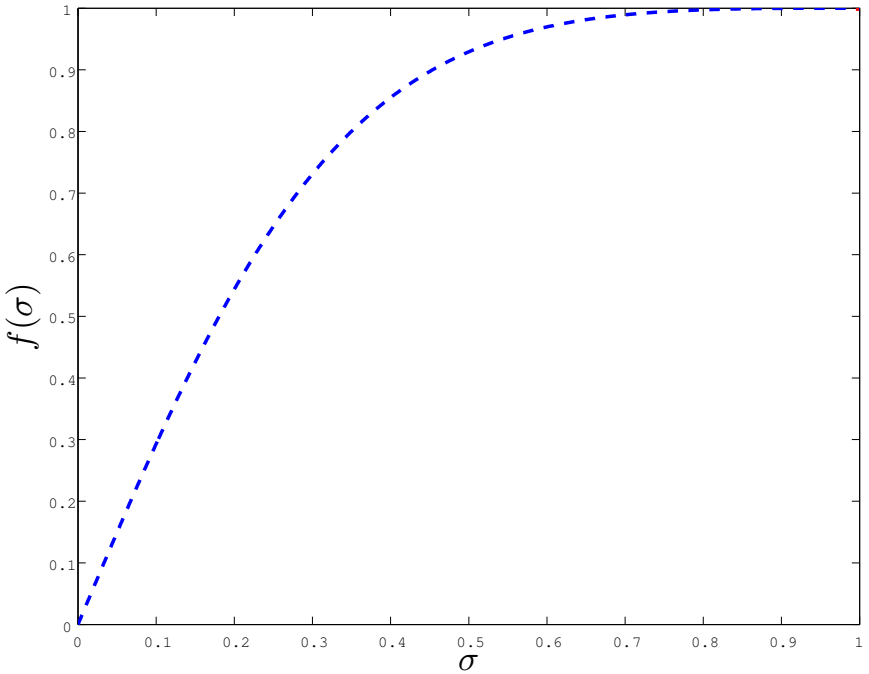
Halley: static parameters

it=4



DWH: dynamical parameters

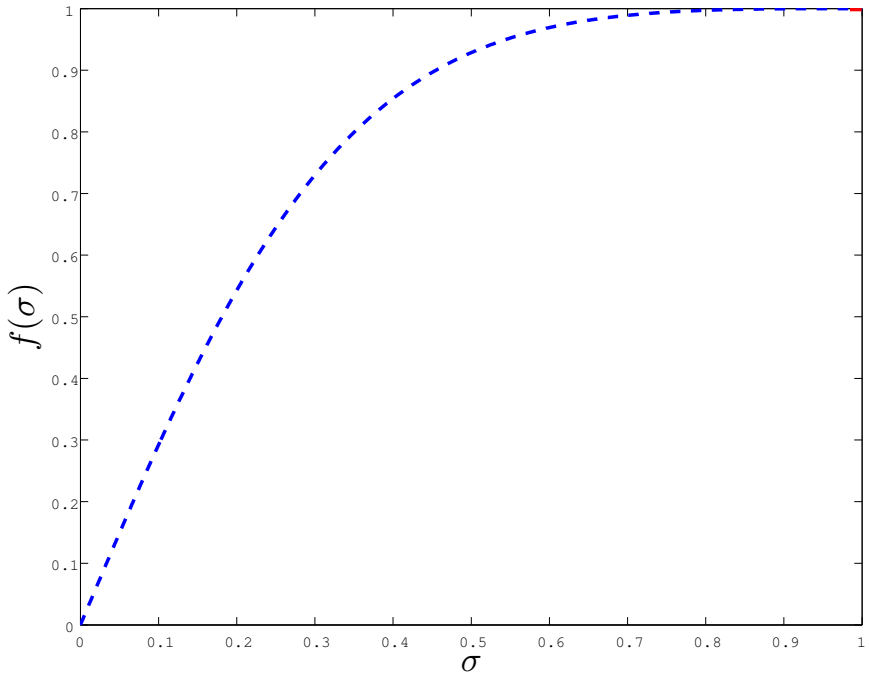
it=2



Halley vs. Dynamical weighted Halley

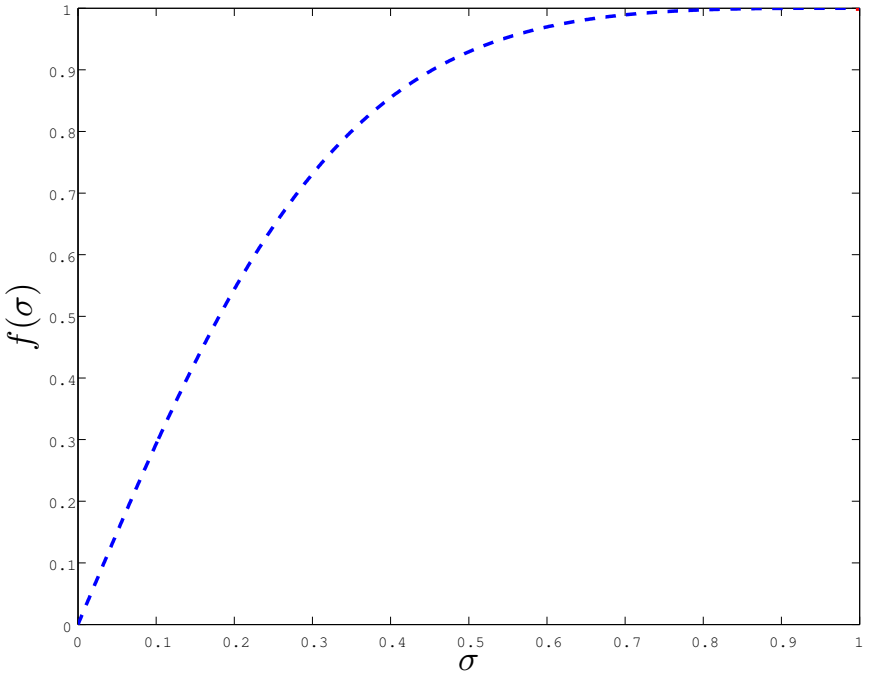
Halley: static parameters

it=5



DWH: dynamical parameters

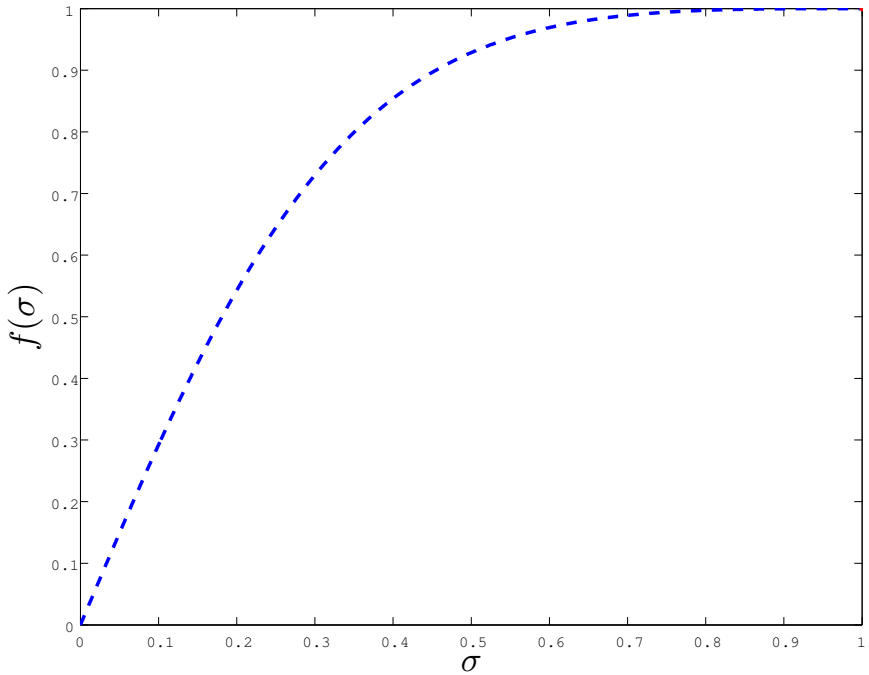
it=2



Halley vs. Dynamical weighted Halley

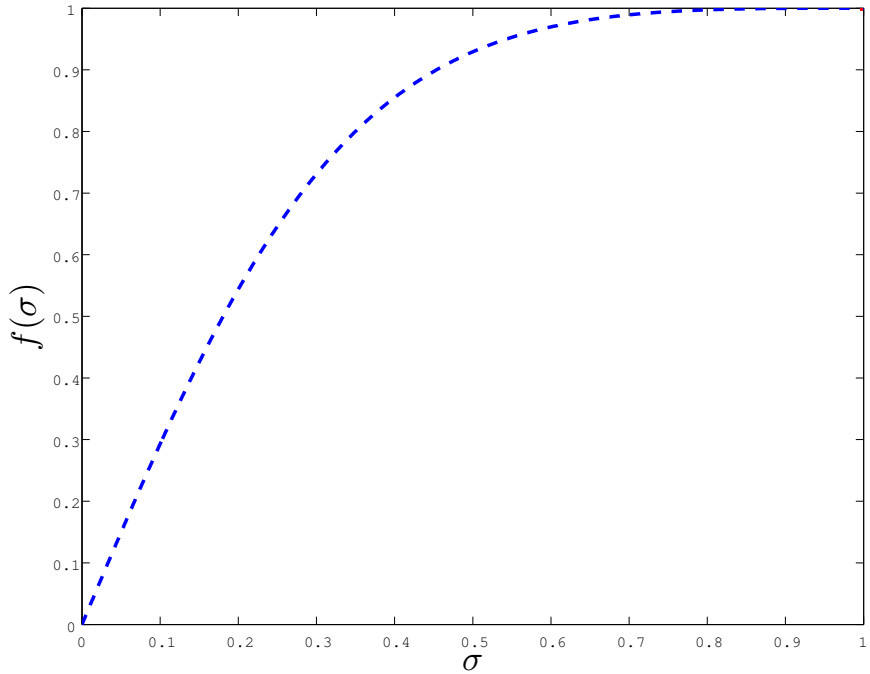
Halley: static parameters

it=6



DWH: dynamical parameters

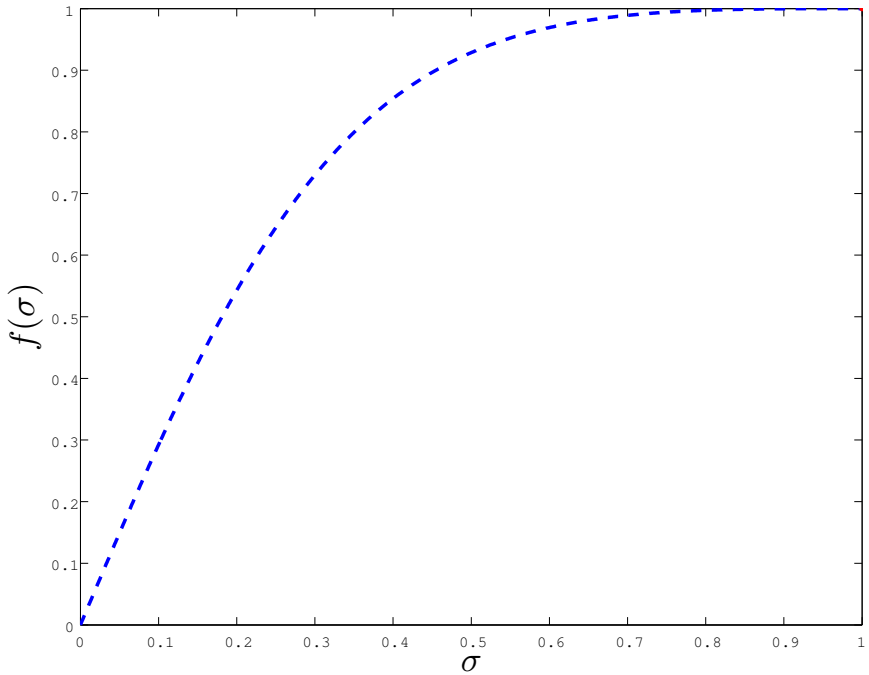
it=2



Halley vs. Dynamical weighted Halley

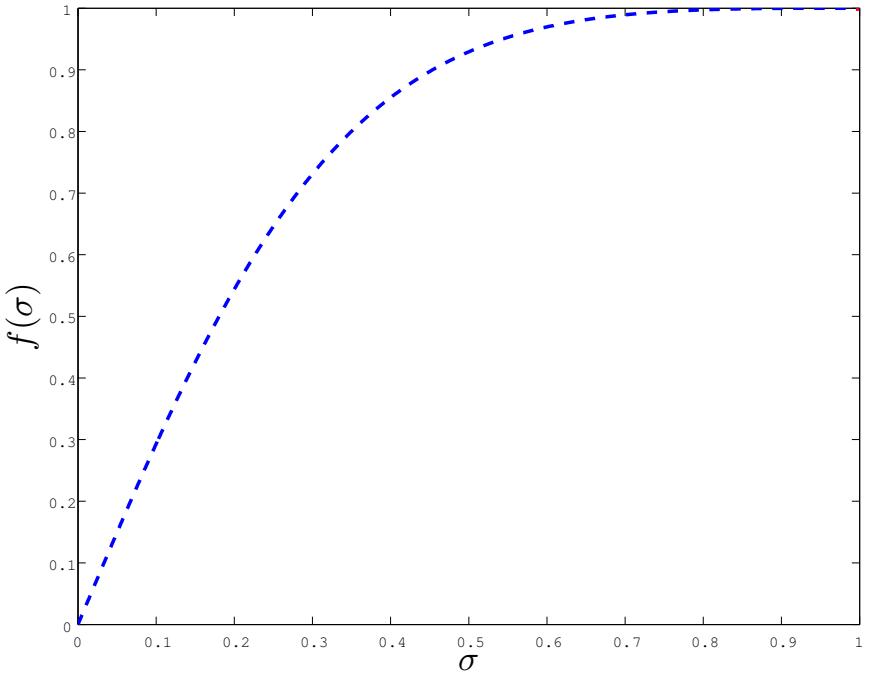
Halley: static parameters

it=6



DWH: dynamical parameters

it=2



dynamical parameters speed up convergence significantly

LAPACK SVD experiments, random matrices

Runtime(s)

	$n = 1000$	$n = 2000$	$n = 3000$	$n = 4000$
QDWH-SVD	6.4	40.9	127	286
Divide-Conquer	2.4	17.1	57.1	133
QR	5.4	40.0	126	287
Jacobi	7.8	60.4	248	579

Backward error $||A - \widehat{U}\widehat{\Sigma}\widehat{V}^T||_F / ||A||_F$

	$n = 1000$	$n = 2000$	$n = 3000$	$n = 4000$
QDWH-SVD	1.9e-15	2.0e-15	2.2e-15	2.3e-15
Divide-Conquer	4.5e-15	5.8e-15	7.0e-15	7.6e-15
QR	1.3e-14	1.8e-14	2.2e-14	2.6e-14
Jacobi	1.2e-14	1.8e-14	2.4e-14	3.0e-14

Distance from orthogonality $\max\{||\widehat{U}^T\widehat{U} - I||_F / \sqrt{n}, ||\widehat{V}^T\widehat{V} - I||_F / \sqrt{n}\}$.

	$n = 1000$	$n = 2000$	$n = 3000$	$n = 4000$
QDWH-SVD	7.4e-16	7.7e-16	7.9e-16	8.0e-16
Divide-Conquer	4.5e-15	5.8e-15	7.0e-15	7.6e-15
QR	1.3e-14	1.8e-14	2.2e-14	2.6e-14
Jacobi	1.2e-14	1.8e-14	2.4e-14	3.0e-14