Coarse Grid Correction for the Neumann–Neumann Waveform Relaxation Method

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Innovative Space-Time Parallel Methods
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Collaborators

- Martin J. Gander (Genève)
- Sébastien Loisel (Heriot-Watt)
- Kévin Santugini (Bordeaux)

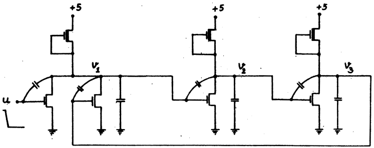
Model problem

Problem : Given spatial domain $\Omega \subset \mathbb{R}^d$, solve in parallel

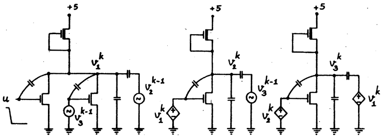
$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f && \text{on } \Omega \times (0, T], \\ u(0,t) &= g && \text{on } \partial\Omega \times (0, T], \\ u(x,0) &= u_0(x) && \text{on } \Omega. \end{aligned}$$

- Discretization in time + domain decomposition in space
- Parallelization in time (Parareal, PFASST, RDIC)
- Waveform relaxation (WR)

Waveform relaxation method



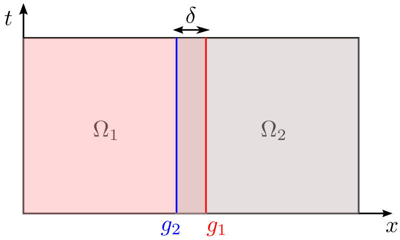
Full circuit



Decoupled circuit

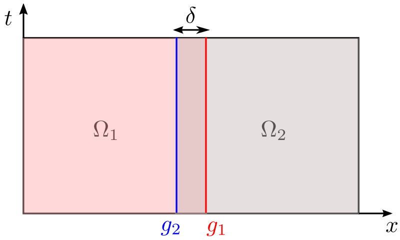
- Originally invented for **systems of ODEs** (Picard 1890, Lindelöf 1894)
- Revived for circuit simulation by Lelarasmee, Ruehli & Sangiovanni-Vincentelli (1982)
- Applied to parabolic PDEs by Gander & Zhao (1997, 2002), Gander & Stuart (1998), Giladi & Keller (2002)

Example : Classical Schwarz WR



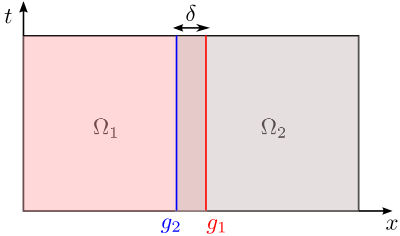
- Two subdomains with overlap δ, Dirichlet transmission conditions

Example : Classical Schwarz WR



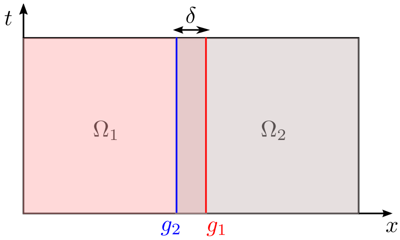
- For $k = 1, 2, 3, \ldots$:
  1. Solve space-time subdomain problems
  2. Exchange interface data

Example : Classical Schwarz WR



- Advantages :
  - Flexible gridding and time-stepping within subdomains
  - Fewer synchronization points
  - Natural incoporation of parallelization in time

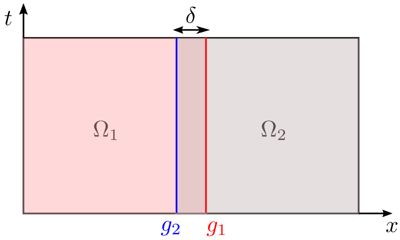
Example : Classical Schwarz WR



- For unbounded time interval (Gander & Stuart 1998)

$$\frac{\|v^k - v^*\|}{\|v^0 - v^*\|} \leq (1 - C\delta)^k \sim e^{-Ck\delta}$$

Example : Classical Schwarz WR



- For bounded time interval T (Giladi & Keller 2002) :

$$\frac{\|v^k - v^*\|}{\|v^0 - v^*\|} \leq C \operatorname{erfc}\left(\frac{k\delta}{\sqrt{T}}\right) \sim \frac{\sqrt{T}}{\sqrt{\pi}k\delta} e^{-(k\delta)^2/T}.$$

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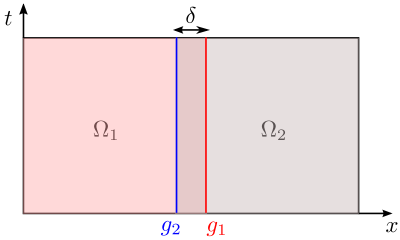
Example : Classical Schwarz WR



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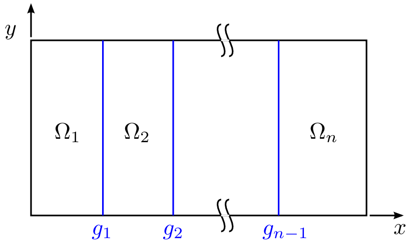
- Convergence depends on $k\delta \approx kh$
- Half the mesh size $\implies$ double the number of iterations !

Neumann–Neumann (NN) Method

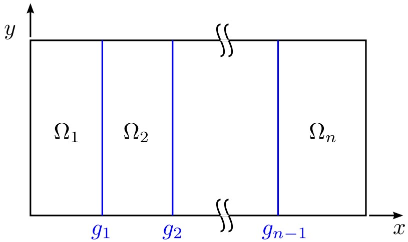


Originally developed for the **elliptic problem**

$$\Delta u = f$$

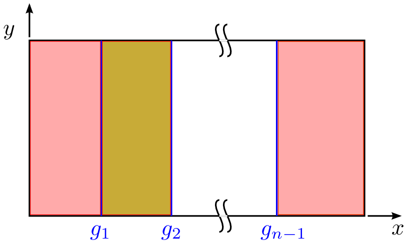
(Bourgat et al. 1989, DeRoeck & Le Tallec 1990, Le Tallec et al. 1991)

Neumann–Neumann (NN) Method



Given Dirichlet traces $g_1^0, \ldots, g_{n-1}^0$:

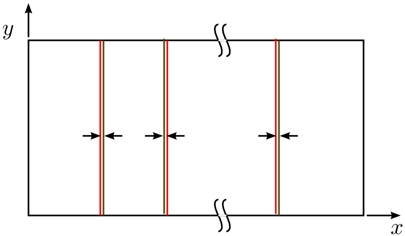
Neumann–Neumann (NN) Method



Given Dirichlet traces $g_1^0, \ldots, g_{n-1}^0$:

1. Solve Dirichlet problems on $\Omega_1, \ldots, \Omega_n$

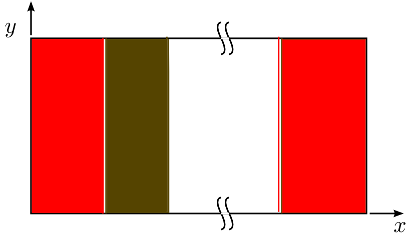
Neumann–Neumann (NN) Method



Given Dirichlet traces $g_1^0, \ldots, g_{n-1}^0$:

1. Solve Dirichlet problems on $\Omega_1, \ldots, \Omega_n$
2. Calculate jumps in Neumann traces

Neumann–Neumann (NN) Method



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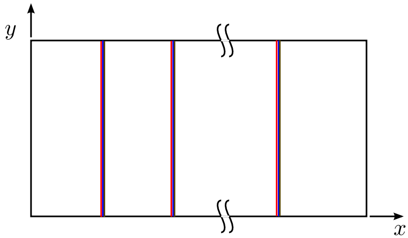
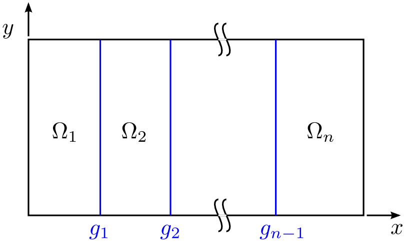


Given Dirichlet traces $g_1^0, \dots, g_{n-1}^0$:

1. Solve Dirichlet problems on $\Omega_1, \dots, \Omega_n$
2. Calculate jumps in Neumann traces
3. Solve Neumann problems on $\Omega_1, \dots, \Omega_n$
4. Update Dirichlet traces :

$$g_i^{k+1} = g_i^k - \theta(\psi_i^k|_{\Gamma_i} + \psi_{i+1}^k|_{\Gamma_i}).$$

Convergence



- Conditioning for steady-state, coercive problems :
  - Without coarse grid (DeRoeck & Le Tallec 1990) :

$$\kappa \leq \frac{C}{H^2}(1 + \log(H/h))^2 \implies \text{Error} \approx e^{-Ck/H(1+\log(H/h))}$$

  - With coarse grid (Dryja & Widlund 1995) :

$$\kappa \leq C(1 + \log(H/h))^2 \implies \text{Error} \approx e^{-Ck(1+\log(H/h))}$$

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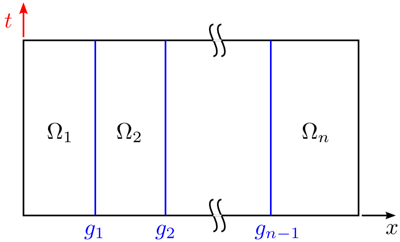
Neumann–Neumann Waveform Relaxation



- Replace second coordinate by time !
- Convergence ?

Convergence of NNWR

Theorem (NNWR convergence, no coarse grid)

Let $g_j^0(t)$ be the initial error along the jth interface. If $\theta = 1/4$, then NNWR with initial error g_j^0 converges superlinearly, with

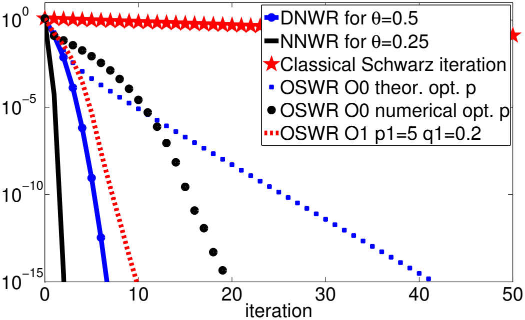
$$\max_j \|g_j^k\|_\infty \leq \left(\frac{\sqrt{6}}{1 - e^{-(2k+1)H^2/T}}\right)^{2k} e^{-(kH)^2/T} \max_j \|g_j^0\|_\infty,$$

where $H =$ minimum subdomain width.

- Proof uses Laplace transforms
- Compare with $e^{-(kh)^2/T}$ for classical Schwarz WR

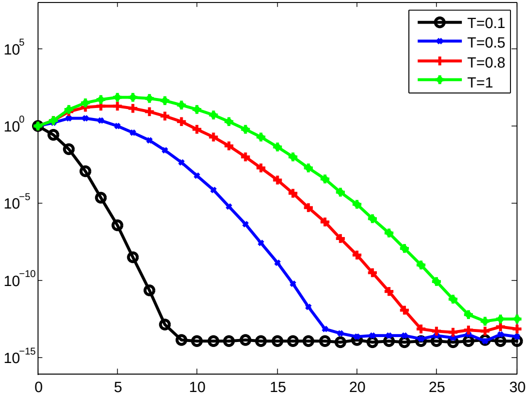
1D example

- Two subdomains, $|\Omega_1| = 3$, $|\Omega_2| = 2$



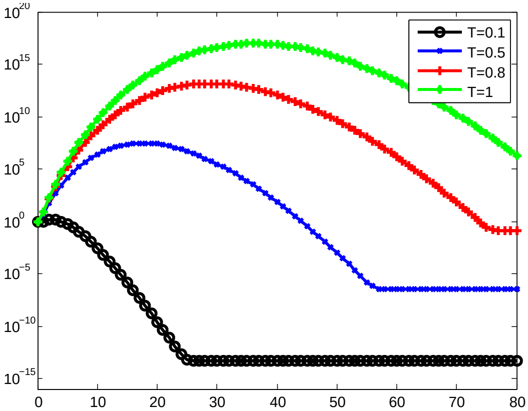
1D example

- Four equal subdomains



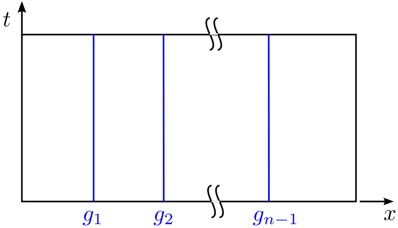
1D example

- Eight equal subdomains

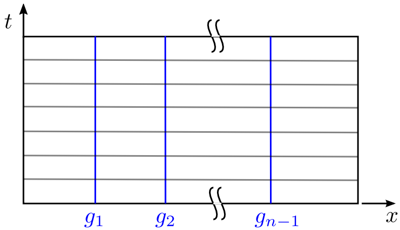


- Algorithm does not scale, needs coarse grid to enable communication between far-away subdomains

Coarse grid for 1D



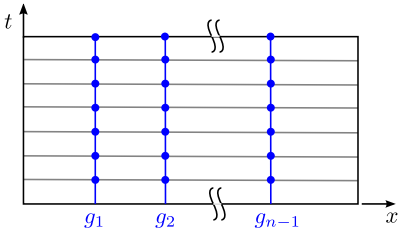
Coarse grid for 1D



- Consider *time discretized* problem, e.g., Backward Euler :

$$\frac{u(x,t_m) - u(x,t_{m-1})}{\Delta t} = \Delta u(x,t_m) + f(x,t_m)$$

Coarse grid for 1D

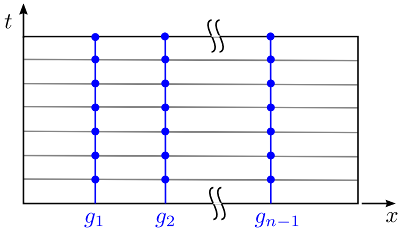


- Consider *time discretized* problem, e.g., Backward Euler :

$$\frac{u(x,t_m)-u(x,t_{m-1})}{\Delta t} = \Delta u(x,t_m) + f(x,t_m)$$

- Solve Dirichlet problems with $g_j^0(t_m)$, $m = 1, \ldots, T/\Delta t$ to get $u^0(x,t_m)$ (with jumps in normal derivative)

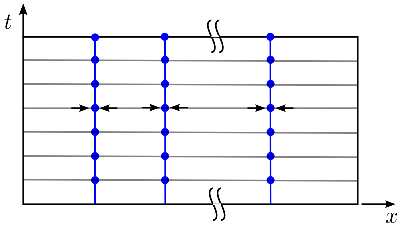
Coarse grid for 1D



- Goal : find interface values of $e^0(x, t_m) = u(x, t_m) - u^0(x, t_m)$
- e^0 satisfies the homogeneous PDE on each subdomain
- e^0 must cancel Neumann jumps introduced by $u^0(x, t_m)$

$$\partial_x e^0(\Gamma_j^-, t_m) - \partial_x e^0(\Gamma_j^+, t_m) = -(\partial_x u^0(\Gamma_j^-, t_m) - \partial_x u^0(\Gamma_j^+, t_m))$$

Coarse grid for 1D



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$$\partial_x e^0(\Gamma_j^-, t_m) - \partial_x e^0(\Gamma_j^+, t_m) = \mathbf{r}$$

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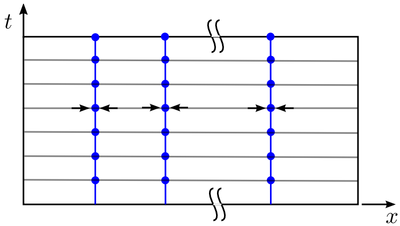
Coarse grid for 1D



- Goal : find interface values of $e^0(x, t_m) = u(x, t_m) - u^0(x, t_m)$
- e^0 satisfies the homogeneous PDE on each subdomain
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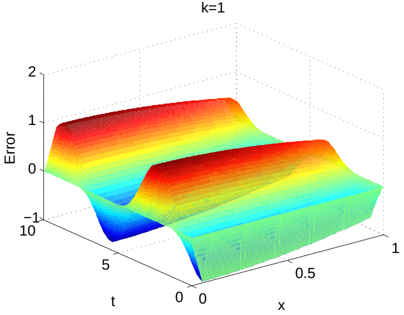
$$L\mathbf{e} = \mathbf{r}$$

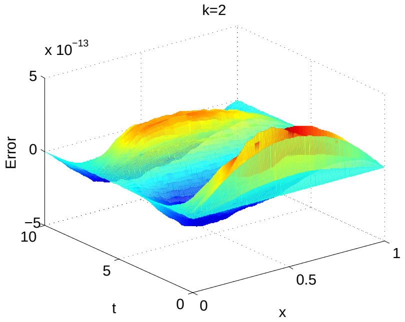
where $\mathbf{e}$ is the interface values of e^0.

1D Example with coarse grid, $N = 16$



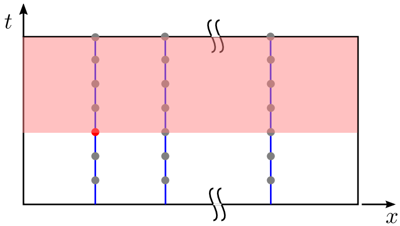
- $u_{ex}(x,t) = \cos(x)\cos(t)$, $g_i^0(t) = 0$

1D Example with coarse grid, $N = 16$



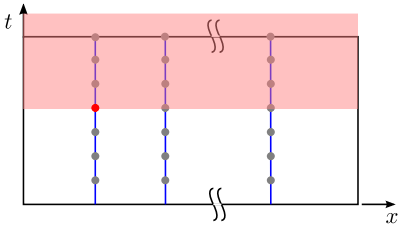
- $u_{ex}(x,t) = \cos(x)\cos(t)$, $g_i^0(t) = 0$

What does L look like ?



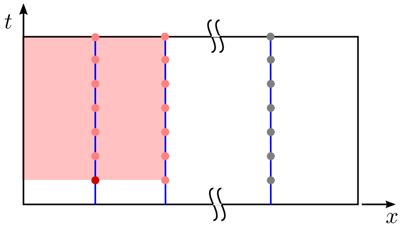
- *Causality :* L is block lower triangular

What does L look like ?



- *Causality :* L is block lower triangular
- *Time invariance :* L has constant block diagonals

What does L look like ?



- *Causality :* L is block lower triangular
- *Time invariance :* L has constant block diagonals
- *Locality :* The blocks of L are sparse

What does L look like ?

$$\begin{bmatrix} A & & & & \\ B_1 & A & & & \\ B_2 & B_1 & A & & \\ \vdots & \ddots & \ddots & \ddots & \\ B_{M-1} & \cdots & B_2 & B_1 & A \end{bmatrix} \begin{pmatrix} \mathbf{e}(t_1) \\ \mathbf{e}(t_2) \\ \mathbf{e}(t_3) \\ \vdots \\ \mathbf{e}(t_M) \end{pmatrix} = \begin{pmatrix} \mathbf{r}(t_1) \\ \mathbf{r}(t_2) \\ \mathbf{r}(t_3) \\ \vdots \\ \mathbf{r}(t_M) \end{pmatrix}.$$

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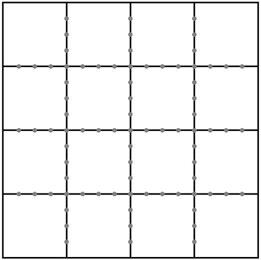
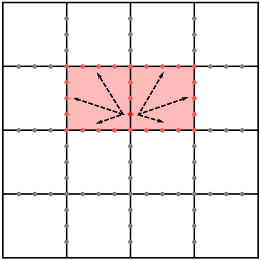
- *Causality :* L is block lower triangular
- *Time invariance :* L has constant block diagonals
- *Locality :* The blocks of L are sparse
- Calculate correction by forward substitution
- Blocks can be computed in parallel and assembled

2D Problem



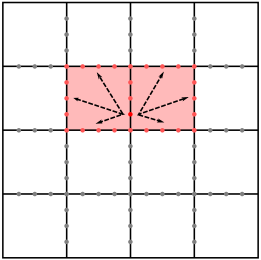
- Coarse grid contains all interface points

2D Problem



- Coarse grid contains all interface points
- Problem : dense coupling between points $\implies$ dense A!

2D Problem



- Coarse grid contains all interface points
- Problem : dense coupling between points $\Longrightarrow$ dense A !
- Must choose a coarse space with fewer points

2D Problem

- Precondition

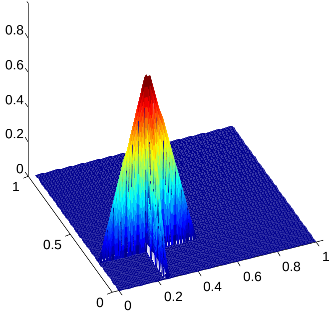
$$\begin{bmatrix} A \\ B_1 & A \\ B_2 & B_1 & A \\ \vdots & \ddots & \ddots & \ddots \\ B_{M-1} & \cdots & B_2 & B_1 & A \end{bmatrix} \quad \text{by} \quad \begin{bmatrix} M \\ B_1 & M \\ B_2 & B_1 & M \\ \vdots & \ddots & \ddots & \ddots \\ B_{M-1} & \cdots & B_2 & B_1 & M \end{bmatrix}$$

  where M is the Neumann–Neumann preconditioner for $-\Delta \mathbf{u}(t_{k+1}) + \frac{1}{\Delta t}\mathbf{u}(t_{k+1}) = \frac{1}{\Delta t}\mathbf{u}(t_k)$ with coarse grid.

- Expect same asymptotic convergence rate as for stationary problem

$$(\eta - \Delta)u = 0 \qquad \text{with } \eta = 1/\Delta t.$$

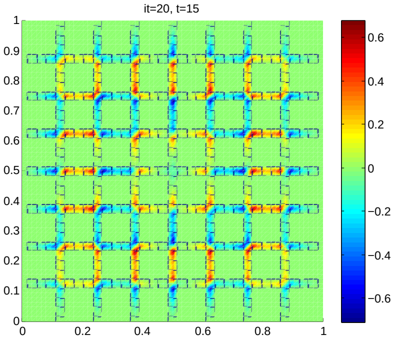
A standard coarse space

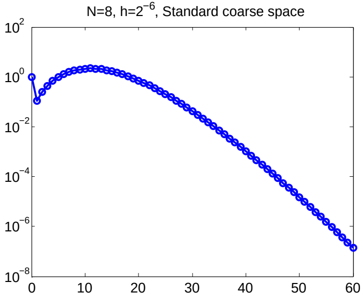


- Hat functions centered at cross points
- Precondition A by projection :

$$P = V(V^T A V)^{-1} V^T A$$

Standard coarse space, 8 × 8 subdomains





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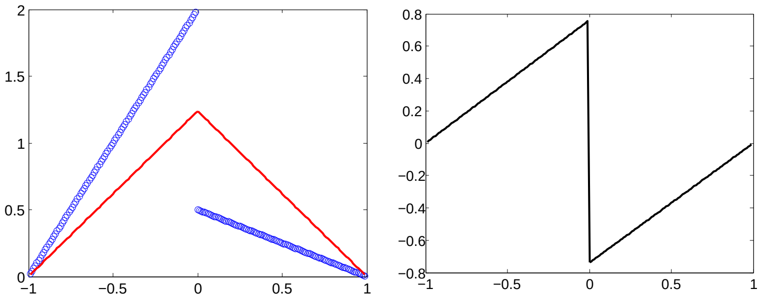
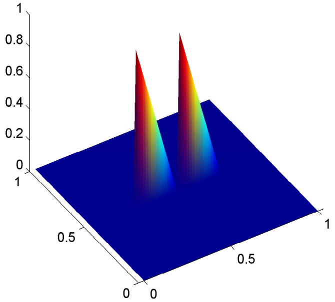


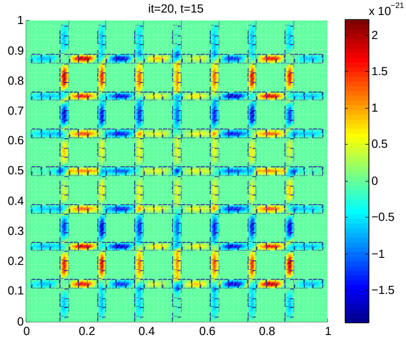
- Hat functions are poor approximations of discontinuous functions !

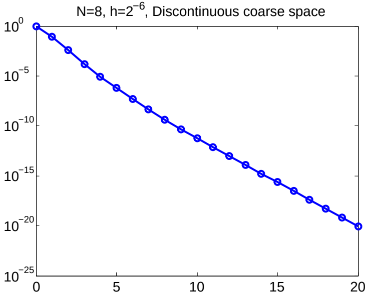
A discontinuous coarse space



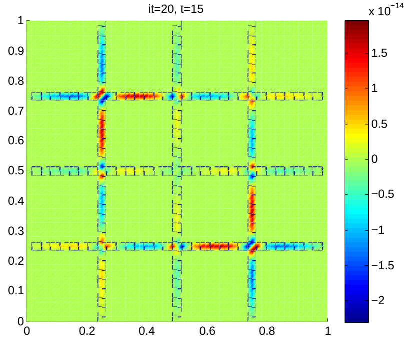
- Piecewise linear functions along edges
- Discontinuous across cross points
- Also used for elliptic problems (Gander, Halpern & Santugini 2012)

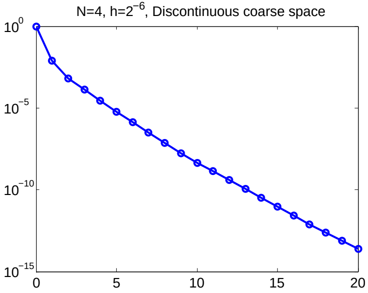
Discontinuous coarse space, 8 × 8 subdomains





Discontinuous coarse space, 4 $\times$ 4 subdomains





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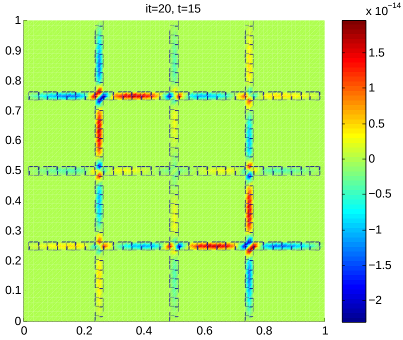

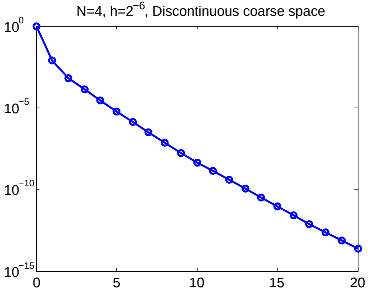

- Slower convergence since

$$\kappa \approx C(1 + \log(H/h))^2$$

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Conclusion

- NNWR with no coarse grid
  - converges superlinearly, BUT...
  - error grows initially for many subdomains.
- Optimal coarse grid in 1D
  - cheap,
  - convergence in 2 iterations.
- Coarse grid in 2D
  - standard coarse grid not enough,
  - use discontinuous, piecewise linear coarse space.
- Ongoing work
  - Coarse space for 3D ?
  - Superlinear convergence for coarse grids ?