Parallel variational space–time methods for the wave equation

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Abstract

The accurate and reliable numerical approximation of the hyperbolic wave equation is of fundamental importance to the simulation of acoustic, electromagnetic and elastic wave propagation phenomena. Here, we present families of variational space–time discretisation methods of higher order for the acoustic wave equation as a protoype model for the three-dimensional elastic wave equation. In temporal domain we present two families of continuous and discontinuous discretisation schemes. For the spatial discretisation a symmetric interior penalty discontinuous Galerkin method is used. From these classes of uniform Galerkin discretisations in space and time an approach of fourth-order accuracy is analysed carefully. The efficient solution of the resulting block-matrix system and inherently parallel numerical simulation through domain decomposition is adressed. The performance properties of the schemes are illustrated by sophisticated and challenging numerical experiments with complex wave propagation phenomena in heterogeneous media.

1 Motivation

Our interest in developing numerical approximation schemes of higher order accuracy for the wave equation comes from mechanical engineering. Multiple layer fibre reinforced composites have become one of the most promising materials to build light-weight structures for several fields of application, for instance in aerospace and automotive fields. These composites are able to combine high strength and rigidity of the reinforced fiber with excellent properties of synthetic resins in best possible way. Nondestructive material inspection by piezoelectric induced ultrasonic waves is a relatively new and challenging technique to monitor the healthiness of such components. Several material damages (delamination of layers, matrix cracks, fibre breaks) may occur and have to be detected by some autonomous structural health monitoring system. For the design of structural health monitoring systems it is strictly necessary to understand phenomenologically and quantitatively wave propagation in layered fibre reinforced composites and the influence of the geometrical and mechanical properties of the system structure. Numerical simulation is a promising way to achieve this goal; cf. [2] and Fig. 1. Therefore, the ability to solve numerically the wave equation in three space dimensions is particularly important from the point of view of physical realism. However, this is still a challenging task and an active area of research.

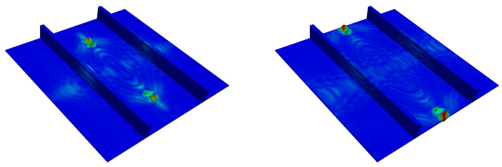

Figure 1: Ultrasonic waves in carbon fibre composite.

2 Introduction to variational space–time methods

In the field of numerical wave propagation the spatial discretisation by some discontinuous Galerkin finite element method (DGM) has attracted the interest of researchers; cf., e.g. [4]. Advantages of the FEM are the flexibility with which it can accommodate discontinuities in the model, material parameters and boundary conditions and the ability to approximate the wavefield with high degree polynomials. The spatial DGM has the further advantage that it can accomodate discontinuities also in the wavefield, it can be energy conservative, and it is suitable for inherently parallel simulations. The mass matrix of the spatial DGM is block-diagonal, where each block size coincides with the degrees of freedom of the associated element, such that its inverse is available at very low computational cost. Recently, variational space–time discretisation schemes were proposed and studied for the parabolic heat equation and for systems of ordinary differential equations [3]. In this contribution we will focus on the presentation of continuous and discontinuous variational temporal discretisation schemes from the variational space–time approach for the hyperbolic wave equation. For the spatial discretisation a symmetric interior penalty discontinuous Galerkin method is used; cf. [4].

From these families of uniform variational discretisations a scheme of fourth order accuracy with respect to the temporal and spatial variables is studied carefully. It will be shown that the block-diagonal structure of the spatial mass matrix, resulting from the discontinuous Galerkin approach, can be used to decouple efficiently the arising temporal block linear system. The performance properties and computational cost of the numerical scheme are illustrated by some numerical convergence studies. Moreover, the schemes are applied to wave propagation phenomena in heterogeneous media admitting mutiple sharp wave fronts; cf. [1] and Fig. 2.

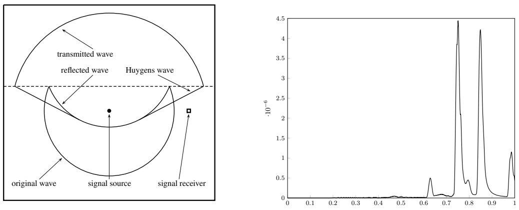

Figure 2: Inherently parallel acoustic wave simulation.

3 Temporal discretisation schemes

Exemplarily, our family of continuous-in-time variational discretisation schemes for the wave equation

$$\begin{aligned} \rho(\boldsymbol{x})\,\partial_t\, v(\boldsymbol{x},t) + a(u(\boldsymbol{x},t)) &= f(\boldsymbol{x},t)\,, \\ \partial_t\, u(\boldsymbol{x},t) - v(\boldsymbol{x},t) &= 0\,, \end{aligned} \tag{1}$$

written as velocity-displacement formulation and equipped with initial conditions $u(0) = u_0$, $v(0) = v_0$ and homogeneous Dirichlet boundary conditions, is presented here. For the discontinuous-in-time counterpart of this approach we refer to [1].

We decompose $I = (0,T]$ into N subintervals $I_n = (t_{n-1}, t_n]$. For some Hilbert space $\mathcal{H}$, let

$$\begin{aligned} \mathcal{X}_{\mathcal{C}}^r(\mathcal{H}) &= \{u \in \mathcal{C}(I,\, \mathcal{H}) : u|_{I_n} \in \mathbb{P}_r(I_n,\, \mathcal{H})\}\,, \\ \mathcal{Y}^r(\mathcal{H}) &= \{w \in L^2(I,\, \mathcal{H}) : w|_{I_n} \in \mathbb{P}_{r-1}(I_n,\, \mathcal{H})\}\,, \\ \mathbb{P}_r(I_n,\mathcal{H}) &= \Big\{u : I_n \to \mathcal{H} : u = \textstyle\sum_{j=0}^r \xi_n^j\, t^j, \xi_n^j \in \mathcal{H}\Big\}\,. \end{aligned}$$

Our continuous-in-time variational approximation of (1) then reads as: *Find* $u_\tau \in \mathcal{X}_{\mathcal{C}}^r(I, H_0^1(\Omega))$, $v_\tau \in \mathcal{X}_{\mathcal{C}}^r(I, L^2(\Omega))$, *such that* $u_\tau(0) = u_0$, $v_\tau(0) = v_0$ *and*

$$\int_0^T \Big\{ \big(\partial_t v_\tau, \widehat{w}_\tau\big)_\Omega + a\big(u_\tau, \widehat{w}_\tau\big) \Big\} \mathrm{d}t = \int_0^T \big(f, \widehat{w}_\tau\big)_\Omega \mathrm{d}t$$

$$\int_0^T \Big\{ \big(\partial_t u_\tau, \widetilde{w}_\tau\big)_\Omega - \big(v_\tau, \widetilde{w}_\tau\big)_\Omega \Big\} \mathrm{d}t = 0$$

for all $\widehat{w}_\tau \in \mathcal{Y}^r(H_0^1(\Omega))$ *and* $\widetilde{w}_\tau \in \mathcal{Y}^r(L^2(\Omega))$. Here, $(\cdot,\cdot)_\Omega$ denotes the $L^2(\Omega)$ inner product. Precisely, we have a temporal Galerkin-Petrov method, since the trial and test spaces do not coincide. Since the test space imposes no continuity constraints between elements, we can rewrite the problem as time marching scheme. Finally, we apply an interior penalty discontinuous Galerkin method in spatial domain and call this approach a cGP(r)-dG(p) method. We obtain the Crank-Nicolson scheme for $r = 1$ and for $r \geqslant 2$ we observe superconvergence in the integration points, cf. [1, 3].

4 Future Prospects

By using variational space–time methods for the discretisation of the wave equation we have a uniform variational approach in space and time which may be advantageous for the future analysis of the fully discrete problem and the construction of simultaneous space–time adaptive methods. Further, it is very natural to construct temporal methods of even higher order then presented here. The well-known finite element stability concepts of the temporal Galerkin-Petrov or discontinuous Galerkin methods can be used to obtain at least A-stable methods. For future developments, the well-known adaptive finite element techniques can be applied for changing the polynomial degree and the length of the time steps, cf. [3, 6, 5].

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