Symbolic representation and Symmetry Integrability

Jing Ping Wang
University of Kent

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What have we learned about the symmetry approach?

It is based on a concept of formal recursion operator.

Advantages: Not sensitive to lacunas in the hierarchy of symmetries; Not assumed equations to be polynomial.

Disadvantages: Difficult to draw a global picture (in all orders); difficult to apply for non-local, non-evolutionary and multi-dimensional equations.

The symmetry approach in symbolic representation

It is based on a symbolic representation of the ring of differential polynomials.

Advantages: Enable to use powerful results from algebraic geometry and number theory; enable to draw a global picture (in all orders); suitable for studying integrability of noncommutative, non-local, non-evolutionary and multi-dimensional equations.

Disadvantages: Restriction to the ring of differential polynomials, which can be amended in some cases by suitable extension of the ring.

Plan of the lecture

- The ring of differential polynomials;
- Introduction of the symbolic representation;
- Symmetries of evolutionary equations in symbolic representation;
- Global classification of scalar homogeneous evolutionary equations;
- Further developments.

The ring of differential polynomials:

- Let $u = u(x,t)$ and $u_i = \partial_x^i u$. A u-monomial is of the form $u_0^{\alpha_0} u_1^{\alpha_1} \cdots u_k^{\alpha_k} (u^\alpha)$ and the total degree is $|\alpha| = \alpha_0 + \alpha_1 + \cdots \alpha_k > 0$.

- A differential polynomial is a finite linear combination of u-monomials with degrees $|\alpha| > 0$ with coefficients in $\mathbb{C}$.

- The set of all such differential polynomials forms a ring $\mathcal{R}$.

- $\mathcal{R}$ is a differential ring and $\mathbb{C} \not\subseteq \mathcal{R}$. The linear operator $D_x = \sum_{k \geq 0} \left(u_{k+1} \frac{\partial}{\partial u_k} \right)$ is a derivation corresponding to the total x-derivative.

Gradations of the ring of differential polynomials:

Monomials u^α are eigenvectors of the commuting linear operators

$$D_u = \sum_{k\geq 0} u_k \frac{\partial}{\partial u_k}, \qquad X_u = \sum_{k\geq 1} k u_k \frac{\partial}{\partial u_k}$$

with $D_u(u^\alpha) = |\alpha| u^\alpha$ and $X_u(u^\alpha) = (\sum_{k\geq 1} k\alpha_k) u^\alpha$.

The ring $\mathcal{R}$ is **graded** and is a direct sum of eigenspaces

$$\mathcal{R} = \bigoplus_{n\in\mathbb{N}} \mathcal{R}^n = \bigoplus_{n,p\in\mathbb{N}} \mathcal{R}^n_{p-1},$$

$$\mathcal{R}^n = \{f \in \mathcal{R} \,|\, D_u(f) = nf\}, \quad \mathcal{R}^n_p = \{f \in \mathcal{R}^n \,|\, X_u(f) = pf\}.$$

Example. $u_1^2 u_7 + 2u_2u_3u_4 - u_0^2 u_9 \in \mathcal{R}^3_9 \subset \mathcal{R}^3$.

Notation. "Little oh": $f = o(\mathcal{R}^n)$ if $f \in \bigoplus_{k>n} \mathcal{R}^k$.

Weighted homogeneous differential polynomials

Assume that dependent variable u has a weight λ. Define

$$W_\lambda = \lambda D_u + X_u.$$

Differential monomials are eigenvectors of W_λ and the spectrum of W_λ is a set $S_\lambda = \{n\lambda + m - 1 \,|\, n, m \in \mathbb{N}\}$.

$$\mathcal{R} = \bigoplus_{\mu \in S_\lambda} \mathcal{W}_\mu \,, \qquad \mathcal{W}_\mu = \{f \in \mathcal{R} \,|\, W_\lambda(f) = \mu f\}\,.$$

Elements of $\mathcal{W}_\mu$ are called λ-**homogeneous polynomials of weight** μ.

Example. $u_3 + 6uu_1$ is a 2-homogeneous polynomial of weight 5.

Note. If $\lambda > 0$, subspaces $\mathcal{W}_\mu$ are finite dimensional.

Lie algebra of differential polynomials

For any $f,\ g \in \mathcal{R}$, define a Lie bracket

$$[f,\ g] = f_*(g) - g_*(f)\ .$$

Here f_* is the **Fréchet derivative**, defined as $h_* = \sum_{k\geq 0} \frac{\partial h}{\partial u_k} D_x^k$.

We say that an element $f \in \mathcal{R}$ has **order** n if the corresponding differential operator h_* is of order n.

The grading of $\mathcal{R}$ induces the grading of the Lie algebra:

$$[\mathcal{R}_p^n, \mathcal{R}_q^m] \subset \mathcal{R}_{p+q}^{n+m-1} \quad \text{and} \quad [\mathcal{W}_\mu, \mathcal{W}_\nu] \subset \mathcal{W}_{\mu+\nu-\lambda}\ .$$

Symmetries and approximate symmetries

$$u_t = K = K^1[u] + K^2[u] + K^3[u] + \cdots, \qquad K^i \in \mathcal{R}^i.$$

A differential polynomial $S \in \mathcal{R}$ is a **symmetry** iff the Lie bracket of K and S vanishes.

A differential polynomial $S \in \mathcal{R}$ is said to be an **approximate symmetry of degree** p if $[K,\ S] = o(\mathcal{R}^p)$.

- Every equation possesses infinitely many approximate symmetries of degree 1;
- Equation $u_t = u_5 + 5uu_1$ has infinitely many approximate symmetry of degree 2, e.g., $u_7 + 7uu_3 + 14u_1u_2$;
- An integrable equation possesses infinitely many approximate symmetries of any degree.

Symbolic representation $\hat{\mathcal{R}}$ of differential polynomials ring $\mathcal{R}$

- Linear monomials u_i: $u_k \longmapsto \hat{u}\xi_1^k$;

- Quadratic monomials $u_i u_j$: $u_i u_j \longmapsto \frac{\hat{u}^2}{2}(\xi_1^i \xi_2^j + \xi_1^j \xi_2^i)$;

- General monomial:

$$u_{i_1} u_{i_2} \cdots u_{i_n} \in \mathcal{R}^n \longmapsto \hat{u}^n \langle \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n} \rangle_{\mathcal{S}_n^\xi} \in \hat{\mathcal{R}}^n,$$

  where $\langle\rangle_{\mathcal{S}_n^\xi}$ means the symmetrisation over the permutation group of n elements.

The symmetrisation over the permutation group defines the symbol **uniquely**.

Some examples and properties

$$D_x(uu_1) = uu_2 + u_1^2 \longmapsto \frac{\hat{u}^2}{2}(\xi_1^2 + \xi_2^2) + \hat{u}^2\xi_1\xi_2 = (\xi_1 + \xi_2)\widehat{uu_1}$$

$$\widehat{D}_u = \hat{u}\frac{\partial}{\partial \hat{u}}, \qquad \widehat{X}_u = \sum_{i=1} \xi_i \frac{\partial}{\partial \xi_i} \ .$$

Basic properties: If $P \in \mathcal{R}^k$, then

$$\begin{aligned} \widehat{D_x P} &= (\xi_1 + \cdots + \xi_k)\, \widehat{P}, \\ \widehat{P_*} &= k\widehat{P}(\xi_1, \ldots, \xi_{k-1}, \eta), \qquad D_x \longmapsto \eta \\ \widehat{P_*[u_n]} &= (\xi_1^n + \cdots + \xi_k^n)\, \widehat{P}. \end{aligned}$$

Example. let $F = u_3 + 6uu_1$, then $F \mapsto \hat{u}\xi_1^3 + 3\hat{u}^2(\xi_1 + \xi_2)$ and

$$F_* \mapsto \eta^3 + 6\hat{u}(\xi_1 + \eta)\,.$$

Statement. A differential operator is a Fréchet derivative of an element of $\mathcal{R}$ iff its symbol is symmetric with respect to all its arguments.

Some immediate results

Let $f \in \mathcal{R}^n,\ f \mapsto \widehat{u}^n a(\xi_1, \ldots, \xi_n)$ and $g \in \mathcal{R}^m,\ g \mapsto \widehat{u}^m b(\xi_1, \ldots, \xi_m)$, then the Lie bracket $[f, g]$ is represented by

$$
\begin{aligned}
[f,\ g] \quad \longmapsto \quad & \widehat{u}^{n+m-1} \\
& \Big\langle na(\xi_1, \ldots, \xi_{n-1}, \xi_n + \cdots + \xi_{n+m-1}) b(\xi_n, \ldots, \xi_{n+m-1}) - \\
& mb(\xi_1, \ldots, \xi_{m-1}, \xi_m + \cdots + \xi_{n+m-1}) a(\xi_m, \ldots, \xi_{n+m-1}) \Big\rangle_{\mathcal{S}^\xi_{n+m-1}}
\end{aligned}
$$

For example, if $f \in \mathcal{R}^1,\ f \mapsto \widehat{u}\omega(\xi_1)$ then

$$[f, g] \longmapsto (\omega(\xi_1 + \cdots + \xi_m) - \omega(\xi_1) - \cdots - \omega(\xi_m))\ \widehat{u}^m b(\xi_1, \ldots, \xi_m).$$

Statements. 1. The symmetries of the linear evolution equation $u_t = u_n (n \geq 2)$ are linear differential polynomials.
2. u_1 is a symmetry for any evolutionary equation $u_t = g \in \mathcal{R}$.

Symmetry Conditions in symbolic representation

Theorem. Consider equation $u_t = F \in \mathcal{R}$, where

$$F \longmapsto \hat{u}\xi_1^n + \hat{u}^2 a_1(\xi_1,\xi_2) + \hat{u}^3 a_2(\xi_1,\xi_2,\xi_3) + \cdots$$

and $n \geq 2$. If

$$G \longmapsto \hat{u}\xi_1^k + \hat{u}^2 A_1(\xi_1,\xi_2) + \hat{u}^3 A_2(\xi_1,\xi_2,\xi_3) + \cdots$$

is a symmetry, then its coefficients can be found recursively

$$\begin{aligned}
&A_1(\xi_1,\xi_2) = \frac{\mathcal{G}_1^k}{\mathcal{G}_1^n} a_1(\xi_1,\xi_2)\\
&A_{m-1}(\xi_1,...,\xi_m) = \frac{1}{\mathcal{G}_{m-1}^n}(\mathcal{G}_{m-1}^k a_{m-1}(\xi_1,...,\xi_m)\\
&+\sum_{j=1}^{m-2}\Big\langle (j+1)A_j(\xi_1,...,\xi_j,\sum_{l=j}^{m-1}\xi_{l+1})a_{m-1-j}(\xi_{j+1},...,\xi_m)\\
&-(m-j)a_{m-1-j}(\xi_1,...,\xi_{m-1-j},\sum_{l=0}^{j}\xi_{m-l})A_j(\xi_{m-j},\ldots,\xi_m)\Big\rangle_{\mathcal{S}_m^\xi}),
\end{aligned}$$

where $\mathcal{G}_{m-1}^n = (\xi_1 + \cdots + \xi_m)^n - (\xi_1^n + \cdots + \xi_m^n)\,.$

Number theoretical methods

The main question now is to determine the common zeros of the $\mathcal{G}_m^n$. There are basically three different cases.

Integrability To prove the existence of infinitely many symmetries, we use the Lech-Mahler theorem, followed by the application of an algorithm by C.J. Smyth.

Finite many symmetries To prove this, we use p–adic analysis.

Completeness This is the most difficult job. In the scalar case, it has been done using Diophantine approximation theory by Frits Beukers. For systems, it is still an open problem.

Irreducibility of polynomials $\mathcal{G}_m^n$ for $m \geq 3$

(F. Beukers) For any positive integer $n \geq 2$ and $m \geq 3$, the polynomials $\mathcal{G}_m^n$ are irreducible over $\mathbb{C}$.

Statement. The following equation has only trivial symmetries:

$$u_t = u_n + f(u_{n-1}, \dots, u)\,, \quad n \geq 2, \quad 0 \neq f \in \bigoplus_{m>3} \bigoplus_{p<n} \mathcal{R}_p^m \ .$$

Question Is equation $u_t = u_3 + u^3 u_1$ integrable?
How about equation $u_t = u_2 + u^2 u_1^2$?

Theorem. Assume that an equation of given form has a nontrivial symmetry. Then for an approximate symmetry $\sum_{j=1}^{3} h_j,\ h_j \in \mathcal{R}^j$ of degree 3, there exists a unique $H = \sum_{j\geq 1} h_j,\ h_j \in \mathcal{R}^j$ such that H is an approximate symmetry of any degree.

Lech–Mahler Theorem

Let $A_1, A_2, \ldots, A_n$ be non-zero complex numbers and similarly for $a_1, a_2, \ldots, a_n$. Suppose that none of the ratios A_i/A_j with $i \neq j$ is a root of unity. Then the equation

$$a_1 A_1^k + a_2 A_2^k + \cdots + a_n A_n^k = 0$$

in the unknown integer k has finitely many solutions.

The Lech-Mahler argument for $\mathcal{G}_1^n$

Notice $\mathcal{G}_1^n = \xi_1^n \left((1+r)^n - 1 - r^n\right), \quad r = \frac{\xi_2}{\xi_1}$. Let r be a root of $\mathcal{G}_1^n$. Integrability implies that there are infinitely many l such that

$$(1+r)^l - 1 - r^l = 0.$$

The Lech-Mahler theorem then implies that r and $1+r$ are root of unity, i.e., $r = e^{\pm\frac{2\pi i}{3}} = \frac{-1\pm i\sqrt{3}}{2} = \zeta_3$, or zero, i.e., $r = 0$ or $r = -1$.

This solves the original equation for all odd l if $r = -1$ and for all l if $r = 0$. Since $1 + \zeta_3$ is a sixth root of unity, ζ_3 is a solution if l is 1 or 5 $\pmod 6$.

Divisibility of polynomials $\mathcal{G}_m^n$ for $m = 1,\ 2$

(F. Beukers) $\mathcal{G}_m^n = t_m^n g_m^n$, where $(g_m^n, g_m^l) = 1$ for all $n < l$, and t_m^n is one of the following cases.

- $m = 1$:
  $n = 0 \pmod 2$: $\xi_1\xi_2$
  $n = 3 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)$
  $n = 5 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)$
  $n = 1 \pmod 6$: $\xi_1\xi_2(\xi_1 + \xi_2)(\xi_1^2 + \xi_1\xi_2 + \xi_2^2)^2$

- $m = 2$:
  $n = 0 \pmod 2$: 1
  $n = 1 \pmod 2$: $(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)$

The Cauchy-Liouville-Mirimanoff polynomials

$$P_k(x) = (1+x)^k - x^k - 1\,.$$

- 1839: Cauchy and Liouville established the periodicity;
- 1903: Mirimanoff conjectured g_1^p is irreducible over $\mathbb{Q}$ for prime p;
- 1997: Beukers conjectured g_1^n is irreducible over $\mathbb{Q}$;
- 2007: Tzermias proved Mirimanoff's conjecture.

Global classification results for scalar equations

Problem. Consider λ-homogeneous equations ($\lambda > 0$) of the form

$$\begin{aligned} & u_t = F = u_n + f_2 + f_3 + \cdots, \quad f_i \in \mathcal{R}^i, \quad n \geq 2 \\ \longmapsto \quad & \hat{u}\xi_1^n + \hat{u}^2 a_1(\xi_1, \xi_2) + \hat{u}^3 a_2(\xi_1, \xi_2, \xi_3) + \cdots, \quad \deg(a_j) = n - j\lambda. \end{aligned}$$

Note that if λ is not integer and $i\lambda \notin \mathbb{N}$, then $a_i = 0$.

Let $G \in \mathcal{R}$ be a nontrivial symmetry. Then it is of the form

$$\begin{aligned} & G = u_m + g_2 + g_3 + \cdots, \quad g_i \in \mathcal{R}^i, \quad m \geq 2 \\ \longmapsto \quad & \hat{u}\xi_1^m + \hat{u}^2 A_1(\xi_1, \xi_2) + \hat{u}^3 A_2(\xi_1, \xi_2, \xi_3) + \cdots, \quad \deg(A_j) = m - j\lambda. \end{aligned}$$

Statement. The equation has nontrivial symmetries when i) $f_2 \neq 0$; ii) $f_2 = 0$, $f_3 \neq 0$ and n is odd.

Observation. Assume n and m are both odd. Then we have

$$A_1 = \frac{a_1 \mathcal{G}_1^m}{\mathcal{G}_1^n} = \frac{a_1\ (\xi_1^2 + \xi_1\xi_2 + \xi_2^2)^{s-s'}\ g_1^m}{g_1^n},$$

where $s' = \frac{n+3}{2} \pmod 3$ and $s = \frac{m+3}{2} \pmod 3$.

The divisibility result of $\mathcal{G}_1^l$ implies that $\lambda \leq 3 + 2\min(s, s')$.

If expression A_1 is a polynomial, then it defines a symmetry $G = u_m + g_2 + \cdots$! The equations defined by G has the same symmetries as $u_t = F$. We consider the equation given by G instead. The lowest possible m is $2s+3$ for $s = 0, 1, 2$. Therefore we only need to consider λ-homogeneous equations with $\lambda \leq 7$ of orders not greater than 7.

Result. A nontrivial symmetry of a λ-homogeneous equation with $\lambda > 0$ is part of a hierarchy starting at order 2, 3, 5 or 7.

Theorem. Suppose a homogeneous polynomial evolution equation

$$u_t = u_n + F(u, u_x, \cdots, u_{n-1}) \quad \lambda > 0$$

has nontrivial symmetries. Then it is a symmetry of one of the following equations (up to a scaling):

$$u_t = u_{xx} + uu_x$$
$$u_t = u_{xxx} + uu_x$$
$$u_t = u_{xxx} + u_x^2$$
$$u_t = u_{xxx} + u^2 u_x$$
$$u_t = u_{xxx} + 3u^2 u_{xx} + 9uu_x^2 + 3u^4 u_x$$
$$u_t = u_{5x} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x$$
$$u_t = u_{5x} + 10uu_{xxx} + 10u_x u_{xx} + 20u^2 u_x$$
$$u_t = u_{5x} + 10u_x u_{xxx} + \frac{15}{2}u_{xx}^2 + \frac{20}{3}u_x^3$$
$$u_t = u_{5x} + 10u_x u_{xxx} + \frac{20}{3}u_x^3$$
$$u_t = u_{5x} + 5u_x u_{xxx} + 5u_{xx}^2 - 5u^2 u_{xxx} - 20uu_x u_{xx} - 5u_x^3 + 5u^4 u_x$$

Further developments

- Non-local equations: Benjamin-Ono and Camassa-Holm types;
- Boussinesq type equations: $u_{tt} = F(u, u_1, \cdots, u_m; u_t, u_{t,1}, \cdots, u_{t,m})$;
- Multi-component systems;
- Symmetry structure of $(2+1)$-dimensional integrable equations.