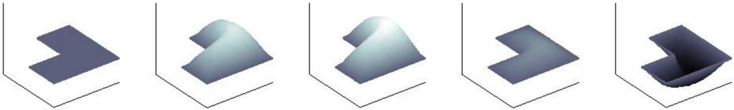
Sparse space-time Petrov-Galerkin discretizations for parabolic evolution equations

Roman Andreev

Seminar for Applied Mathematics / ETH Zürich
(academic guest)
andreevr@math.ethz.ch



Jun 18, 2013

Support

- Swiss NSF 127034 & ERC AdG 247277
  during PhD thesis, advisor: Ch. Schwab, ETH Zürich
- Partly joint work with C. Tobler
- Travel support: LMS R115785 ERZ (S. Güttel)

References

- arXiv, **2012**: Space-time discretization of the heat equation. A concise Matlab implementation (and further references).

Space-time simultaneous finite element discretizations for

$$\begin{aligned}(\partial_t - \nabla\cdot a(t,x)\nabla)u(t,x) &= f(t,x) \quad \text{in} \quad \mathrm{J}\times D,\\ u(0,x) &= g(x) \quad \text{in} \quad D,\\ u(t,x) &= 0 \quad \text{on} \quad \mathrm{J}\times\partial D,\end{aligned}$$

and **preconditioning** of the resulting linear algebraic system.

- $\mathrm{J} = (0,\mathrm{T})$ temporal interval
- $D \subset \mathbb{R}^d$ bounded domain
- a = diffusion coefficient ($a \in L^\infty(\mathrm{J}\times D)$)
- f = source ($f \in L^2(\mathrm{J}; H^{-1}(D))$)
- g = initial value ($g \in L^2(D)$)

Space-time simultaneous finite element discretizations for

$$
\begin{aligned}
(\partial_t - \nabla \cdot a(t,x)\nabla)u(t,x) &= f(t,x) \quad \text{in} \quad \mathrm{J} \times D, \\
u(0,x) &= g(x) \quad \text{in} \quad D, \\
u(t,x) &= 0 \quad \text{on} \quad \mathrm{J} \times \partial D,
\end{aligned}
$$

and **preconditioning** of the resulting linear algebraic system.

- various space-time variational formulations are available
- quasi-optimality (cf. Céa's lemma) from space-time stability
- massively parallel space-time compressive algorithms
- low regularity assumptions on a, f and g, but none on u
- high order approximation in time and space possible
- space-time error control; towards space-time adaptivity
- novel parabolic multilevel preconditioners
- application:

Space-time simultaneous finite element discretizations for

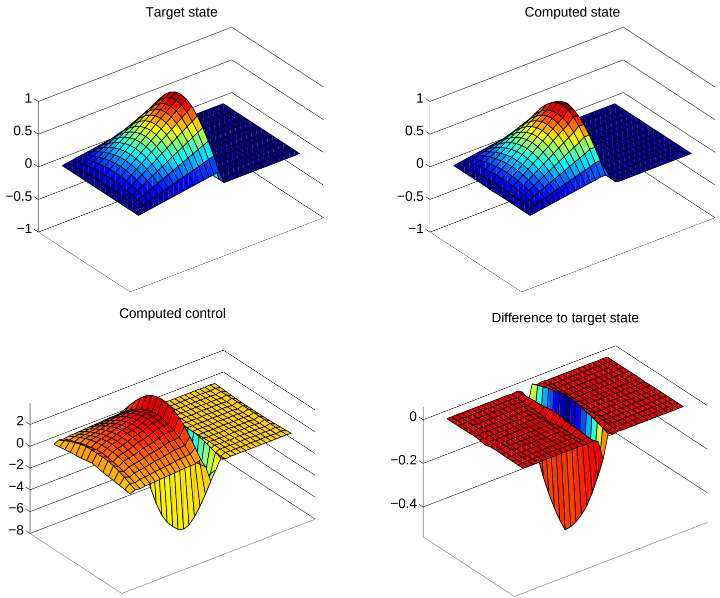
$$\begin{aligned}(\partial_t - \nabla\cdot a(t,x)\nabla)u(t,x) &= f(t,x) \quad \text{in} \quad \mathrm{J}\times D,\\ u(0,x) &= g(x) \quad \text{in} \quad D,\\ u(t,x) &= 0 \quad \text{on} \quad \mathrm{J}\times\partial D,\end{aligned}$$

and **preconditioning** of the resulting linear algebraic system.

Example: given $u^\star \in L^2(\mathrm{J}\times D)$ and $\alpha > 0$, minimize

$$\|u - u^\star\|^2_{L^2(\mathrm{J}\times D)} + \alpha\|(f,g)\|^2$$

where u solves the parabolic PDE. The mapping $(f,g)\mapsto u$ enters the optimality conditions. $\rightsquigarrow$ “correct” discretization?



Target state
1
0.5
0
−0.5
−1
Computed state
1
0.5
0
−0.5
−1
Computed control
2
0
−2
−4
−6
−8
Difference to target state
0
−0.2
−0.4

Space-time simultaneous finite element discretizations for

$$
\begin{aligned}
(\partial_t - \nabla \cdot a(t,x)\nabla)u(t,x) &= f(t,x) \quad \text{in} \quad \mathrm{J} \times D, \\
u(0,x) &= g(x) \quad \text{in} \quad D, \\
u(t,x) &= 0 \quad \text{on} \quad \mathrm{J} \times \partial D,
\end{aligned}
$$

and **preconditioning** of the resulting linear algebraic system.

Suppose $a = 1$, $f = 0$, and $-\Delta g = \lambda^2 g$, then

$$
u(t,x) = w(t)g(x), \quad (\partial_t + \lambda^2)w(t) = 0 \quad \text{on} \quad \mathrm{J}, \quad w(0) = 1.
$$

This motivates the variational formulation: find $w \in H^1(\mathrm{J})$ s.t.

$$
\int_{\mathrm{J}} (\partial_t w + \lambda^2 w) v_1 \mathrm{d}t + w(0) v_0 \stackrel{!}{=} 1 v_0 \quad \forall v_1 \in L^2(\mathrm{J}), v_0 \in \mathbb{R}.
$$

This set-up will suffice to exhibit the main difficulty with $u \mapsto u_L$.

For $w \in H^1(\mathrm{J})$ and $v = (v_0, v_1) \in \mathbb{R} \times L^2(\mathrm{J})$ define B_λ and F by

$$\langle B_\lambda w, v \rangle := \int_{\mathrm{J}} (\partial_t w + \lambda^2 w) v_1 \mathrm{d}t + w(0) v_0 \quad \text{and} \quad \langle F, v \rangle := 1 v_0.$$

The variational formulation thus reads:

$$\text{Find } u \in X : \quad \langle B_\lambda u, v \rangle = \langle F, v \rangle \quad \forall v \in Y$$

where $X := H^1(\mathrm{J})$ and $Y := \mathbb{R} \times L^2(\mathrm{J})$.

It is well-posed only if any perturbation $B_\lambda(u + w) - B_\lambda u$ can be detected by some test function v:

$$\gamma := \inf_{w \neq 0} \sup_{v \neq 0} \frac{\langle B_\lambda w, v \rangle}{\|w\| \|v\|} > 0.$$

For $w \in H^1(\mathrm{J})$ and $v = (v_0, v_1) \in \mathbb{R} \times L^2(\mathrm{J})$ define B_λ and F by

$$\langle B_\lambda w, v \rangle := \int_{\mathrm{J}} (\partial_t w + \lambda^2 w) v_1 \mathrm{d}t + w(0) v_0 \quad \text{and} \quad \langle F, v \rangle := 1 v_0.$$

First attempt at a **discrete** variational formulation:

$$\text{Find } u_L \in X_L : \quad \langle B_\lambda u_L, v_L \rangle = \langle F, v_L \rangle \quad \forall v_L \in Y_L$$

where $X_L \subset H^1(\mathrm{J})$ and $Y_L \subset \mathbb{R} \times L^2(\mathrm{J})$ finite-dimensional.

It is well-posed only if any perturbation $B_\lambda(u_L + w_L) - B_\lambda u_L$ can be detected by some test function v_L:

$$\gamma_L := \inf_{w_L \neq 0} \sup_{v_L \neq 0} \frac{\langle B_\lambda w_L, v_L \rangle}{\|w_L\| \|v_L\|} > 0.$$

For $w \in H^1(\mathrm{J})$ and $v = (v_0, v_1) \in \mathbb{R} \times L^2(\mathrm{J})$ define B_λ and F by

$$\langle B_\lambda w, v \rangle := \int_{\mathrm{J}} (\partial_t w + \lambda^2 w) v_1 \mathrm{d}t + w(0) v_0 \quad \text{and} \quad \langle F, v \rangle := 1 v_0.$$

First attempt at a **discrete** variational formulation:

$$\text{Find } u_L \in X_L : \quad \langle B_\lambda u_L, v_L \rangle = \langle F, v_L \rangle \quad \forall v_L \in Y_L$$

where $X_L \subset H^1(\mathrm{J})$ and $Y_L \subset \mathbb{R} \times L^2(\mathrm{J})$ finite-dimensional.

A family of subspaces X_L, Y_L, parameterized by "level" L,
is called **space-time stable** if $\gamma_L \geq \gamma > 0$ in

$$\gamma_L := \inf_{w_L \neq 0} \sup_{v_L \neq 0} \frac{\langle B_\lambda w_L, v_L \rangle}{\|w_L\| \|v_L\|} > 0.$$

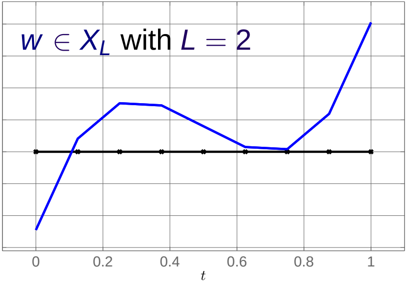
Outline of the talk

- The "natural" X_L and Y_L are not space-time stable.
- Abstract stable MinRes discrete variational formulation.
- Space-time (sparse) discretization of the heat equation.
- Novel parabolic multilevel preconditioners.

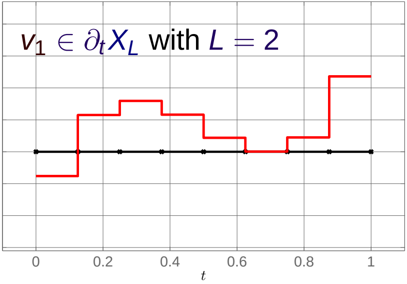
The "natural" X_L and Y_L are not space-time stable

Take ↓ finite set of nodes

- temporal mesh $\mathcal{T}_L = \{\inf J =: t_0 < t_1 < \ldots < t_{2^L+1} := \sup J\}$
- $X_L \subset H^1(J)$ continuous piecewise affine on $\mathcal{T}_L$
- $Y_L := \mathbb{R} \times \partial_t X_L \subset \mathbb{R} \times L^2(J)$ piecewise constant on $\mathcal{T}_L$



$\dim X_L = 9$



$\dim Y_L = 1 + 8$

The “natural” X_L and Y_L are not space-time stable

Take $\downarrow$ finite set of nodes

- temporal mesh $\mathcal{T}_L = \{\inf \mathrm{J} =: t_0 < t_1 < \ldots < t_{2^{L+1}} := \sup \mathrm{J}\}$
- $X_L \subset H^1(\mathrm{J})$ continuous piecewise affine on $\mathcal{T}_L$
- $Y_L := \mathbb{R} \times \partial_t X_L \subset \mathbb{R} \times L^2(\mathrm{J})$ piecewise constant on $\mathcal{T}_L$

This is equivalent to Crank-Nicolson time-stepping, see

- B.L. Hulme,
  One-step pw. polynomial Galerkin [...], **1972**
- G. Akrivis, Ch.G. Makridakis, R.H. Nochetto,
  Galerkin and Runge-Kutta [...], **2011**

The "natural" X_L and Y_L are not space-time stable

Take ↓ finite set of nodes

- temporal mesh $\mathcal{T}_L = \{\inf \mathrm{J} =: t_0 < t_1 < \ldots < t_{2^{L+1}} := \sup \mathrm{J}\}$
- $X_L \subset H^1(\mathrm{J})$ continuous piecewise affine on $\mathcal{T}_L$
- $Y_L := \mathbb{R} \times \partial_t X_L \subset \mathbb{R} \times L^2(\mathrm{J})$ piecewise constant on $\mathcal{T}_L$

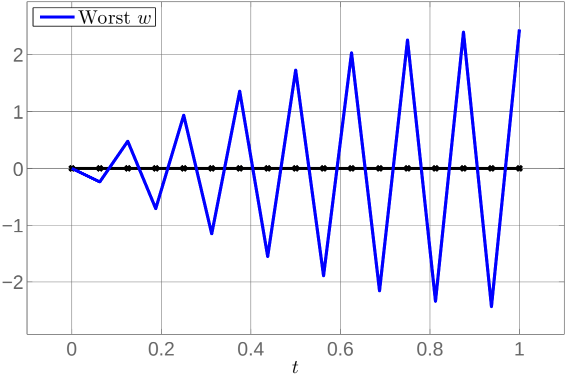
Do we have stability

$$\gamma_L := \inf_{w_L \neq 0} \sup_{v_L \neq 0} \frac{\langle B_\lambda w_L, v_L \rangle}{\|w_L\|_\lambda \|v_L\|_\lambda} \geq \gamma > 0$$

uniformly in $\lambda > 0$ (and independently of u)?

The norms $\|\cdot\|_\lambda$ allow to generalize to the parabolic PDE.

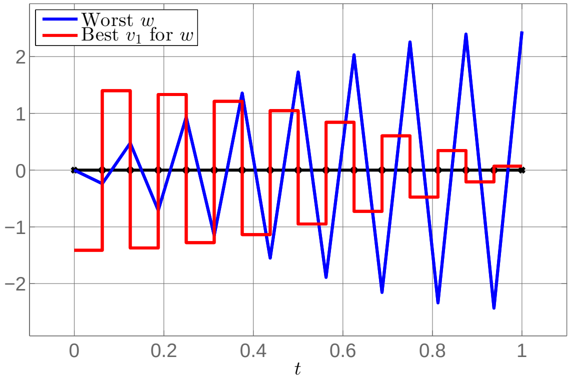
The “natural” X_L and Y_L are not space-time stable



A unit-norm $w \in X_L$ that realizes the worst case $\gamma_L \approx 0.0848$

cf. I. Babuška and T. Janik, The h-p [...] for parabolic equations. II., **1990**, also for the discussion of the Crank-Nicolson time-stepping method.

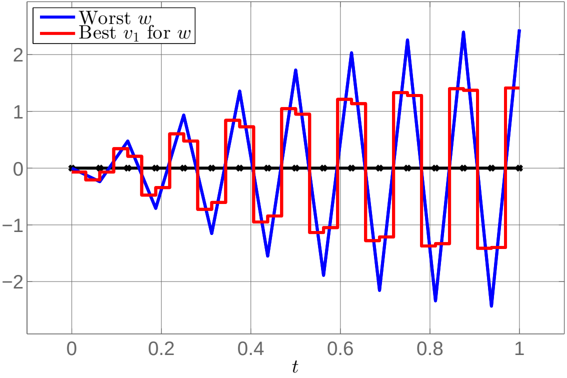
The “natural” X_L and Y_L are not space-time stable



A unit-norm $w \in X_L$ that realizes the worst case $\gamma_L \approx 0.0848$
and its best-detecting unit-norm $v_1 \in \partial_t X_L$

May get arbitrarily small for some temporal mesh

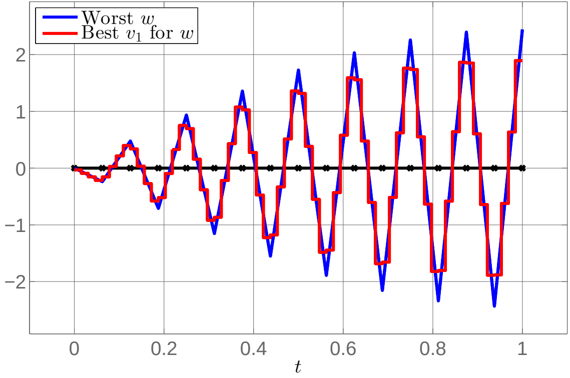
The "natural" X_L and Y_L are not space-time stable



A unit-norm $w \in X_L$ that realizes the worst case $\gamma_L \approx 0.8671 \geq \sqrt{3/4}$
and its best-detecting unit-norm $v_1 \in \partial_t X_{L+1}$

Cannot get much worse for any temporal mesh

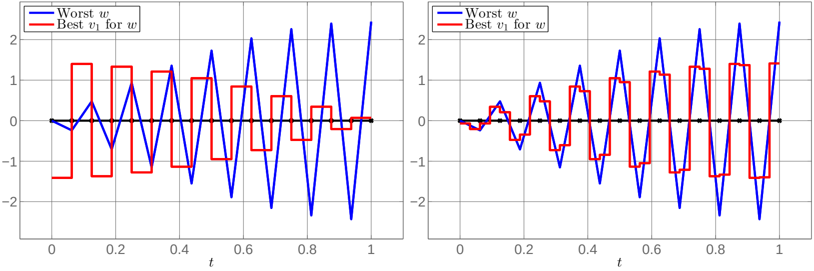
The “natural” X_L and Y_L are not space-time stable



A unit-norm $w \in X_L$ that realizes the worst case $\gamma_L \approx 0.9685 \geq \sqrt{15/16}$
and its best-detecting unit-norm $v_1 \in \partial_t X_{L+2}$

<u>Cannot</u> get much worse for any temporal mesh

The “natural” X_L and Y_L are not space-time stable



How to incorporate the finer test space ↑
in the discrete variational formulation?

Abstract stable MinRes discrete variational formulation

Let

- X and Y be real Hilbert spaces,
- $B : X \to Y'$ be a bounded linear operator,
- $X_L \subset X$ and $Y_L \subset Y$ be finite-dimensional subspaces,
- the pair X_L, Y_L satisfy the discrete inf-sup condition $\gamma_L > 0$,

$$\gamma_L := \inf_{w_L \neq 0} \sup_{v_L \neq 0} \frac{\langle B w_L, v_L \rangle}{\|w_L\| \|v_L\|} > 0.$$

Abstract stable MinRes discrete variational formulation

Let

- X and Y be real Hilbert spaces,
- $B : X \to Y'$ be a bounded linear operator,
- $X_L \subset X$ and $Y_L \subset Y$ be finite-dimensional subspaces,
- the pair X_L, Y_L satisfy the discrete inf-sup condition $\gamma_L > 0$.

Let $u \in X$. Given Bu, the goal is to

$$\text{find} \quad u_L \in X_L \quad \text{such that} \quad Bu \approx Bu_L.$$

Remarks

- No assumption on B^{-1}.
- $\dim X_L < \dim Y_L$ is ok.

Abstract stable MinRes discrete variational formulation

Let

- X and Y be real Hilbert spaces,
- $B : X \to Y'$ be a bounded linear operator,
- $X_L \subset X$ and $Y_L \subset Y$ be finite-dimensional subspaces,
- the pair X_L, Y_L satisfy the discrete inf-sup condition $\gamma_L > 0$.

Thm. For each $u \in X$ there exists a unique $T_L u \in X_L$ such that

$$R_L(T_L u) \leq \inf_{w_L \in X_L} R_L(w_L), \quad R_L(w_L) := \sup_{v_L \neq 0} \frac{|\langle Bu - Bw_L, v_L \rangle|}{\|v_L\|}.$$

The map T_L is linear, $T_L^2 = T_L$, and $\|T_L\| \leq \gamma_L^{-1} \|B\|$. Therefore,

$$\|u - T_L u\| \leq \frac{\|B\|}{\gamma_L} \inf_{w_L \in X_L} \|u - w_L\| \qquad \text{(quasi-optimality)}$$

Abstract stable MinRes discrete variational formulation

Given $X_L \subset X$ and $Y_L \subset Y$ with $\gamma_L > 0$, the MinRes solution to

$$Bu = F \in Y'$$

is therefore well-defined as the residual minimizer,

$$u_L := \operatorname*{arg\,min}_{w_L \in X_L} R_L(w_L), \quad R_L(w_L) := \sup_{v_L \neq 0} \frac{|\langle F - Bw_L, v_L \rangle|}{\|v_L\|}.$$

Operator preconditioning: With bases for X_L and Y_L, the sol'n u_L is approximated by iterating on the linear algebraic system

$$\mathbf{M}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{N}^{-1}\mathbf{B}\mathbf{u} = \mathbf{M}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{N}^{-1}\mathbf{F},$$

where the matrices **N** and **M** measure the Y and the X norms.
The condition number is controlled by γ_L^{-1}.

On the heat equation: a selection of related efforts

- I. Babuška and T. Janik, The h-p version [...], **1990**
- G. Horton and S. Vandewalle, A space-time multigrid method [...], **1995**
  W. Hackbusch, Parabolic multi-grid methods, **1984**
- D. Sheen, I.H. Sloan, and V. Thomée, A parallel method for [...] parabolic equations based on Laplace transformation [...], **2003**
- M. Griebel and D. Oeltz, A sparse grid space-time [...], **2007**
- M.J. Gander and S. Vandewalle, Analysis of the parareal [...], **2007**
- Y. Maday and E.M. Rønquist, Parallelization in time through tensor-product space-time solvers, **2008**
- Ch. Schwab and R. Stevenson, Space-time adaptive wavelet methods for parabolic evolution problems, **2009**
- N.G. Chegini and R. Stevenson, Adaptive wavelet schemes [...], **2011**
- L. Banjay and D. Peterseim, Parallel multistep methods [...], **2011**
- M. Neumüller and O. Steinbach, Refinement of flexible space-time finite element meshes and discontinuous Galerkin methods, **2011**
- A. Chernov and Ch. Schwab, Sparse space-time Galerkin BEM for the nonstationary heat equation, **2012**
- S.V. Dolgov, B.N. Khoromskij, and I.V. Oseledets, Fast solution of parabolic problems in the TT/QTT format, **2012**
- ...

On the heat equation: variational formulation

Define the spaces

- $X := L^2(\mathrm{J}; V) \cap H^1(\mathrm{J}; V')$, where $V := H_0^1(D)$,
- $Y := H \times L^2(\mathrm{J}; V)$, where $H := L^2(D)$.

Encode the heat eq'n in a **space-time variational formulation**

$$\text{Find} \quad u \in X : \quad \langle Bu, v\rangle = \langle F, v\rangle \quad \forall v \in Y$$

with the bounded linear operators $B : X \to Y'$ and $F \in Y'$:

$$\langle Bw, v\rangle := \int_{\mathrm{J}} \langle \partial_t w - \nabla \cdot a \nabla w, v_1\rangle \mathrm{d}t + (w(0), v_0)_{L^2(D)},$$

$$\langle F, v\rangle := \int_{\mathrm{J}} \langle f, v_1\rangle \mathrm{d}t + (g, v_0)_{L^2(D)}.$$

Studied by Ch. Schwab and R. Stevenson, Space-time adaptive wavelet methods for parabolic evolution problems, **2009**

On the heat equation: variational formulation

Define the spaces

- $X := L^2(\mathrm{J}; V) \cap H^1(\mathrm{J}; V')$, where $V := H_0^1(D)$,
- $Y := H \times L^2(\mathrm{J}; V)$, where $H := L^2(D)$.

Encode the heat eq'n in a **space-time variational formulation**

$$\text{Find} \quad u \in X : \quad \langle Bu, v \rangle = \langle F, v \rangle \quad \forall v \in Y.$$

Default norms on X and Y:

$$\|w\|_X^2 := \|\partial_t w\|_{L^2(\mathrm{J};V')}^2 + \|w\|_{L^2(\mathrm{J};V)}^2, \qquad w \in X,$$
$$\|v\|_Y^2 := \|v_0\|_H^2 + \|v_1\|_{L^2(\mathrm{J};V)}^2, \qquad v = (v_0, v_1) \in Y.$$

The norm on V is the H^1-seminorm.

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ such that the compatibility condition

$$\tau := \inf_{L \in \mathbb{N}} \left[\inf_{e \in \partial_t E_L + E_L \setminus \{0\}} \sup_{f \in F_L \setminus \{0\}} \frac{(e, f)_{L^2(\mathrm{J})}}{\|e\|_{L^2(\mathrm{J})} \|f\|_{L^2(\mathrm{J})}} \right] > 0$$

  holds. For example,
  - $E_L = F_L = \{\text{polynomials up to degree } L\}$, (Babuška & Janik)
  - $E_L = F_L = \{\sin + \cos \text{ up to frequency } L\}$, (Langer & Wolfmayr)
  - $F_L = \partial_t E_L + E_L$,
  - $E_L = \{\sim 2^L \text{ hats on uniform mesh}\}$, $F_L = E_{L+1}$. (R.A.)
- $V_L \subseteq V_{L+1} \subseteq V$
- $X_L \times Y_L \subseteq X \times Y$ as

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$
- $X_L \times Y_L \subseteq X \times Y$ as

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ such that the "approximate self-duality" condition

$$\kappa := \inf_{L \in \mathbb{N}} \left[\inf_{\chi' \in V_L \setminus \{0\}} \sup_{\chi \in V_L \setminus \{0\}} \frac{(\chi', \chi)_H}{\|\chi'\|_{V'} \|\chi\|_V} \right] > 0$$

  holds. For example,
  - wavelet Riesz bases for V', H, and V, (Schwab & Stevenson)
  - P1 FEM on quasi-uniform meshes, (Babuška & Janik)
  - if the H-projection is V-stable. (cf. Bramble, Pasciak & Steinbach)
- $X_L \times Y_L \subseteq X \times Y$ as

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ with $\kappa > 0$,
- $X_L \times Y_L \subseteq X \times Y$ as

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ with $\kappa > 0$,
- $X_L \times Y_L \subseteq X \times Y$ as the space-time **full tensor product**

$$X_L := E_L \otimes V_L, \quad Y_L := V_L \times [F_L \otimes V_L]$$

discrete trial and test spaces, **or** ...

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ with $\kappa > 0$,
- $X_L \times Y_L \subseteq X \times Y$ as the space-time **full tensor product**

$$X_L := E_L \otimes V_L, \quad Y_L := V_L \times [F_L \otimes V_L]$$

or the **sparse tensor product**

$$X_L := \sum_{0 \le k+\ell \le L} E_k \otimes V_\ell, \quad Y_L := V_L \times \sum_{0 \le k+\ell \le L} [F_k \otimes V_\ell]$$

discrete trial and test spaces.

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(J)$ and $F_L \subseteq F_{L+1} \subseteq L^2(J)$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ with $\kappa > 0$,
- $X_L \times Y_L \subseteq X \times Y$ as the **FTP** or the **STP** subspaces.

On the heat equation: Space-time stability

Take nested sequences of closed nontrivial subspaces ($L \in \mathbb{N}_0$)

- $E_L \subseteq E_{L+1} \subseteq H^1(\mathrm{J})$ and $F_L \subseteq F_{L+1} \subseteq L^2(\mathrm{J})$ with $\tau > 0$,
- $V_L \subseteq V_{L+1} \subseteq V$ with $\kappa > 0$,
- $X_L \times Y_L \subseteq X \times Y$ as the **FTP** or the **STP** subspaces.

Theorem

There exists $c > 0$ *such that*

$$\inf_{L \in \mathbb{N}} \gamma_L \geq c\tau\kappa.$$

In fact, $c = 1$ if $a \equiv 1$ and the norm on X is taken as

$$\|w\|_X^2 = \|\partial_t w\|^2_{L^2(\mathrm{J};V')} + \|w\|^2_{L^2(\mathrm{J};V)} + \|w(\mathrm{T})\|^2_H.$$

On the heat equation: Conditional space-time stability

Take $(L \in \mathbb{N}_0)$

- $E_L \subset H^1(\mathrm{J})$ pw. polynomial on any temporal mesh $\mathcal{T}_L$,
- $V_L \subset V$ any nontrivial finite-dimensional subspace,
- $X_L := E_L \otimes V_L$ and $Y_L := V_L \times [\partial_t E_L \otimes V_L]$ ($\equiv$ cG method).

Theorem

There exists $c > 0$ such that

$$\gamma_L \geq c\kappa \min\{1, \mathrm{CFL}_L^{-1}\}$$

where

$$\mathrm{CFL}_L := \max \Delta\mathcal{T} \times \sup_{\chi \in V_L \setminus \{0\}} \frac{\|\chi\|_V}{\|\chi\|_{V'}} \qquad \sim \Delta t \times \frac{1}{\Delta x^2}.$$

The dependence on CFL_L cannot be improved, in general.

On the heat equation: Example 1

Set-up

- equidistant temporal mesh $\mathcal{T}_L = \{k2^{-L} : k = 0, \ldots, 2^{L+1}\}$
- L-shaped domain $D \subset \mathbb{R}^2$
- $V_0 \subset H_0^1(D)$ P1 FEM on a simplicial triangulation (pdetool)
- X_L continuous piecewise affine on $\mathcal{T}_L$ with values in V_0
- $Y_L := V_0 \times \partial_t X_L$ piecewise constant on $\mathcal{T}_L$
- source $f(t,x) := \sin(t)$, initial condition $g(x) := 0$

Corollary.

- The family X_L, Y_{L+1} is space-time stable.
- The family X_L, Y_L (C-N) is conditionally space-time stable.

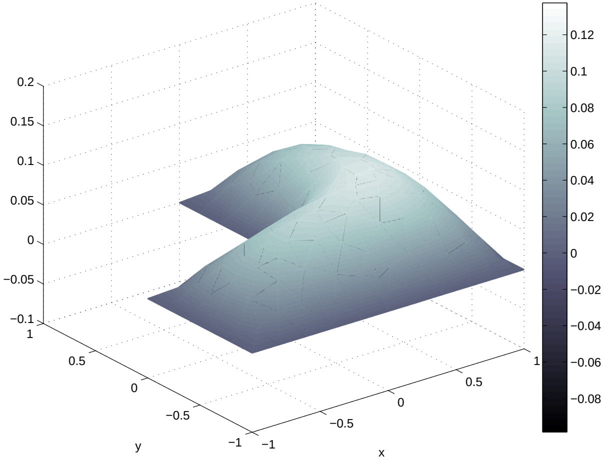


Figure: Solution to the heat equation on the L-shaped domain (snapshot, almost uniform triangular mesh, $32'705$ spatial dof's).

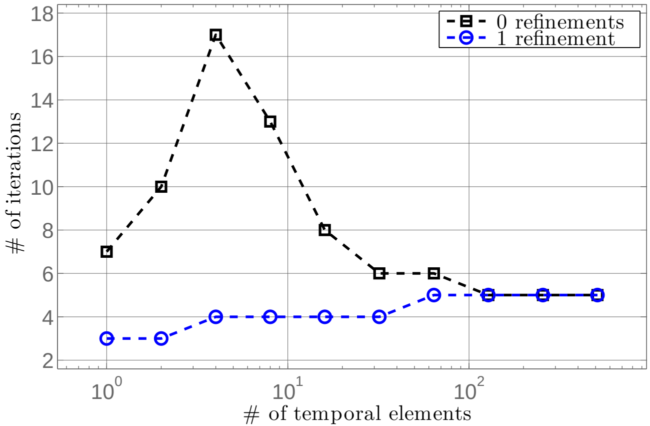


Figure: Number of iterations for the “operator preconditioned LSQR” for Y_L vs. Y_{L+1} as test space (GNE tol. 10^{-4}). The effect of γ_L is seen.

On the heat equation: Example 2

Consider the semi-linear parabolic PDE

$$\partial_t u(t,x) - \partial_{xx} u(t,x) + 10\, u(t,x)^3 = f(t,x), \quad (t,x) \in \mathrm{J} \times D,$$

in $\mathrm{J} \times D = (0,2) \times (-1,1)$, with zero I.C. and zero Dirichlet B.C.

The problem is of the form

$$Bu + G(u) = F$$

which we solve using the fixed point iteration

$$u_L^i := [w \mapsto T_L B^{-1}(F - G(w))]^i(0), \quad i = 0, 1, \ldots .$$

$H_0^1(D) \hookrightarrow L^4(D)$

On the heat equation: Example 2

Consider the semi-linear parabolic PDE

$$\partial_t u(t,x) - \partial_{xx} u(t,x) + 10\, u(t,x)^3 = f(t,x), \quad (t,x) \in \mathrm{J} \times D,$$

We define

- $E_L \subset H^1(\mathrm{J})$ pw. affine on uniform mesh with $\Delta t = 2^{-L}$,
- $V_L \subset H_0^1(D)$ pw. affine on uniform mesh with $\Delta x = 2^{-L}$,

and the space-time **full tensor product** trial & test spaces

$$X_L := E_L \otimes V_L, \quad Y_L := V_L \times [E_{L+1} \otimes V_L].$$

On the heat equation: Example 2

Consider the semi-linear parabolic PDE

$$\partial_t u(t,x) - \partial_{xx} u(t,x) + 10\, u(t,x)^3 = f(t,x), \quad (t,x) \in \mathrm{J} \times D,$$

We define

- $E_L \subset H^1(\mathrm{J})$ pw. affine on uniform mesh with $\Delta t = 2^{-L}$,
- $V_L \subset H_0^1(D)$ pw. affine on uniform mesh with $\Delta x = 2^{-L}$,

and the space-time **sparse tensor product** trial & test spaces

$$X_L := \sum_{0 \le k+\ell \le L} E_k \otimes V_\ell, \quad Y_L := V_L \times \sum_{0 \le k+\ell \le L} [E_{k+1} \otimes V_\ell].$$

On the heat equation: Example 2

Summary of parameters:

- level of discretization: $L = 0, 1, \ldots, 7$
- number of fixed point iterations: $i = 0, 1, \ldots, 8$
- full tensor product (**FTP**) solution: u_L^i
- sparse tensor product (**STP**) solution: $\widehat{u}_L^i$
- reference solution: FTP with $L = 8$ and $i = 10$

The PDE data:

- $\partial_t u(t,x) - \partial_{xx} u(t,x) + 10\, u(t,x)^3 = f(t,x)$
- $f(t,x) = \sin(\pi t/2)^2 \cos(\cos(\pi t/2) + x)$
- is posed on $\mathrm{J} \times D = (0,2) \times (-1,1)$
- zero initial value
- homogeneous Dirichlet boundary conditions

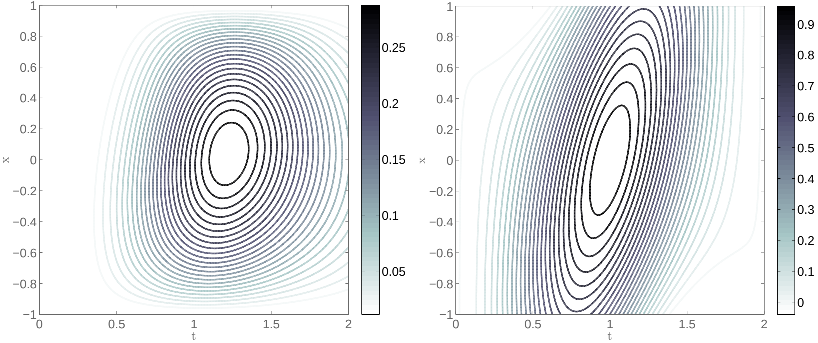


Figure: The solution u (**left**) and the source f (**right**)

Note: $\|10u^3\|_{L^\infty(\mathrm{J}\times D)} \approx \frac{1}{6}\|f\|_{L^\infty(\mathrm{J}\times D)}$.

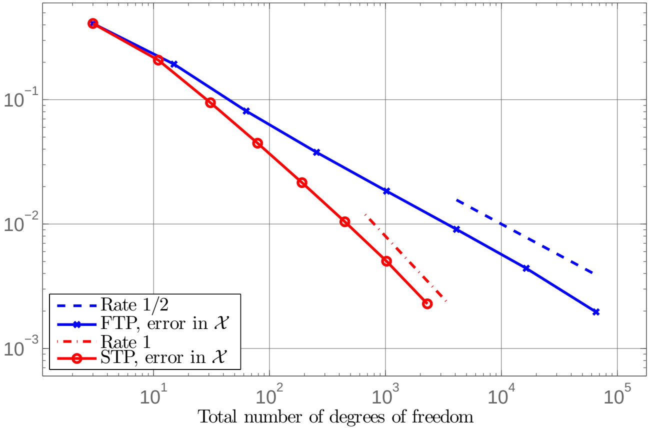


Figure: Error of the FTP u_L^i and the STP $\widehat{u}_L^i$ solutions for fixed $i = 10$ as a function of the total number of degrees of freedom, $L = 0, \ldots, 7$

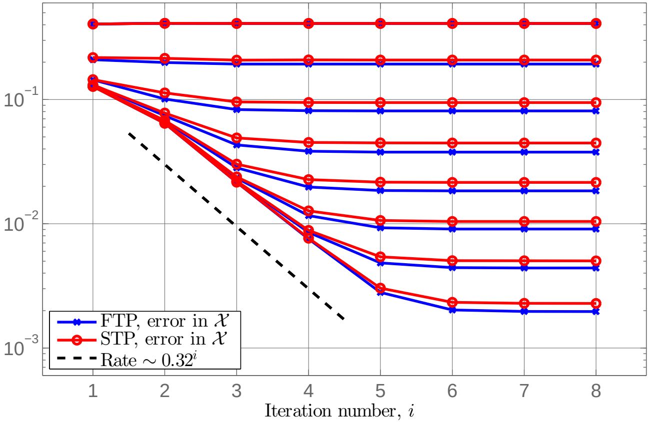


Figure: Error of the FTP u_L^i and the STP $\widehat{u}_L^i$ solutions for $L = 0, \ldots, 7$ as a function of the iteration number i

Parabolic multilevel preconditioners

Operator preconditioning: With bases for X_L and Y_L, the sol'n u_L is approximated by iterating on the linear algebraic system

$$\mathbf{M}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{N}^{-1}\mathbf{B}\mathbf{u} = \mathbf{M}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{N}^{-1}\mathbf{F},$$

where the matrices $\mathbf{N}$ and $\mathbf{M}$ measure the Y and the X norms, possibly only approximately.

Parabolic multilevel preconditioners: variant A

Proposition

An s.p.d. isomorphism $M : X \to X'$ *on* $X = L^2(\mathrm{J}; V) \cap H^1(\mathrm{J}; V')$ *is defined by*

$$\langle Mw, w\rangle := \sum_{k,\ell\in\mathbb{N}} \{2^{0k} 2^{2\ell} + 2^{2k} 2^{-2\ell}\} \left\|(P_k^\Delta \otimes Q_\ell^\Delta) w\right\|^2_{L^2(\mathrm{J};H)}$$

where P_k^Δ *and* Q_ℓ^Δ *are suitable projections on* $L^2(\mathrm{J})$ *and* H.

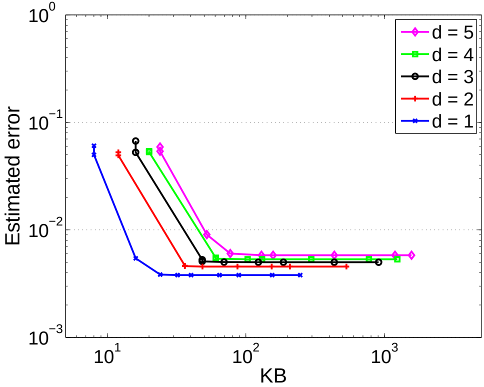
Then, $\mathbf{M}^{-1}$ is obtained from M^{-1}.

An s.p.d. isomorphism

$$N : Y \to Y' \quad \text{on} \quad H \times Y = L^2(\mathrm{J}; V)$$

can be defined analogously.

Parabolic multilevel preconditioners: variant A



R.A. and C. Tobler, Multilevel preconditioning and low rank tensor iteration for space-time simultaneous discretizations of parabolic PDEs, (**2012**)

Figure: Space-time MinRes PG coupled with the `htucker` toolbox: error of the solution computed in the **hierarchical Tucker** low rank format for a space-time problem of full size of up to 127PB

Parabolic multilevel preconditioners: variant B

The matrix

$$\mathbf{M} = \mathbf{M}_t \otimes \mathbf{A}_x + \mathbf{A}_t \otimes (\mathbf{M}_x \mathbf{A}_x^{-1} \mathbf{M}_x)$$

measures the X norm (via $\mathbf{w}^\mathsf{T}\mathbf{M}\mathbf{w}$), where $\mathbf{M}_{t,x}$ is the mass and $\mathbf{A}_{t,x}$ is the stiffness matrix, subscript indicating *time* or *space*.

Diagonalize $\mathbf{M}_t$ and $\mathbf{A}_t$ simultaneously by taking $\mathbf{V}_t$ such that

$$\mathbf{V}_t^\mathsf{T} \mathbf{M}_t \mathbf{V}_t = \mathbf{I}_t \quad \text{and} \quad \mathbf{V}_t^\mathsf{T} \mathbf{A}_t \mathbf{V}_t = \mathbf{D}_t$$

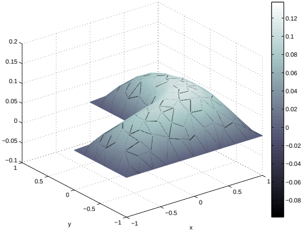
are diagonal. Then

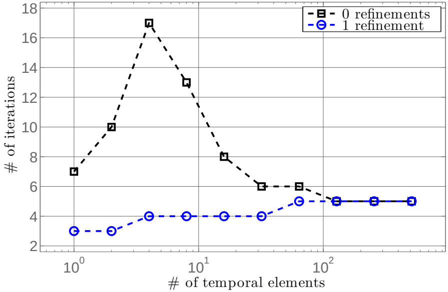
$$\mathbf{M}^{-1} = (\mathbf{V}_t \otimes \mathbf{I}_x)(\mathbf{I}_t \otimes \mathbf{A}_x + \mathbf{D}_t \otimes (\mathbf{M}_x \mathbf{A}_x^{-1} \mathbf{M}_x))^{-1}(\mathbf{V}_t^\mathsf{T} \otimes \mathbf{I}_x).$$

Need to solve (many, approx., in parallel) problems of the form

$$(\mathbf{A}_x + k^2 \mathbf{M}_x \mathbf{A}_x^{-1} \mathbf{M}_x)\mathbf{w} = \ldots$$

Parabolic multilevel preconditioners: variant B





Summary

- The "natural" X_L and Y_L are not space-time stable.
- Abstract stable MinRes discrete variational formulation.
- Space-time (sparse) discretization of the heat equation.
- Novel parabolic multilevel preconditioners.

References

- arXiv, **2012**: Space-time discretization of the heat equation. A concise Matlab implementation (and further references).

Thank you!

andreevr@math.ethz.ch

Summary

- The "natural" X_L and Y_L are not space-time stable.
- Abstract stable MinRes discrete variational formulation.
- Space-time (sparse) discretization of the heat equation.
- Novel parabolic multilevel preconditioners.

References

- arXiv, **2012**: Space-time discretization of the heat equation. A concise Matlab implementation (and further references).

Thank you!

andreevr@math.ethz.ch

this slide is intentionally left blank