The Mathematics Behind Google's PageRank

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Two Factors

Determine where Google displays a web page on the *Search Engine Results Page*:

1. PageRank (links)

   A page has high PageRank
   if many pages with high PageRank link to it

2. Hypertext Analysis (page contents)

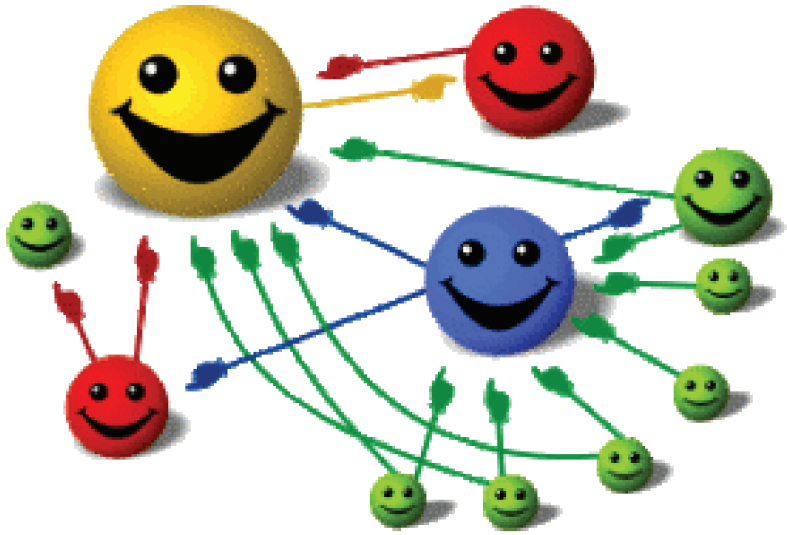
   Text, fonts, subdivisions, location of words, contents of neighbouring pages

PageRank

An objective measure of the citation importance of a web page [Brin & Page 1998]

- Assigns a rank to every web page
- Influences the order in which Google displays search results
- Based on link structure of the web graph
- Does not depend on contents of web pages
- Does not depend on query

More PageRank More Visitors



PageRank

. . . continues to provide the basis for all of our web search tools http://www.google.com/technology/

- “Links are the currency of the web”
- Exchanging & buying of links
- BO (backlink obsession)
- Search engine optimization

Overview

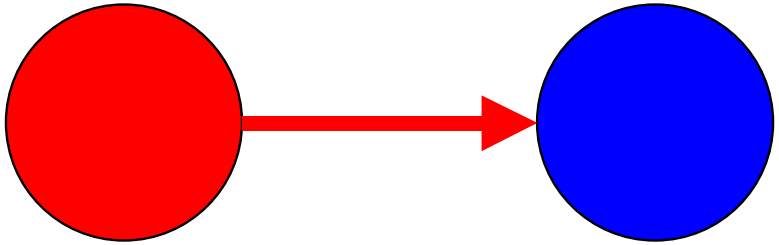
- Mathematical Model of Internet
- Computation of PageRank
- Is the Ranking Correct?
- Floating Point Arithmetic Issues

Mathematical Model of Internet

1. Represent internet as graph
2. Represent graph as stochastic matrix
3. Make stochastic matrix more convenient
   $\Longrightarrow$ Google matrix
4. dominant eigenvector of Google matrix
   $\Longrightarrow$ PageRank

The Internet as a Graph

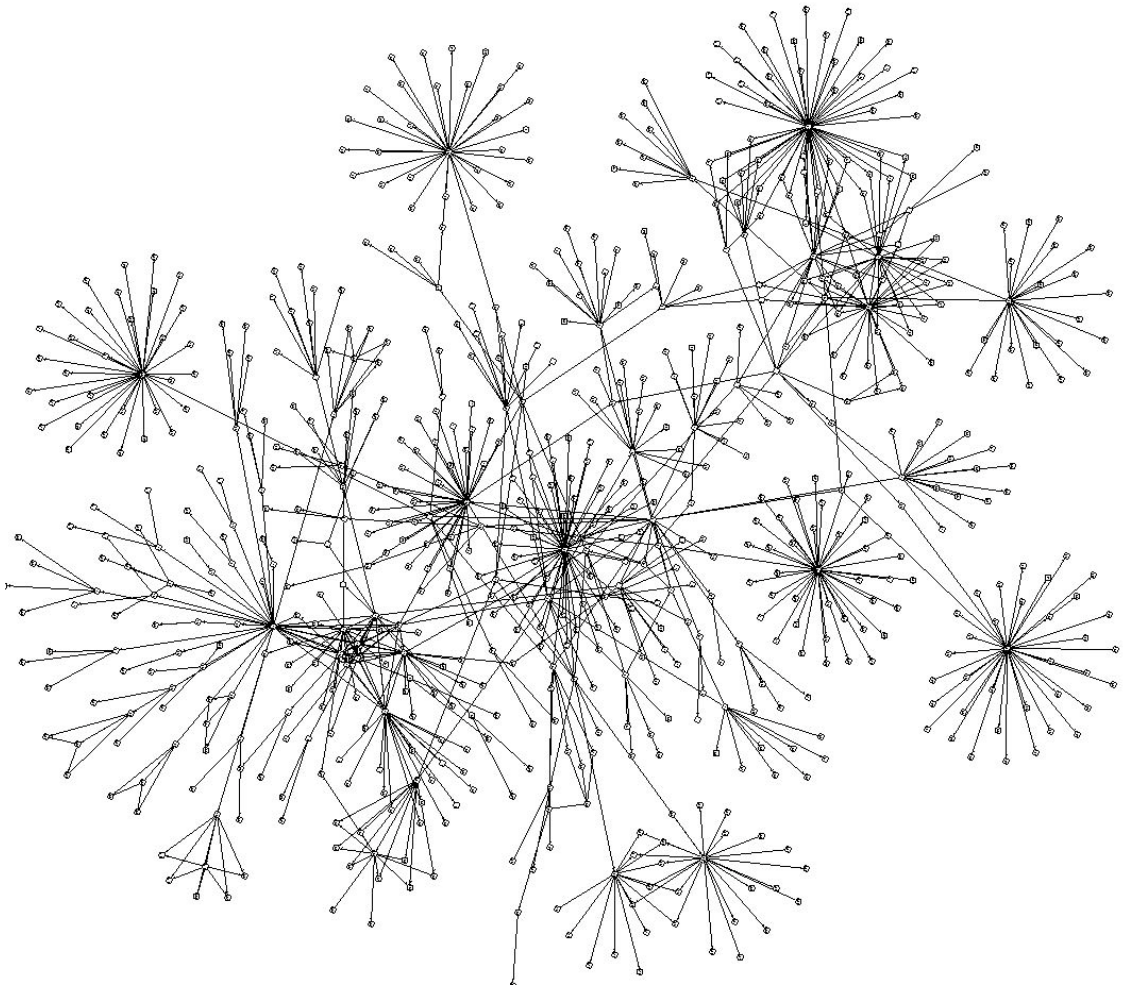
Link from one web page to another web page



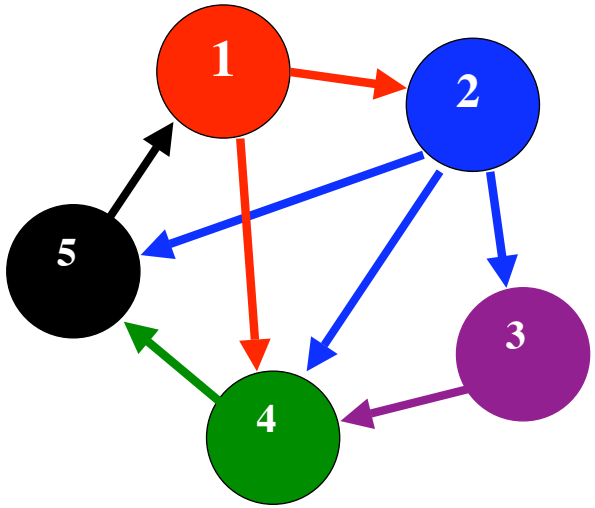
Web graph:

Web pages = nodes
Links = edges

The Internet as a Graph



The Web Graph as a Matrix



$$S = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Links = nonzero elements in matrix

Properties of Matrix S

- Row i of S: Links from page i to other pages
- Column i of S: Links into page i
- S is a stochastic matrix:
  All elements in $[0, 1]$
  Elements in each row sum to 1
- Dominant left eigenvector:

$$\omega^T S = \omega^T \qquad \omega \geq 0 \qquad \|\omega\|_1 = 1$$

- ω_i is probability of visiting page i
- But: ω not unique

Google Matrix

Convex combination

$$G = \alpha S + \underbrace{(1-\alpha)\mathbb{1}v^T}_{\text{rank } 1}$$

- Stochastic matrix S
- Damping factor $0 \leq \alpha < 1$
  e.g. $\alpha = .85$
- Column vector of all ones $\mathbb{1}$
- Personalization vector $v \geq 0 \quad \|v\|_1 = 1$
  Models teleportation

PageRank

$$G = \alpha S + (1 - \alpha) \mathbb{1} v^T$$

- G is stochastic, with eigenvalues:

$$1 > \alpha |\lambda_2(S)| \geq \alpha |\lambda_3(S)| \geq \dots$$

- Unique dominant left eigenvector:

$$\pi^T G = \pi^T \qquad \pi \geq 0 \qquad \|\pi\|_1 = 1$$

- π_i is PageRank of web page i

[Haveliwala & Kamvar 2003, Eldén 2003, Serra-Capizzano 2005]

How Google Ranks Web Pages

- Model:
  Internet $\longrightarrow$ web graph $\longrightarrow$ stochastic matrix G
- Computation:
  PageRank π is eigenvector of G

  π_i is PageRank of page i
- Display:

  If $\pi_i > \pi_k$ then
  page i may* be displayed before page k

  * depending on hypertext analysis

Facts

- The anatomy of a large-scale hypertextual web search engine [Brin & Page 1998]
- US patent for PageRank granted in 2001
- Google indexes 10's of billions of web pages (1 billion = 10^9)
- Google serves ≥ 200 million queries per day
- Each query processed by ≥ 1000 machines
- All search engines combined process more than 500 million queries per day

[Desikan, 26 October 2006]

Computation of PageRank

The world's largest matrix computation
[Moler 2002]

- Eigenvector
- Matrix dimension is 10's of billions
- The matrix changes often
  250,000 new domain names every day
- Fortunately: Matrix is sparse

Power Method

Want: π such that $\pi^T G = \pi^T$

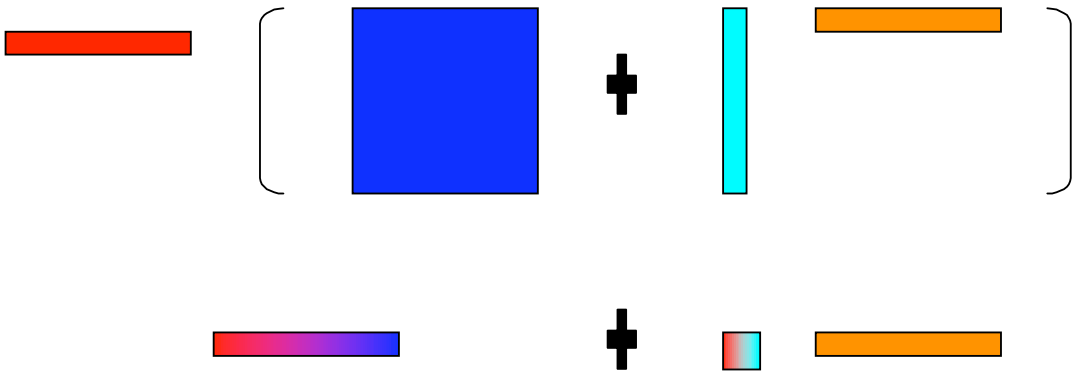
Power method:

Pick an initial guess $x^{(0)}$
Repeat
$$[x^{(k+1)}]^T := [x^{(k)}]^T G$$
until "termination criterion satisfied"

Each iteration is a matrix vector multiply

Matrix Vector Multiply

$$x^T G = x^T \left[\alpha S + (1-\alpha)\mathbb{1}v^T\right]$$



Cost: # non-zero elements in S

A power method iteration is cheap

Error Reduction in 1 Iteration

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

$$\begin{aligned} [x^{(k+1)} - \pi]^T &= [x^{(k)}]^T G - \pi^T G \\ &= \alpha [x^{(k)} - \pi]^T S \end{aligned}$$

Error: $\underbrace{\|x^{(k+1)} - \pi\|_1}_{\text{iteration } k+1} \leq \alpha \underbrace{\|x^{(k)} - \pi\|_1}_{\text{iteration } k}$

Error in Power Method

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1}v^T$$

Error after k iterations:

$$\|x^{(k)} - \pi\|_1 \leq \alpha^k \underbrace{\|x^{(0)} - \pi\|_1}_{\leq 2}$$

[Bianchini, Gori & Scarselli 2003]

Error bound does not depend on matrix dimension

Advantages of Power Method

- Simple implementation (few decisions)
- Cheap iterations (sparse matvec)
- Minimal storage (a few vectors)
- Robust convergence behaviour
- Convergence rate independent of matrix dimension
- Numerically reliable and accurate (no subtractions, no overflow)

But: can be slow

PageRank Computation

- Power method

  Page, Brin, Motwani & Winograd 1999
  Bianchini, Gori & Scarselli 2003

- Acceleration of power method

  Kamvar, Haveliwala, Manning & Golub 2003

  Haveliwala, Kamvar, Klein, Manning & Golub 2003

  Brezinski & Redivo-Zaglia 2004, 2006
  Brezinski, Redivo-Zaglia & Serra-Capizzano 2005

- Aggregation/Disaggregation

  Langville & Meyer 2002, 2003, 2006
  Ipsen & Kirkland 2006

PageRank Computation

- Methods that adapt to web graph

  Broder, Lempel, Maghoul & Pedersen 2004 Kamvar, Haveliwala & Golub 2004

  Haveliwala, Kamvar, Manning & Golub 2003

  Lee, Golub & Zenios 2003

  Lu, Zhang, Xi, Chen, Liu, Lyu & Ma 2004
  Ipsen & Selee 2006

- Krylov methods

  Golub & Greif 2004
  Del Corso, Gullí, Romani 2006

PageRank Computation

- Schwarz & asynchronous methods

  Bru, Pedroche & Szyld 2005
  Kollias, Gallopoulos & Szyld 2006

- Linear system solution

  Arasu, Novak, Tomkins & Tomlin 2002

  Arasu, Novak & Tomkins 2003

  Bianchini, Gori & Scarselli 2003

  Gleich, Zukov & Berkin 2004

  Del Corso, Gullí & Romani 2004
  Langville & Meyer 2006

PageRank Computation

- Surveys of numerical methods:
  Langville & Meyer 2004
  Berkhin 2005
  Langville & Meyer 2006 (book)

Is the Ranking Correct?

$$\pi^T = \begin{pmatrix} .23 & .24 & .26 & .27 \end{pmatrix}$$

- $x^T = \begin{pmatrix} .27 & .26 & .24 & .23 \end{pmatrix}$

  $\|x - \pi\|_\infty = .04$

  Small error, but incorrect ranking

- $y^T = \begin{pmatrix} 0 & .001 & .002 & .997 \end{pmatrix}$

  $\|y - \pi\|_\infty = .727$

  Large error, but correct ranking

What is Important?

Numerical value $\leftrightarrow$ ordinal rank

ordinal rank:
position of an element in an ordered list

Very little research on ordinal ranking
Rank-stability, rank-similarity

[Lempel & Moran, 2005]
[Borodin, Roberts, Rosenthal & Tsaparas 2005]

Ordinal Ranking

Largest element gets $\mathbf{Orank}\ 1$

$$\pi^T = (.23 \quad .24 \quad .26 \quad .27)$$

$$\mathbf{Orank}(\pi_4) = 1, \mathbf{Orank}(\pi_1) = 4$$

- $x^T = (.27 \quad .26 \quad .24 \quad .23)$

  $$\mathbf{Orank}(x_1) = 1 \neq \mathbf{Orank}(\pi_1) = 4$$

- $y^T = (0 \quad .001 \quad .002 \quad .997)$

  $$\mathbf{Orank}(y_4) = 1 = \mathbf{Orank}(\pi_4)$$

Problems with Ordinal Ranking

When done with power method:

- Popular termination criteria
  do not guarantee correct ranking
- Additional iterations can destroy ranking
- Rank convergence depends on:
  α, v, initial guess, matrix dimension,
  structure of web graph
- Even if successive iterates have the same ranking, their ranking may not be correct

[Wills & Ipsen 2007]

Ordinal Ranking Criterion

Given:

Approximation x, $x \geq 0$

Error bound $\beta \geq \|x - \pi\|_1$

Criterion: $x_i > x_j + \beta \Longrightarrow \pi_i > \pi_j$

Why?

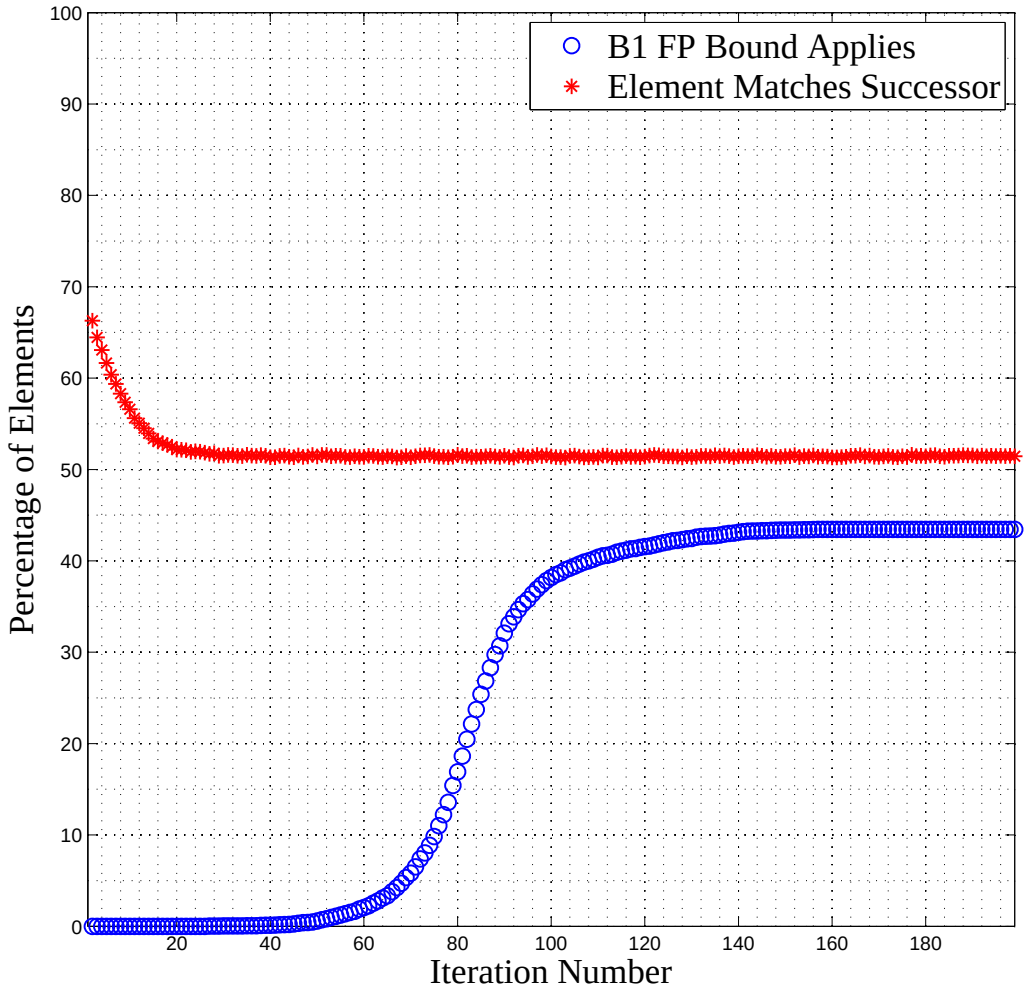
$$(x_i - \pi_i) - (x_j - \pi_j) \leq \|x - \pi\|_1 \leq \beta$$

$$x_i - (x_j + \beta) \leq \pi_i - \pi_j$$

$$0 < x_i - (x_j + \beta) \Longrightarrow 0 < \pi_i - \pi_j$$

[Kirkland 2006]

Applicability of Criterion



Properties of Ranking Criterion

- Applies to any approximation, provided error bound is available
- Requires well-separated elements
- Tends to identify ranks of larger elements
- Determines partial ranking
- Top-k, bucket and exact ranking
- Easy to use with power method

Top-k Ranking

Given:

Approximation x to PageRank π

Permutation P so that $\tilde{x} = Px$ with $\tilde{x}_1 \geq \ldots \geq \tilde{x}_n$

Write: $\tilde{\pi} = P\pi$

Suppose $\tilde{x}_k > \tilde{x}_{k+1} + \beta$

Top-k Ranking

$$\begin{array}{ll} \tilde{\boldsymbol{x}}_1 & \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_1) \leq \boldsymbol{k} \\ \vdots & \vdots \\ \tilde{\boldsymbol{x}}_k & \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_k) \leq \boldsymbol{k} \\ & \uparrow \\ & \boldsymbol{\beta} \\ & \downarrow \\ \tilde{\boldsymbol{x}}_{k+1} & \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_{k+1}) \geq \boldsymbol{k}+1 \\ \vdots & \vdots \\ \tilde{\boldsymbol{x}}_n & \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_n) \geq \boldsymbol{k}+1 \end{array}$$

Exact Ranking

Given:

Approximation x to PageRank π
Permutation P so that $\tilde{x} = Px$ with
$\tilde{x}_1 \geq \ldots \geq \tilde{x}_n$

Write: $\tilde{\pi} = P\pi$

If $\tilde{x}_{k-1} > \tilde{x}_k + \beta$ and $\tilde{x}_k > \tilde{x}_{k+1} + \beta$ then

$$\mathbf{Orank}(\tilde{\pi}_k) = k$$

Exact Ranking

$$
\begin{array}{l}
\tilde{\boldsymbol{x}}_1 \\
\vdots \\
\tilde{\boldsymbol{x}}_{k-1} \\
\qquad \updownarrow \quad \beta \qquad \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_k) \geq \boldsymbol{k} \\
\tilde{\boldsymbol{x}}_k \\
\qquad \updownarrow \quad \beta \qquad \mathbf{Orank}(\tilde{\boldsymbol{\pi}}_k) \leq \boldsymbol{k} \\
\tilde{\boldsymbol{x}}_{k+1} \\
\vdots \\
\tilde{\boldsymbol{x}}_n
\end{array}
$$

Bucket Ranking

Given:

Approximation x to PageRank π
Permutation P so that $\tilde{x} = Px$ with
$\tilde{x}_1 \geq \ldots \geq \tilde{x}_n$

Write: $\tilde{\pi} = P\pi$

Suppose $\tilde{x}_{k-i} > \tilde{x}_k + \beta$ and $\tilde{x}_k > \tilde{x}_{k+j} + \beta$

$\tilde{\pi}_k$ is in bucket of width $i + j - 1$

Bucket Ranking

$\tilde{x}_1$

$\vdots$

$\tilde{x}_{k-i}$

$\updownarrow \quad \beta \qquad \mathbf{Orank}(\tilde{\pi}_k) \geq k - i + 1$

$\tilde{x}_k$

$\updownarrow \quad \beta \qquad \mathbf{Orank}(\tilde{\pi}_k) \leq k + j - 1$

$\tilde{x}_{k+j}$

$\vdots$

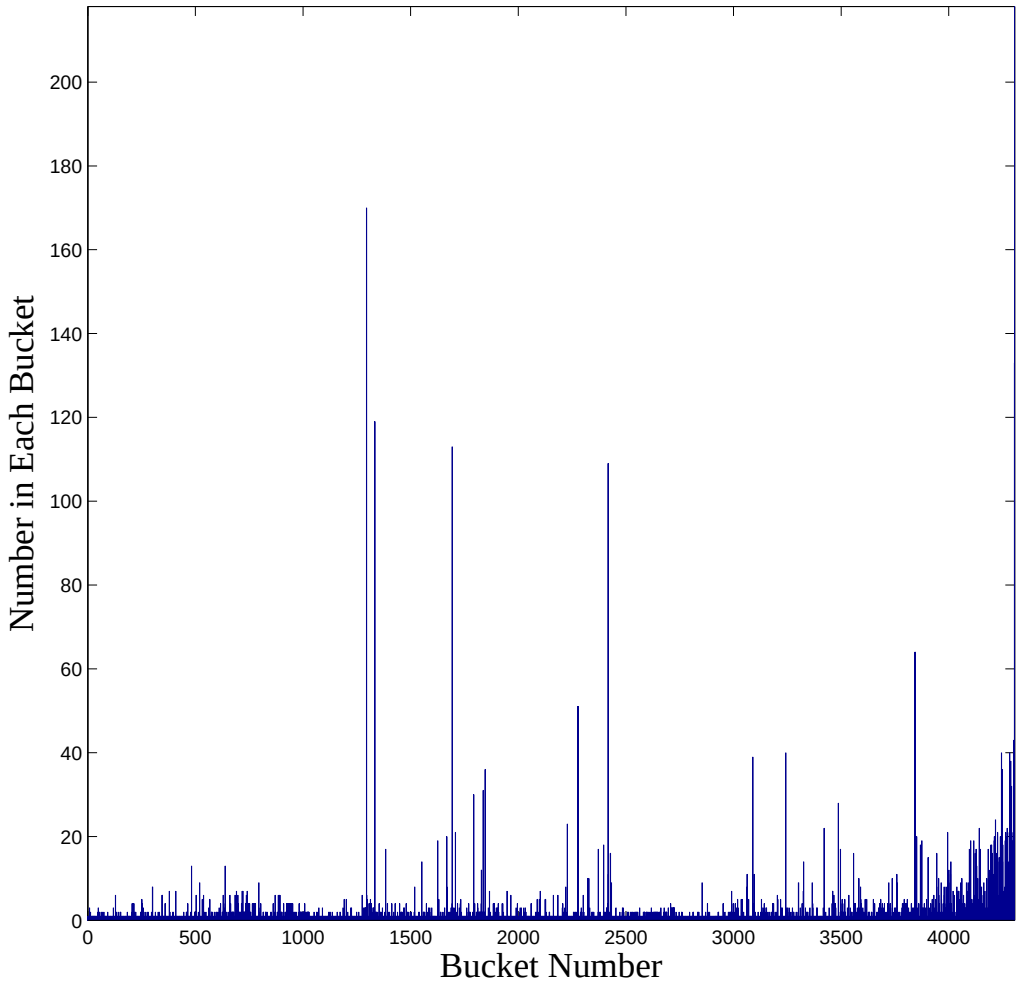
$\tilde{x}_n$

Experiments

n	# buckets	1. bucket	last bucket
9,914	4,307	1	7%
3,148,440	34,911	1	95%

n	exact rank	exact top 100	lowest rank
9,914	32%	79	9,215
3,148,440	0.76%	100	151,794

Buckets for Small Matrix



Power Method Ranking

Simple error bound:

$$\|x^{(k)} - \pi\|_1 \leq \underbrace{2\alpha^k}_{\beta}$$

Simple ranking criterion:

If $x_i^{(k)} > x_j^{(k)} + 2\alpha^k$ then $\pi_i > \pi_j$

But: $2\alpha^k$ is too pessimistic (not tight enough)

Power Method Ranking

Tighter error bound:

$$\|x^{(k)} - \pi\|_1 \leq \underbrace{\frac{\alpha}{1-\alpha}\|x^{(k)} - x^{(k-1)}\|_1}_{\beta}$$

More effective ranking criterion:

If $x_i^{(k)} > x_j^{(k)} + \beta$ then $\pi_i > \pi_j$

Floating Point Ranking

If $x_i^{(k)} > x_j^{(k)} + \beta$ then $\pi_i > \pi_j$

$$\beta = \frac{\alpha}{1-\alpha}\|x^{(k)} - x^{(k-1)}\|_1 + e$$

e is round off error from single matvec

IEEE double precision floating point arithmetic:

$$e \approx cm10^{-16}$$

$m \approx$ max # links into any web page

Expensive Implementation

To prevent accumulation of round off

- Explicit normalization of iterates
  $x^{(k+1)} = x^{(k+1)} / \|x^{(k+1)}\|_1$
- Compute norms, inner products, matvecs with compensated summation
- Limited by round off error from single matvec
- Analysis for matrix dimensions $n < 10^{14}$ in IEEE arithmetic ($\epsilon \approx 10^{-16}$)

Difficulties for XLARGE Problems

- Catastrophic cancellation when computing

$$\beta = \frac{\alpha}{1-\alpha}\|x^{(k)} - x^{(k-1)}\|_1 + e$$

- Bound β dominated by round off e
- Compensated summation insufficient to reduce higher order round off $\mathcal{O}(n\epsilon^2)$
- Doubly compensated summation too expensive: $\mathcal{O}(n \log n)$ flops

Possible Remedies

- Lump dangling nodes [Ipsen & Selee 2006]
  Web pages w/o outlinks:
  pdf & image files, protected pages, web frontier
  Up to 50%-80% of all web pages
- Remove unreferenced web pages
- Use faster converging method
  Then 1 power method iteration for ranking
- Relative ranking criteria?

Summary

- Google orders web pages according to: PageRank and hypertext analysis
- PageRank = left eigenvector of G

$$G = \alpha S + (1 - \alpha)\mathbb{1}v^T$$

- Power method: simple, robust, accurate
- Convergence rate depends on α but not on matrix dimension
- Criterion for ordinal ranking
- Round off serious for XLarge problems

User-Friendly Resources

- Rebecca Wills:
  Google's PageRank: The Math Behind the Search Engine
  Mathematical Intelligencer, 2006
- Amy Langville & Carl Meyer:
  Google's PageRank and Beyond
  The Science of Search Engine Rankings
  Princeton University Press, 2006
- Amy Langville & Carl Meyer:
  Broadcast of On-Air Interview, November 2006
  Carl Meyer's web page