Deferred correction from equispaced data based on efficient high-order rational integration

Georges Klein* with Stefan Güttel†

In this talk, we present *rational deferred correction* (RDC) methods for the solution of initial value problems. Inspired by *spectral deferred correction* (SDC) methods from [1] by Dutt, Greengard & Rokhlin, we demonstrate that similar accuracy and stability can be achieved with equispaced points instead of Gauss–Legendre points if one resorts to the linear barycentric rational interpolants [2] constructed by Floater and Hormann.

To be more specific, we are interested in solving initial-value problems for a function $u : [0, T] \to \mathbb{C}^N$,

$$u'(t) = f(t, u(t)), \qquad u(0) = u_0 \in \mathbb{C}^N. \tag{1}$$

Assume we have already computed a discrete low order approximation $\widetilde{u} \approx u$, e.g., with a forward or backward Euler scheme. In order to achieve higher precision, $\widetilde{u}$ is iteratively corrected as follows.

The problem (1) is reformulated as a Picard integral to avoid numerical differentiation,

$$u(t) = u(0) + \int_0^t f(\tau, u(\tau))\,\mathrm{d}\tau, \tag{2}$$

or equivalently, with $e = u - \widetilde{u}$ the approximation error,

$$\widetilde{u}(t) + e(t) = u(0) + \int_0^t f(\tau, \widetilde{u}(\tau) + e(\tau))\,\mathrm{d}\tau. \tag{3}$$

Using (2) to define the residual

$$r(t) = u(0) + \int_0^t f(\tau, \widetilde{u}(\tau))\,\mathrm{d}\tau - \widetilde{u}(t), \tag{4}$$

we immediately find from (3)

$$e(t) = r(t) + \int_0^t f(\tau, \widetilde{u}(\tau) + e(\tau)) - f(\tau, \widetilde{u}(\tau))\,\mathrm{d}\tau, \tag{5}$$

which is a Picard-type formulation for the error e. Equation (5) can then be solved with the same time-stepping method which was used to obtain the initial approximation $\widetilde{u}$. With this estimation of the error $\widetilde{u}$ is corrected via $\widetilde{u}_{\mathrm{new}} = \widetilde{u} + e$, so as to conclude one deferred correction sweep. The procedure can be iterated until stagnation occurs, typically when the precision of the collocation solution of (1) is reached.

Since the approximation $\widetilde{u}$ is available only as $\widetilde{u}_j$ at discrete values of the time variable once (2) is solved, for (4) to make sense, a continuous approximation must be constructed from the $\widetilde{u}_j$, e.g., via interpolation. The procedure proposed in [1] involves polynomial interpolants of degree n, where n can be quite large to achieve sufficient accuracy. To prevent this polynomial interpolation from being unstable, it was advocated in [1] to interpolate the integrand in (4) at Gauss–Legendre points

*Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, United Kingdom. Supported by the Swiss National Science Foundation under Grant No. PBFRP2-142826

†School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, United Kingdom

or Chebyshev points. Because such an interpolation process will achieve spectral accuracy in time, the resulting method is called spectral deferred correction (SDC).

Recent investigations have revealed that linear rational interpolation [2] with equispaced points can be stable and achieve high accuracy even with large numbers of points. We define rational deferred correction (RDC) as the analogue of the above described SDC with linear *rational* interpolation and *equispaced* points.

Before we present the RDC integrator in more detail and compare its performance with SDC, we review the construction and properties of the linear rational interpolation scheme. Assume we are given $n+1$ points $t_0 < t_1 < \ldots < t_n$ in a closed interval $[a,b]$ and corresponding values of a function $f_0, f_1, \ldots, f_n$. Each choice of the nonnegative parameter $d \le n$ defines a member of the family of rational interpolants,

$$r_n(t) = \frac{\sum_{i=0}^{n-d} \lambda_i(t) p_i(t)}{\sum_{i=0}^{n-d} \lambda_i(t)}, \qquad \lambda_i(t) = \frac{(-1)^i}{(t-t_i)\cdots(t-t_{i+d})},$$

as a blend of polynomial interpolants of degree d. With equispaced points in particular and an adequate choice of d, which we will explain, these interpolants are well-conditioned and lead to approximations of order $d+1$ of sufficiently smooth functions. Moreover the linearity in the data conveniently leads to quadrature rules of order $d+2$, and makes them very appealing for the solution of initial value problems as described above.

We review theoretical results, present numerical examples and compare some properties of RDC with those of SDC. Our investigations are initial studies of integration methods based on linear rational interpolation and can be extended in numerous ways.

References

[1] A. Dutt, L. Greengard, and V. Rokhlin. Spectral deferred correction methods for ordinary differential equations. *BIT*, 40:241–266, 2000.

[2] M. S. Floater and K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. *Numer. Math.*, 107:315–331, 2007.