Frobenius Manifolds and Evolution Equations of Hydrodynamic Type

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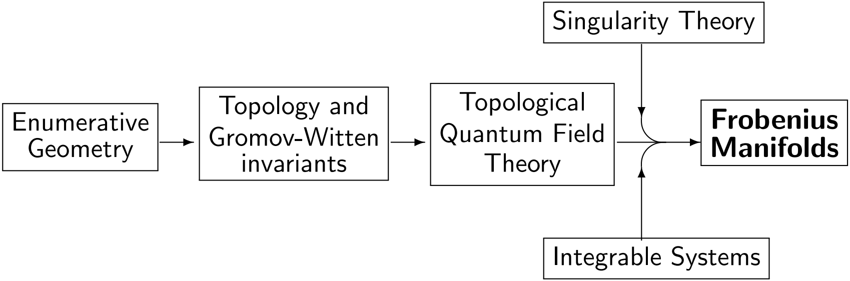
Plan

1. A (very) brief introduction to the theory of Frobenius manifolds.
2. Getting from Frobenius manifolds to evolution equations of hydrodynamic type.
3. Symmetries of Frobenius manifolds lifted to hydrodynamic systems.

This is joint work with my supervisor Dr. Ian Strachan.

Why Study Frobenius Manifolds?

Frobenius manifolds lie at the cross roads between many branches of mathematics...



Frobenius Algebras

Definition. The triple $(A, \circ, \langle \cdot\,,\,\cdot \rangle)$ is said to constitute a (commutative, associative) Frobenius algebra if:

- $(A, \circ)$ is a commutative associative algebra over $\mathbb{C}$ with unity e;
- The bilinear pairing $\langle \cdot\,,\,\cdot \rangle$ and mutiplication $\circ$ satisfy the following *Frobenius condition*

$$\langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle, \quad X, Y, Z \in A.$$

Example. Let $G = \{g_1, g_2, ..., g_n\}$ be a finite (commutative) group with identity g_1, and let $\mathbb{C}G$ be the group ring over G. Then $\langle \cdot\,,\,\cdot \rangle$ defined on the basis by

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } g_i g_j = g_1, \\ 0 & \text{else} \end{cases}$$

endows $\mathbb{C}G$ with the structure of a (commutative) Frobenius algebra.

Frobenius Manifolds

Definition. Let M be an N-dimensional smooth manifold with a smoothly varying Frobenius algebra structure on each tangent space. We say the data $(M, \circ, \langle\,\cdot\,,\,\cdot\,\rangle, e, E)$ define a *Frobenius manifold* if

(i) $\eta := \langle\,\cdot\,,\,\cdot\,\rangle$ defines a flat metric on M;

(ii) $\nabla e = 0$;

(iii) $\nabla_W c(X, Y, Z)$ is a totally symmetric $(0,4)$-tensor, where $c(X, Y, Z) := \langle X \circ Y, Z\rangle$;

(iv) $\exists E \in \Gamma(TM)$ such that

$$\nabla(\nabla(E)) = 0, \quad \mathcal{L}_E \eta = (2-d)\eta, \quad \mathcal{L}_E \circ = \circ, \quad \mathcal{L}_E e = -e,$$

for some constant d.

Frobenius Manifolds and the WDVV Equations

There is a correspondence between Frobenius manifolds and solutions of the WDVV equations, which is established as follows.

- Since η is flat, one may choose flat coordinates $(t^1, ..., t^N)$ such that the functions $\eta_{\alpha\beta} = \langle \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta} \rangle$ are constants.
- The condition (ii) $\nabla e = 0$ means this can be done in such a way that $e = \frac{\partial}{\partial t^1}$.
- Symmetry of ∇c reads in t-coordinates $c_{\alpha\beta\gamma,\kappa} = c_{\alpha\beta\kappa,\gamma}$, and so by the Poincaré lemma we may introduce a potential $A_{\alpha\beta}$ s.t.

$$c_{\alpha\beta\gamma} = \frac{\partial A_{\alpha\beta}}{\partial t^\gamma}. \tag{1}$$

Further symmetry of c implies the existence of F, called the *free energy* of the Frobenius manifold such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{2}$$

Frobenius Manifolds and the WDVV Equations contd.

- The components of $\circ$ are then given by $c^{\alpha}_{\beta\gamma} = \eta^{\alpha\varepsilon} c_{\varepsilon\beta\gamma}$,

$$\partial_\alpha \circ \partial_\beta|_t := c^{\gamma}_{\alpha\beta}(t)\partial_\gamma.$$

  The associativity condition gives the following system of nonlinear PDEs for the function F:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma}, \quad (3)$$

  called the WDVV equations. Note: This is a strong, indeed *quadratic* condition on $c_{\alpha\beta\gamma}$!

- The conditions (iv) lead to demanding F be a quasihomogeneous function

$$\mathcal{L}_E F = (3-d)F \text{ modulo quadratic terms in } t. \quad (4)$$

To Summarize

Frobenius Manifold

Introduce flat coords.

$$\begin{gathered} c_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma F \\ \eta_{\alpha\beta} = c_{1\alpha\beta} \\ e = \partial_1 \end{gathered}$$

Assoc. $\Leftrightarrow$ F satisfies WDVV

An Example

Consider the following three dimensional example:

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16}\gamma(t^3); \quad E = t^1\partial_1 + \frac{1}{2} t^2 \partial_2, \quad (5)$$

where γ is some unknown 1-periodic function. In order for F to satisfy WDVV, γ must satisfy Chazy's equation,

$$\gamma'''(t^3) = 6\gamma(t^3)\gamma''(t^3) - 9(\gamma'(t^3))^2. \quad (6)$$

Here $d = 1$:

$$\mathcal{L}_E(F) = 2F. \quad (7)$$

The main property of the Chazy equation is an $SL(2,\mathbb{C})$ invariance:

$$\begin{aligned} t^3 &\mapsto \frac{at^3 + b}{ct^3 + d}, \quad ad - bc = 1, \\ \gamma(t^3) &\mapsto (ct^3 + d)^2 \gamma(t^3) + 2c(ct^3 + d). \end{aligned} \quad (8)$$

An Example contd.

The metric in flat coordinates is

$$\partial_1\partial_\alpha\partial_\beta F = (c_{1\alpha\beta}) = (\eta_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{9}$$

The remaining non-zero entries of the (covariant) multiplication tensor are

$$\begin{aligned} c_{222} &= -\tfrac{3}{2}\gamma(t^3)t^2, & c_{223} &= -\tfrac{3}{4}(t^2)^2\gamma'(t^3), \\ c_{233} &= -\tfrac{1}{4}(t^2)^3\gamma''(t^3), & c_{333} &= -\tfrac{1}{16}(t^2)^4\gamma'''(t^3). \end{aligned} \tag{10}$$

(Recall $c_{\alpha\beta\gamma} = \partial_\alpha\partial_\beta\partial_\gamma F$.)

The Dubrovin Connection

We may define on a Frobenius manifold a one parameter family of flat connections:
Let ∇ be a flat connection, and $\circ$ define a multiplication of tangent vectors on a manifold. Then we may define a 1-parameter family of connections via

$$\tilde{\nabla}^{\lambda}_{X} Y = \nabla_X Y + \lambda X \circ Y. \tag{11}$$

Now, $\circ$ commutative $\Rightarrow \tilde{\nabla}^{\lambda}$ is torsion free $\forall\lambda$. Further, requiring zero curvature $\forall\lambda$,

$$\left\{ \tilde{R}^{\lambda} = 0 \right\} \equiv \left\{ \begin{array}{c} \circ \text{ is associative, and the tensor} \\ \nabla_W c(X, Y, Z) := \nabla_W \langle X \circ Y, Z \rangle \text{ totally symm.} \end{array} \right\}.$$

So, on a Frobenius manifold, $\tilde{\nabla}^{\lambda}$ is flat for all λ.

The Gauss-Manin Equation

Then we may choose flat coordinates $\tilde{t}^i(t^1, ..., t^N, \lambda)$ satisfying $\tilde{\nabla}^\lambda d\tilde{t}^i = 0$. In coordinates $\{t^\alpha\}$ this reads:

$$\frac{\partial^2 \tilde{t}^k}{\partial t^\alpha \partial t^\beta} = \lambda c^\nu{}_{\alpha\beta} \frac{\partial \tilde{t}^k}{\partial t^\nu}. \tag{12}$$

Note that for $\lambda = 0$ these are just the flat coordinates for $\langle \cdot\,, \cdot \rangle$. In this spirit, we look for solutions of the form

$$\tilde{t}^i(t^1, ..., t^N, z) = \sum_{n=0}^{\infty} \lambda^n h_{(n,i)}(t^1, ..., t^N). \tag{13}$$

Equating powers of λ we have,

$$\frac{\partial^2 h_{(n,i)}}{\partial t^\alpha \partial t^\beta} = c^\nu{}_{\alpha\beta} \frac{\partial h_{(n-1,i)}}{\partial t^\nu}. \tag{14}$$

We start the recursion off by suitably defining $h_{(0,\alpha)} := t_\alpha$. (12) is called the Gauss-Manin equation.

An Example contd.

Solving the Gauss-Manin equation for the above mentioned example, we have

$$\begin{aligned} h_{(0,1)} &= t^3, & h_{(1,1)} &= t^1 t^3 + \tfrac{1}{2}(t^2)^2, \\ h_{(0,2)} &= t^2, & h_{(1,2)} &= t^1 t^2 - \tfrac{1}{4}\gamma(t^3)(t^2)^3, \\ h_{(0,3)} &= t^1, & h_{(1,3)} &= \tfrac{1}{2}(t^1)^2 - \tfrac{(t^2)^4}{16}\gamma'(t^3), \end{aligned}$$

$$\begin{aligned} h_{(2,1)} &= \frac{1}{2}t^1(t^2)^2 - \frac{1}{8}\gamma(t^3)(t^2)^4 + \frac{1}{2}(t^1)^2 t^3 - \frac{(t^2)^4 t^3}{16}\gamma'(t^3), \\ h_{(2,2)} &= \frac{1}{2}(t^1)^2 t^2 - \frac{1}{4}t^1(t^2)^2\gamma(t^3) + \frac{9}{160}(t^2)^5\gamma(t^3)^2 - \frac{1}{20}(t^2)^5\gamma'(t^3), \\ h_{(2,3)} &= \frac{1}{6}(t^1)^3 - \frac{1}{16}t^1(t^2)^4\gamma'(t^3) + \frac{1}{80}\gamma(t^3)\gamma'(t^3)(t^2)^6 - \frac{1}{480}(t^2)^6\gamma''(t^3). \end{aligned}$$

Equations of Hydrodynamic Type

An evolution equation of *hydrodynamic type* is of the form

$$u_{,T} = M(u)u_{,X} \,, \tag{15}$$

where $M(u)$ is some $n \times n$ matrix of functions of u, and *not* its derivatives. The eigenvalues of the matrix M are known as the *characteristic speeds*.
An example of such an equation is the *dispersionless* or *quasiclassical limit* of the Korteweg de Vries equation:

$$u_{,T} = \frac{3}{2} u u_{,X} \,. \tag{16}$$

In this case, $M(u) = \frac{3}{2}u$.

Let M be a manifold with a set of local coordinates $(t^1, ..., t^N)$. Regard the coordinates as functions of some parameter X, $t^\alpha = t^\alpha(X)$. We define on M a *Poisson bracket* of *differential geometric type*,

$$\{H_{(n,\varepsilon)}, H_{(m,\nu)}\} = \int_{S^1} \frac{\delta H_{(n,\varepsilon)}}{\delta t^\alpha} \left(\eta^{\alpha\beta} \frac{d}{dX} + b^{\alpha\beta}_{\kappa} t^{\kappa},_X \right) \frac{\delta H_{(m,\nu)}}{\delta t^\beta} dX, \tag{17}$$

where the functional densities for $H_{(n,\varepsilon)}$ depend on t and not its derivatives. Indeed, we define

$$H_{(n,\varepsilon)} = \int_{S^1} h_{(n,\varepsilon)}(t) dX.$$

Such brackets were first studied by Dubrovin and Novikov (≈'89). As the name suggests, they realized that properties of the brackets depended on differential geometric structure on the target space M.

In particular:

- For $\eta^{\alpha\beta}$ non-degenerate on M, the inverse must define a flat metric on M.
- The coefficients $b^{\alpha\beta}_{\kappa}$ are related to the Christoffel symbols for this metric via $b^{\alpha\beta}_{\kappa} = -\eta^{\alpha\nu}\Gamma^{\beta}_{\nu\kappa}$.

Using the recursion relation (14) and (17), we construct the equations of hydrodynamic type

$$\frac{dt^{\alpha}}{dT_{(n,\varepsilon)}} = \{t^{\alpha}, H_{(n,\varepsilon)}\} = \underbrace{\eta^{\alpha\nu} c^{\kappa}_{\nu\sigma}\partial_{\kappa} h_{(n-1,\varepsilon)}}_{(M_{(n,\varepsilon)}(t))^{\alpha}_{\sigma}} \frac{dt^{\sigma}}{dX}. \tag{18}$$

A *symmetry* of WDVV is a map from one solution F to another $\hat{F}$. Of interest to us here will be the so-called *inversion* symmetry I:

$$\hat{t}^1 = \frac{1}{2}\frac{t_\sigma t^\sigma}{t^N}, \quad \hat{t}^\alpha = \frac{t^\alpha}{t^N} \text{ (for } \alpha \neq 1, N), \quad \hat{t}^N = -\frac{1}{t^N},$$

$$\hat{F}(\hat{t}) = (\hat{t}^N)^2 F\left(\frac{1}{2}\frac{\hat{t}_\sigma \hat{t}^\sigma}{\hat{t}^N}, -\frac{\hat{t}^2}{\hat{t}^N}, ..., -\frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N}\right) + \frac{1}{2}\hat{t}^1\hat{t}_\sigma\hat{t}^\sigma, \quad (19)$$

$$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.$$

I is an involution up to sign. Health warning: These transformations are not tensorial! But they are almost: One can show that, for example

$$\hat{c}_{\alpha\beta\gamma} = (t^N)^{-2}\frac{\partial t^\lambda}{\partial \hat{t}^\alpha}\frac{\partial t^\mu}{\partial \hat{t}^\beta}\frac{\partial t^\nu}{\partial \hat{t}^\gamma}c_{\lambda\mu\nu}. \quad (20)$$

Within the solution space of WDVV, there are functions F that lie at a fixed point of I. For example, consider the free energy of the above mentioned example:

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16}\gamma(t^3). \tag{21}$$

Using (19), one can show that

$$\hat{F} = \frac{1}{2}(\hat{t}^1)^2 \hat{t}^3 + \frac{1}{2} \hat{t}^1 (\hat{t}^2)^2 - \frac{(\hat{t}^2)^4}{16}\gamma(\hat{t}^3).$$

A natural question may be to ask how the Hamiltonian densities, and therefore systems of hydrodynamic type are affected by the inversion. Consider our explicit example:

$$
\begin{aligned}
h_{(1,1)}(t^1,t^2,t^3) &= t^1t^3+\frac{1}{2}(t^2)^2 \\
\Rightarrow h_{(1,1)}\left(\hat{t}^1+\frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3},\frac{\hat{t}^2}{\hat{t}^3},-\frac{1}{\hat{t}^3}\right) &= \left(\hat{t}^1+\frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3}\right)\left(-\frac{1}{\hat{t}^3}\right)+\frac{1}{2}\left(\frac{\hat{t}^2}{\hat{t}^3}\right)^2 \\
&= -\frac{\hat{t}^1}{\hat{t}^3} = -\frac{1}{\hat{t}^3}h_{(0,3)}(\hat{t}^1,\hat{t}^2,\hat{t}^3).
\end{aligned}
$$

$$
\begin{aligned}
&h_{(1,2)}(t^1,t^2,t^3) = t^1t^2 - \frac{1}{4}\gamma(t^3)(t^2)^3\\
&\Rightarrow h_{(1,2)}(\hat{t}^1 + \frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3}, \frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3}) = \left(\hat{t}^1 + \frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3}\right)\left(\frac{\hat{t}^2}{\hat{t}^3}\right) - \frac{1}{4}\gamma\left(-\frac{1}{\hat{t}^3}\right)\left(\frac{\hat{t}^2}{\hat{t}^3}\right)^3\\
&= \left(\hat{t}^1 + \frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3}\right)\left(\frac{\hat{t}^2}{\hat{t}^3}\right) - \frac{1}{4}\left((\hat{t}^3)^2\gamma(\hat{t}^3) + 2\hat{t}^3\right)\left(\frac{\hat{t}^2}{\hat{t}^3}\right)^3 \text{ Using (8)}\\
&= \frac{\hat{t}^1\hat{t}^2}{\hat{t}^3} - \frac{1}{4}(\hat{t}^3)^2\gamma(\hat{t}^3) = \frac{1}{\hat{t}^3}h_{(1,2)}(\hat{t}^1,\hat{t}^2,\hat{t}^3).
\end{aligned}
$$

Lifting the Inversion Symmetry contd.

In general, for the current example, we observed the following pattern:

$$h_{(n,\alpha)}(\hat{t}^1+\frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^3},\frac{\hat{t}^2}{\hat{t}^3},-\frac{1}{\hat{t}^3})=\pm\frac{1}{\hat{t}^3}h_{(\tilde{n},\tilde{\alpha})}(\hat{t}^1,\hat{t}^2,\hat{t}^3), \qquad (22)$$

where

$$\tilde{n}=\begin{cases} n+1, & \text{if } \alpha=3, \\ n, & \text{if } \alpha=2, \\ n-1, & \text{if } \alpha=1, \end{cases} \quad \tilde{\alpha}=\begin{cases} 1, & \text{if } \alpha=3, \\ \alpha, & \text{if } \alpha=2, \\ 3, & \text{if } \alpha=1, \end{cases} \quad \pm=\begin{cases} +, & \text{if } \alpha=2, \\ -, & \text{else.} \end{cases} \qquad (23)$$

A Diagram...

$$(14):\ \partial_\alpha\partial_\beta h_{(n,\kappa)} = c^{\sigma}_{\alpha\beta}\partial_\sigma h_{(n-1,\kappa)} \ \longrightarrow$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
\alpha = 3 & \bullet \arrow[r] \arrow[ddr, leftrightarrow, "I"] & \bullet \arrow[r] \arrow[ddr, leftrightarrow, "I"] & \bullet \\
\alpha = 2 & \bullet \arrow[r] \arrow[loop below, "I"] & \bullet \arrow[r] \arrow[loop below, "I"] & \bullet \arrow[loop below, "I"] \\
\alpha = 1 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
& n-1 & n & n+1
\end{tikzcd}$$

Lifting the Inversion Symmetry contd.

We found this pattern to hold in general, not just for solutions of WDVV lying at fixed points:

$$h_{(n,\alpha)}(\hat{t}^1 + \frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^N}, \frac{\hat{t}^2}{\hat{t}^N}, \ldots, \frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N}) = \pm\frac{1}{\hat{t}^N} h_{(\tilde{n},\tilde{\alpha})}(\hat{t}^1, \hat{t}^2, \ldots, \hat{t}^N), \tag{24}$$

where

$$\tilde{n} = \begin{cases} n+1, & \text{if } \alpha = N, \\ n, & \text{if } \alpha \neq 1, N, \\ n-1, & \text{if } \alpha = 1, \end{cases} \quad \tilde{\alpha} = \begin{cases} 1, & \text{if } \alpha = N, \\ \alpha, & \text{if } \alpha \neq 1, N, \\ 3, & \text{if } \alpha = 1, \end{cases} \quad \pm = \begin{cases} +, & \text{if } \alpha \neq 1, N, \\ -, & \text{else.} \end{cases} \tag{25}$$

Proof consists of showing that the formulae (24) satisfies the inverted version of the recursion relations (14).

Given that the respective flows were defined via

$$\frac{dt^\alpha}{dT_{(n,\varepsilon)}} = \{t^\alpha, H_{(n,\varepsilon)}\} = \underbrace{\eta^{\alpha\nu} c^\kappa_{\nu\sigma} \partial_\kappa h_{(n-1,\varepsilon)}}_{(M_{(n,\varepsilon)}(t))^\alpha_\sigma} \frac{dt^\sigma}{dX}, \tag{26}$$

and we now know how the Hamiltonian densities are mapped under I, the symmetry lifts also to the flows:

$$M_{(n,\alpha)}((\hat{t}^1 + \frac{1}{2}\frac{(\hat{t}^2)^2}{\hat{t}^N}, \frac{\hat{t}^2}{\hat{t}^N}, \ldots, \frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N})) = \pm \hat{M}_{(\tilde{n},\tilde{\alpha})}(\hat{t}) \mp \hat{h}_{(\tilde{n}-1,\tilde{\alpha})}(\hat{t})\mathbf{1}, \tag{27}$$

where $\tilde{n}$, $\tilde{\alpha}$ and $\pm = -(\mp)$ are as above.

Further questions

- Do these symmetries exist for other hydrodynamic equations, not just those constructed from a Frobenius manifold?
- Would like to start adding dispersive terms to equations using a recipe of Dubrovin & Zhang. Does the symmetry extend to dispersive equations?
- Do these shifts in the Hamiltonian densities, or change in characteristic speeds look familiar?