Space-Time Methods for Wave Equations

Discretizations and Convergence Analysis

Christian Wieners

Karlsruhe Institute of Technology
Department of Mathematics
Kaiserstr. 89-93, 76128 Karlsruhe, Germany
christian.wieners@kit.edu

We discuss three full space-time discretizations for linear wave equations: an adaptive discontinuous Galerkin method in space and time (dG), a discontinuous Petrov-Galerkin method (DPG), and a new hybrid variant (hDPG).

The general setting Let $\Omega \subset \mathbb{R}^D$ be a bounded Lipschitz domain, and let $V \subset \mathrm{L}_2(\Omega)^J$ be a Hilbert space with weighted inner product $(\mathbf{v}, \mathbf{w})_V = (M\mathbf{v}, \mathbf{w})_{0,\Omega}$, where $M \in \mathrm{L}_\infty(\Omega)^{J\times J}$ is uniformly positive and symmetric. We study the evolution equation

$$M\partial_t \mathbf{u}(t) + A\mathbf{u}(t) \;=\; \mathbf{f} \qquad t \in [0, T]\,, \qquad \mathbf{u}(0) = \mathbf{u}_0\,, \tag{1}$$

where A is a linear operator in V with domain $\mathcal{D}(A) \subset V$ corresponding to a hyperbolic linear system, i.e. $(A\mathbf{v}, \mathbf{v})_{0,\Omega} = 0$ for $\mathbf{v} \in \mathcal{D}(A)$. For simplicity, we consider only homogeneous boundary conditions on $\partial\Omega$ which are included in the domain of the operator.

For the existence and uniqueness of the solution we require that $V(A) \subset V$ is the closure of $\mathcal{D}(A)$ with respect to the topology in V, that A is a closed operator in $V(A)$ and that the operator $M^{-1}A$ generates a semigroup in $V(A)$. This setting applies, e.g., to acoustic waves with $M(\mathbf{q}, p) = (\mathbf{q}, \rho p)$ and $A(\mathbf{q}, p) = (\nabla p, \operatorname{div} \mathbf{q})$, to elastic waves with $M(\boldsymbol{\sigma}, \mathbf{v}) = (\mathbf{C}^{-1}\boldsymbol{\sigma}, \rho\mathbf{v})$ and $A(\boldsymbol{\sigma}, \mathbf{v}) = (-\boldsymbol{\varepsilon}(\mathbf{v}), -\operatorname{div} \boldsymbol{\sigma})$, and to electro-magnetic waves with $M(\mathbf{H}, \mathbf{E}) = (\mu\mathbf{H}, \varepsilon\mathbf{E})$, where the operator $A(\mathbf{H}, \mathbf{E}) = (\operatorname{curl} \mathbf{E}, -\operatorname{curl} \mathbf{H})$ is defined, e.g., in $\mathcal{D}(A) = \big\{(\mathbf{H}, \mathbf{E}) \in \mathrm{H}(\mathrm{curl}, \Omega) \times \mathrm{H}_0(\mathrm{curl}, \Omega)\colon \operatorname{div}(\mu\mathbf{H}) = 0,\ \operatorname{div}(\varepsilon\mathbf{E}) = 0\big\}$ for a perfect conducting boundary.

Let $Q = \Omega \times (0, T)$ be the space-time cylinder. We define $H = \mathrm{L}_2(0, T; V(A)) \subset \mathrm{L}_2(Q)$, and we consider the space-time operator $L = M\partial t + A$ defined on $U = \mathcal{D}(L)$, where we assume that U is closed with respect to the graph norm of L. Our analysis is based on the a-priori bound

$$\|\mathbf{u}\|_H \le 2T\, \|L\mathbf{u}\|_H\,,$$

which shows that solution of the equation $L\mathbf{u} = \mathbf{f}$ is unique. Note that L is skew-adjoint, i.e., $(L\mathbf{u}, \mathbf{v})_Q = -(\mathbf{u}, L\mathbf{v})_Q$.

For a given decomposition of the space-time cylinder $\bar{Q} = \bigcup \bar{\tau}$ into space-time cells $\tau = K \times I$ we discuss three discretizations.

A discontinuous Galerkin method (dG) Let $V_h \subset \mathrm{L}_2(\Omega)^J$ be a discontinuous space of polynomials with variable degree, and let A_h be the discontinuous Galerkin operator with full upwind flux, see [1]. In every time slice $\Omega \times (t_{n-1}, t_n)$ we use the implicit mid-point rule; note that this is unconditionally stable. This allows for an adaptive local refinement of the space-time discretization and for a parallel multigrid preconditioner on the refinement hierarchy, where the full problem on the coarsest level is solved with a parallel direct solver [2].

A discontinuous Petrov-Galerkin method (DPG) On every space-time cell τ, integration by parts yields

$$(L\mathbf{u}, \mathbf{v})_\tau = -(\mathbf{u}, L\mathbf{v})_\tau + \langle \gamma_\tau \mathbf{u}, \gamma_\tau^* \mathbf{v} \rangle_\tau \,,$$

where γ_τ is the trace operator of $U|_\tau$ to values on $\partial\tau$, and γ_τ^* is its adjoint. Let $\hat{U} = \prod \gamma_\tau(U|_\tau)$ be the global trace space. The DPG method computes an approximation $(\hat{\mathbf{u}}_\tau) \in \hat{U}$ of the trace and local discontinuous approximations $\mathbf{u}_\tau \in \mathrm{L}_2(\tau)^J$ such that

$$-(\mathbf{u}_\tau, L\mathbf{v}_\tau)_\tau + \langle \hat{\mathbf{u}}_\tau, \gamma_\tau^* \mathbf{v}_\tau \rangle_\tau = (\mathbf{f}, \mathbf{v}_\tau)_\tau$$

for all test functions $\mathbf{v}_\tau$ in an optimal test space. In this setting, optimal space-time a priori estimates exist [4], and the trace approximation is determined by a symmetric positive definite Schur complement problem which can be preconditioned efficiently by multigrid methods [3].

A hybrid discontinuous Petrov-Galerkin method (hDPG) A hybrid variant of the DPG method is formally obtained by choosing a nonconforming approximation of the trace space. This makes the data structure and a parallel communication more flexible. Again this allows for a symmetric positive definite Schur complement reduction.

References

[1] M. Hochbruck, T. Pazur, A. Schulz, E. Thawinan, and C. Wieners. Efficient time integration for discontinuous Galerkin approximations of linear wave equations. 2013. In preparation.

[2] D. Maurer and C. Wieners. A parallel block LU decomposition method for distributed finite element matrices. Parallel Computing, 37:742–758, 2011.

[3] C. Wieners. A geometric data structure for parallel finite elements and the application to multigrid methods with block smoothing. Computing and Visualization in Science 13:161-175, 2010.

[4] C. Wieners and B. Wohlmuth. Robust operator estimates and the application to substructuring methods for first-order systems. 2013. Submitted.