Karlsruhe Institute of Technology

Space-Time Methods for Wave Equations

Discretizations and Convergence Analysis

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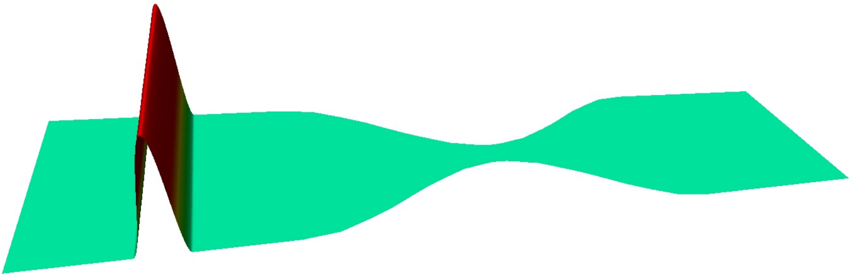
Objectives

We aim for

- efficient implicit high order adaptive discretizations in space and time;
- reliable error estimation with criteria for refinement in space and/or time;
- optimal solution methods for the linear systems;
- parallel strategies for the potential use of supercomputers;
- applications to inverse problems, optimal control, or model reduction.

Here, we discuss a suitable finite element basis for these tasks.



electro-magnetic wave propagation (2D)

Linear Hyperbolic Problems

We consider a wave in the bounded domain $\Omega \subset \mathbb{R}^D$ and for the time interval $[0, T]$.

Acoustic waves

Find a potential $\psi = \psi(t, \mathbf{x})$ such that

$$\rho \partial_t^2 \psi - \Delta \psi = g$$

in $\Omega \times (0, T)$ subject to initial and boundary conditions.

Elastic waves

Find a displacement vector $\boldsymbol{\varphi}$ such that

$$\rho \partial_t^2 \boldsymbol{\varphi} - \operatorname{div} \mathbb{C} \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) = \mathbf{g}\,.$$

Electro-magnetic waves

Find an electric field $\mathbf{E}$ such that

$$\varepsilon \partial_t^2 \mathbf{E} + \operatorname{curl}\left(\mu^{-1} \operatorname{curl} \mathbf{E}\right) = \mathbf{g}\,.$$

Linear Hyperbolic Systems

We consider a wave in the bounded domain $\Omega \subset \mathbb{R}^D$ and for the time interval $[0, T]$.

Acoustic waves

Find p and $\mathbf{q}$ such that

$$\begin{aligned} \partial_t \mathbf{q} + \nabla p &= 0\,, \\ \rho \partial_t p + \operatorname{div} \mathbf{q} &= g \end{aligned}$$

in $\Omega \times (0, T)$ subject to initial and boundary conditions.

Elastic waves

Find a velocity vector $\mathbf{v}$ and a stress tensor $\boldsymbol{\sigma}$ such that

$$\begin{aligned} \partial_t \boldsymbol{\sigma} - \mathbb{C}\boldsymbol{\varepsilon}(\mathbf{v}) &= 0\,, \\ \rho \partial_t \mathbf{v} - \operatorname{div} \boldsymbol{\sigma} &= \mathbf{g}\,. \end{aligned}$$

Electro-magnetic waves

Find an electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ such that

$$\begin{aligned} \varepsilon \partial_t \mathbf{E} - \operatorname{curl} \mathbf{H} &= 0\,, \\ \mu \partial_t \mathbf{H} + \operatorname{curl} \mathbf{E} &= \mathbf{h}\,. \end{aligned}$$

Linear Hyperbolic Operators

We consider a Hilbert space $V \subset \mathrm{L}_2(\Omega)^J$ with inner product $(\cdot,\cdot)_V = (M\cdot,\cdot)_\Omega$ and an operator A defined on $D(A) \subset V$ such that the wave equation takes the form

$$M\partial_t \mathbf{u} + A\mathbf{u} = \mathbf{f} \qquad \text{in } [0,T]\,.$$

Acoustic waves

$V = \mathrm{L}_2(\Omega)^D \times \mathrm{L}_2(\Omega)$, $\quad M(\mathbf{q},p) = (\mathbf{q},\rho p)$,
$A(\mathbf{q},p) = (\nabla p, \operatorname{div}\mathbf{q})$, $\quad D(A) = \mathrm{H}(\operatorname{div},\Omega) \times \mathrm{H}^1_0(\Omega)$

Elastic waves

$V = \mathrm{L}_2(\Omega)^{D\times D}_{\mathrm{sym}} \times \mathrm{L}_2(\Omega)^D$, $\quad M(\boldsymbol{\sigma},\mathbf{v}) = (\mathbb{C}^{-1}\boldsymbol{\sigma},\rho\mathbf{v})$,
$A(\boldsymbol{\sigma},\mathbf{v}) = -(\boldsymbol{\varepsilon}(\mathbf{v}), \operatorname{div}\boldsymbol{\sigma})$, $\quad D(A) = \mathrm{H}(\operatorname{div},\Omega)^D \times \mathrm{H}^1_0(\Omega)^D$

Electro-magnetic waves

$V = \mathrm{L}_2(\Omega)^D \times \mathrm{L}_2(\Omega)^D$, $\quad M(\mathbf{E},\mathbf{H}) = (\varepsilon\mathbf{E},\mu\mathbf{H})$, $\quad A(\mathbf{E},\mathbf{H}) = (-\operatorname{curl}\mathbf{H}, \operatorname{curl}\mathbf{E})$,
$D(A) = \{(\mathbf{E},\mathbf{H}) \in \mathrm{H}_0(\operatorname{curl},\Omega) \times \mathrm{H}(\operatorname{curl},\Omega)\colon\ \operatorname{div}(\varepsilon\mathbf{E}) = 0,\ \operatorname{div}(\mu\mathbf{H}) = 0\}$

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Electro-magnetic waves

$V = \mathrm{L}_2(\Omega)^D \times \mathrm{L}_2(\Omega)^D$, $\quad M(\mathbf{E},\mathbf{H}) = (\varepsilon\mathbf{E}, \mu\mathbf{H})$, $\quad A(\mathbf{E},\mathbf{H}) = (-\operatorname{curl}\mathbf{H}, \operatorname{curl}\mathbf{E})$,
$D(A) = \{(\mathbf{E},\mathbf{H}) \in \mathrm{H}_0(\operatorname{curl},\Omega) \times \mathrm{H}(\operatorname{curl},\Omega) \colon \operatorname{div}(\varepsilon\mathbf{E}) = 0,\ \operatorname{div}(\mu\mathbf{H}) = 0\}$

We have $(A\mathbf{u},\mathbf{v})_\Omega = -(\mathbf{u},A\mathbf{v})_\Omega$ for $\mathbf{u},\mathbf{v} \in D(A)$, with implies conservation of the energy $\mathcal{E}(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|_V^2$, i.e., the solution satisfies $\partial_t\mathcal{E}(\mathbf{u}(t)) = 0$.

The Semigroup Setting

Let $V \subset \mathrm{L}_2(\Omega)^J$ be a Hilbert space with inner product $(\cdot,\cdot)_V = (M\cdot,\cdot)_\Omega$.

Let A be a linear operator in V with domain $D(A) \subset V$.

Theorem

Assume

- $D(A)$ *is dense in* V*;*
- $\omega \geq 0$ *exists with* $(A\mathbf{v},\mathbf{v})_\Omega \leq \omega\,(M\mathbf{v},\mathbf{v})_\Omega$ *for all* $\mathbf{v} \in D(A)$*;*
- $\lambda_0 > \omega$ *exists such that* $A - \lambda_0 M$ *is onto.*

Then, $M^{-1}A$ *generates a semigroup with* $\|\exp(tM^{-1}A)\| \leq \exp(\omega t)$.

Renardy-Rogers: An Introduction to Partial Differential Equations
Evans: Partial Differential Equations

Linear Evolution Equations

In the applications $(M + A)^{-1}$ exists and is bounded, and $M + A$ maps onto V.

Then, the operator $-M^{-1}A$ generates a semigroup in V.
The linear evolution equation $M\partial_t\mathbf{u} + A\mathbf{u} = 0$ is solved by

$$\mathbf{u}(t) = \exp(-tM^{-1}A)\mathbf{u}(0)\,.$$

Acoustic waves $(M + A)(\mathbf{q}, p) = (\mathbf{f}, g)$ implies

$$-\Delta p + \rho p = -\operatorname{div}\mathbf{f} + \rho g$$

and $\mathbf{q} = \mathbf{f} - \nabla p$.

Elastic waves $(M + A)(\boldsymbol{\sigma}, \mathbf{v}) = (\mathbf{f}, \mathbf{g})$ implies

$$-\operatorname{div}\mathbb{C}\boldsymbol{\varepsilon}(\mathbf{v}) + \rho\mathbf{v} = \operatorname{div}\mathbb{C}\mathbf{f} + \rho\mathbf{g}$$

and $\sigma = \mathbf{f} - \mathbb{C}\boldsymbol{\varepsilon}(\mathbf{v})$.

Electro-magnetic waves $(M + A)(\mathbf{E}, \mathbf{H}) = (\mathbf{f}, \mathbf{g})$ implies

$$\operatorname{curl}\mu^{-1}\operatorname{curl}\mathbf{E} + \varepsilon\mathbf{E} = -\operatorname{curl}\mu^{-1}\mathbf{g} + \mathbf{f}$$

and $\mathbf{H} = \mu^{-1}(\mathbf{g} - \operatorname{curl}\mathbf{E})$.

The Babuška-Nečas Setting

Theorem

Let U, H be Hilbert spaces, and let $b\colon U \times H \longrightarrow \mathbb{R}$ be a bilinear form. Assume

- $C > 0$ *exists such that* $|b(\mathbf{u},\mathbf{v})| \le C\,\|\mathbf{u}\|_U\|\mathbf{v}\|_H$*;*
- $\alpha \ge 0$ *exists such that* $\sup_{\mathbf{v}\in H} \dfrac{b(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_H} \ge \alpha\,\|u\|_U$ *for all* $\mathbf{u} \in U$*;*
- *for all* $\mathbf{v} \in V$, $\mathbf{v} \ne 0$, *exists some* $\mathbf{u} \in U$ *such that* $b(\mathbf{u},\mathbf{v}) \ne 0$.

Then, for all $\mathbf{f} \in H$ *a unique solution* $\mathbf{u} \in U$ *of* $b(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_H$ *for* $\mathbf{v} \in H$ *exists.*

Let $U_h \subset U$ *and* $H_h \subset H$ *be discrete subspaces. Assume for* $\alpha_0 \ge 0$

$$\sup_{\mathbf{v}_h\in H_h} \frac{b(\mathbf{u}_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_H} \ge \alpha_0\,\|u_h\|_U\,, \qquad \mathbf{u}_h \in U_h\,.$$

Then, for all $\mathbf{f} \in H$ *a unique discrete solution* $\mathbf{u}_h \in U_h$ *exists solving*

$$b(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h)_H\,, \qquad \mathbf{v}_h \in H_h$$

satisfying $\|\mathbf{u} - \mathbf{u}_h\|_U = \dfrac{C}{\alpha_0} \inf_{\mathbf{w}_h\in U_h} \|\mathbf{u} - \mathbf{w}_h\|_U$.

A Space-Time Setting

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We consider $L = M\partial_t + A$ on the space-time cylinder $Q = \Omega \times (0, T)$ with domain $U = D(L)$, where U is the closure of $\{\mathbf{u} \in \mathrm{C}^1(0, T; D(A)) \colon \mathbf{u}(0) = 0\}$ with respect to the weighted graph norm $\|\mathbf{u}\|_U^2 = (M\mathbf{u}, \mathbf{u})_Q^2 + (M^{-1}L\mathbf{u}, L\mathbf{u})_Q^2$.

Then we define $H = \overline{L(U)} \subset \mathrm{L}_2(0, T; V)$ with norm $\|\mathbf{u}\|_H^2 = (M\mathbf{u}, \mathbf{u})_Q^2$.

Lemma

For given $\mathbf{f} \in \mathrm{L}_2(Q)^J$ *a unique solution* $\mathbf{u} \in U$ *exists solving the variational problem*

$$(L\mathbf{u}, \mathbf{v})_Q = (\mathbf{f}, \mathbf{v})_Q \,, \qquad \mathbf{v} \in H$$

For the proof we define $b \colon U \times H \longrightarrow \mathbb{R}$ with $b(\mathbf{u}, \mathbf{v}) = (L\mathbf{u}, \mathbf{v})_Q$.

We observe for all $\mathbf{u} \in \mathrm{C}^1(0, T; D(A))$ with $\mathbf{u}(0) = 0$

$$\|\mathbf{u}\|_H \leq 2T\, \|M^{-1}L\mathbf{u}\|_H \,.$$

This extends to all $\mathbf{u} \in U$ and shows $L(U) = \overline{L(U)}$. Inserting $\mathbf{v} = M^{-1}L\mathbf{u}$ yields

$$\inf_{\mathbf{u}\in U} \sup_{\mathbf{v}\in H} \frac{b(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_U \|\mathbf{v}\|_H} \geq \inf_{\mathbf{u}\in U} \frac{b_{\mathrm{dG}}(\mathbf{u}, M^{-1}L\mathbf{u})}{\|\mathbf{u}\|_U \|M^{-1}L\mathbf{u}\|_H} = \inf_{\mathbf{u}\in U} \frac{\|M^{-1}L\mathbf{u}\|_H}{\sqrt{\|\mathbf{u}\|_H^2 + \|M^{-1}L\mathbf{u}\|_H^2}} \geq \frac{1}{\sqrt{1 + 4T^2}} \,.$$

Discrete Space-Time Settings

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An implicit space-time discontinuous Galerkin approximation

ansatz space: discontinuous in space and continuous in time
test space: discontinuous in space and time
approximate continuity across faces by the choice of a numerical flux

$\Longrightarrow$ results in non-symmetric linear problems
$\Longrightarrow$ low regularity requirements

The space-time discontinuous Petrov-Galerkin method

ansatz space: discontinuous in space and time and conforming traces
optimal test space by solving local problems: discontinuous in space and time

$\Longrightarrow$ allows for symmetric Schur complements for the trace values
$\Longrightarrow$ more regularity required

A hybrid space-time discontinuous Petrov-Galerkin method

ansatz space: discontinuous in space and time and non-conforming traces
optimal test space by solving local problems: discontinuous in space and time

$\Longrightarrow$ allows for a hierarchy of symmetric Schur complements for the traces

A Space-Time Discontinuous Galerkin Approximation

Let $\bar{Q} = \bigcup_{\tau \in \mathcal{T}} \bar{\tau}$ be a decomposition into space-time cells $\tau = K_\tau \times I_\tau$, with mesh sizes $h_{\tau,K} = \operatorname{diam}(K_\tau)$ and $h_{\tau,I} = |I_\tau|$ for the local time interval $I_\tau = (t_\tau^{\min}, t_\tau^{\max})$.

For every τ choose polynomial degrees p_τ and q_τ for the ansatz in space and time, and define the local test spaces $H_{\tau,h} = (\mathbb{P}_{p_\tau}(K_\tau) \times \mathbb{P}_{q_\tau - 1})^J$ and the test space

$$H_h = \left\{ \mathbf{v}_h \in \mathrm{L}_2(0, T; V) \colon \mathbf{v}_{\tau,h} \in H_{\tau,h} \right\}.$$

For the ansatz space, we define

$$U_h = \Big\{ \mathbf{u}_h \in \mathrm{H}^1(0, T; V) \colon \mathbf{u}_h(0) = 0 \text{ and for all } \tau \in \mathcal{T} \text{ and } (\mathbf{x}, t) \in \tau$$
$$\mathbf{u}_h(\mathbf{x}, t) = \frac{t_\tau^{\max} - t}{t_\tau^{\max} - t_\tau^{\min}} \mathbf{w}_{\tau,h}(\mathbf{x}, t_\tau^{\min}) + \frac{t - t_\tau^{\min}}{t_\tau^{\max} - t_\tau^{\min}} \mathbf{v}_{\tau,h}(\mathbf{x}, t),$$
$$\text{where } \mathbf{w}_{\tau,h} \in U_h|_{[0, t_\tau^{\min}]} \text{ and } \mathbf{v}_{\tau,h} \in H_{\tau,h} \Big\}.$$

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$$\mathbf{u}_h(\mathbf{x},t) = \frac{t_\tau^{\max} - t}{t_\tau^{\max} - t_\tau^{\min}} \mathbf{w}_{\tau,h}(\mathbf{x}, t_\tau^{\min}) + \frac{t - t_\tau^{\min}}{t_\tau^{\max} - t_\tau^{\min}} \mathbf{v}_{\tau,h}(\mathbf{x},t)\,,$$

$$\text{where } \mathbf{w}_{\tau,h} \in U_h|_{[0,t_\tau^{\min}]} \text{ and } \mathbf{v}_{\tau,h} \in H_{\tau,h} \Big\}.$$

Let A_h be the discontinuous Galerkin operator with upwind flux approximating A.

Lemma

Let $L_h = M\partial_t + A_h$ and $\mathbf{f} \in \mathrm{L}_2(Q)^J$. A unique discrete solution $\mathbf{u}_h \in U_h$ exists:

$$(L_h \mathbf{u}_h, \mathbf{v}_h)_Q = (\mathbf{f}, \mathbf{v}_h)_Q\,, \qquad \mathbf{v}_h \in H_h\,.$$

A Weak Space-Time Setting

Let $U_\tau = U|_\tau$ be the restriction to $\tau = K \times (t_{\min}, t_{\max})$. Let $\hat{\mathbf{u}}_\tau = \gamma_\tau \mathbf{u}_\tau$ be the trace

$$\hat{\mathbf{u}}_\tau(\mathbf{x}, t) = \begin{cases} -\mathbf{u}_\tau(\mathbf{x}, t_{\min}) & \mathbf{x} \in K\,, \\ \mathbf{u}_\tau(\mathbf{x}, t_{\max}) & \mathbf{x} \in K\,, \\ \gamma_K \mathbf{u}_\tau(\mathbf{x}, t) & \mathbf{x} \in \partial K \text{ and } t \in (t_{\min}, t_{\max})\,. \end{cases}$$

Let $\gamma_\tau^{\mathrm{ad}}$ be the adjoint trace mapping and U^{ad} the adjoint space with $\mathbf{u}(T) = 0$. Integration by part yields $L^{\mathrm{ad}} = -L$ and

$$(L\mathbf{u}, \mathbf{v})_\tau = (\mathbf{u}, L^{\mathrm{ad}}\mathbf{v})_\tau + \langle \gamma_\tau \mathbf{u}, \gamma_\tau^{\mathrm{ad}} \mathbf{v} \rangle\,, \qquad \mathbf{u} \in U_\tau\,,\ \mathbf{v} \in U_\tau^{\mathrm{ad}}\,.$$

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Let $\gamma_\tau^{\mathrm{ad}}$ be the adjoint trace mapping and U^{ad} the adjoint space with $\mathbf{u}(T) = 0$. Integration by part yields $L^{\mathrm{ad}} = -L$ and

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Define $U_{\mathcal{T}}^{\mathrm{ad}} = \prod U_\tau^{\mathrm{ad}}$, $\gamma_{\mathcal{T}} = (\gamma_\tau)$, the trace space $\hat{U} = \gamma_{\mathcal{T}}(U) \subset \prod U_\tau / \mathcal{N}(\gamma_\tau)$, and

$$b_{\mathrm{dPG}}\colon \hat{U} \times H \times U_{\mathcal{T}}^{\mathrm{ad}} \longrightarrow \mathbb{R}\,, \qquad b_{\mathrm{dPG}}(\hat{\mathbf{u}}, \mathbf{u}, \mathbf{v}) = \sum_\tau (\mathbf{u}, L^{\mathrm{ad}}\mathbf{v})_\tau + \langle \hat{\mathbf{u}}_\tau, \gamma_\tau^{\mathrm{ad}} \mathbf{v} \rangle\,.$$

Lemma

For $\mathbf{f} \in \mathrm{L}_2(Q)^J$, a unique solution $(\hat{\mathbf{u}}, \mathbf{u}) \in \hat{U} \times H$ exists solving

$$b_{\mathrm{dPG}}(\hat{\mathbf{u}}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_Q\,, \qquad \mathbf{v} \in U_{\mathcal{T}}^{\mathrm{ad}}\,.$$

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$$\hat{\mathbf{u}}_\tau(\mathbf{x}, t) = \begin{cases} -\mathbf{u}_\tau(\mathbf{x}, t_{\min}) & \mathbf{x} \in K\,, \\ \mathbf{u}_\tau(\mathbf{x}, t_{\max}) & \mathbf{x} \in K\,, \\ \gamma_K \mathbf{u}_\tau(\mathbf{x}, t) & \mathbf{x} \in \partial K \text{ and } t \in (t_{\min}, t_{\max})\,. \end{cases}$$

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For $\mathbf{f} \in \mathrm{L}_2(Q)^J$, a unique solution $(\hat{\mathbf{u}}, \mathbf{u}) \in \hat{U} \times H$ exists solving

$$b_{\mathrm{dPG}}(\hat{\mathbf{u}}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_Q\,, \qquad \mathbf{v} \in U_{\mathcal{T}}^{\mathrm{ad}}\,.$$

Example: Acoustic waves — For $L(\mathbf{q}, \mathbf{q}) = (\partial_t \mathbf{q} + \nabla p, \rho \partial_t p + \nabla \cdot \mathbf{q})$ we have

$$\begin{aligned}(\partial_t \mathbf{q} + \nabla p, \tilde{\mathbf{q}})_\tau + (\rho \partial_t p + \nabla \cdot \mathbf{q}, \tilde{q})_\tau &= -(\mathbf{q}, \partial_t \tilde{\mathbf{q}} + \nabla \tilde{p})_\tau - (p, \rho \partial_t \tilde{p} + \nabla \cdot \tilde{\mathbf{q}})_\tau \\ &+ \big(\mathbf{q}(t_{\max}), \tilde{\mathbf{q}}(t_{\max})\big)_K - \big(\mathbf{q}(t_{\min}), \tilde{\mathbf{q}}(t_{\min})\big)_K + (\nabla \mathbf{q} \cdot \mathbf{n}, \tilde{p})_{\partial K \times (t_{\min}, t_{\max})} \\ &+ \big(p(t_{\max}), \tilde{p}(t_{\max})\big)_K - \big(p(t_{\min}), p(t_{\min})\big)_K + (p, \nabla \tilde{\mathbf{q}} \cdot \mathbf{n})_{\partial K \times (t_{\min}, t_{\max})}\,.\end{aligned}$$

The Discontinuous Petrov-Galerkin Method

Let $\hat{U}_h \subset \hat{U}_h$ be a discrete trace space, let $H_h = \prod H_{\tau,h} \subset H$ be a discrete ansatz space, and let $U^{\mathrm{ad}}_{\mathcal{T},h} = \prod U^{\mathrm{ad}}_{\tau,h} \subset U^{\mathrm{ad}}_{\mathcal{T}}$ be a discrete broken test space such that

$$\sup_{\mathbf{v}_{\tau,h} \in U^{\mathrm{ad}}_{\tau,h}} \frac{b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_{\tau,h}, \mathbf{v}_{\tau,h})}{\|\mathbf{v}_{\tau,h}\|_{U^{\mathrm{ad}}_{\tau}}} \geq \beta_0 \sup_{\mathbf{v}_{\tau} \in U^{\mathrm{ad}}_{\tau}} \frac{b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_{\tau,h}, \mathbf{v}_{\tau})}{\|\mathbf{v}_{\tau}\|_{U^{\mathrm{ad}}_{\tau}}}$$

for all $(\hat{\mathbf{u}}_h, \mathbf{u}_{\tau,h}) \in \hat{U}_h \times H_{\tau,h}$ and all $\tau \in \mathcal{T}$. Define the optimal test space

$$U^{\mathrm{opt}}_{\mathcal{T},h} = \Big\{ \mathbf{v}_h \in U^{\mathrm{ad}}_{\mathcal{T},h} \colon (\hat{\mathbf{u}}_h, \mathbf{u}_h) \in \hat{U}_h \times H_h \text{ exists such that } (\mathbf{v}_h, \mathbf{w}_h)_{U^{\mathrm{ad}}_{\mathcal{T}}} = b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{w}_h) \text{ for all } \mathbf{w}_h \in U^{\mathrm{ad}}_{\mathcal{T},h} \Big\}\,.$$

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for all $(\hat{\mathbf{u}}_h, \mathbf{u}_{\tau,h}) \in \hat{U}_h \times H_{\tau,h}$ and all $\tau \in \mathcal{T}$. Define the optimal test space

$$U^{\mathrm{opt}}_{\mathcal{T},h} = \Big\{\mathbf{v}_h \in U^{\mathrm{ad}}_{\mathcal{T},h} \colon (\hat{\mathbf{u}}_h, \mathbf{u}_h) \in \hat{U}_h \times H_h \text{ exists such that} \\ (\mathbf{v}_h, \mathbf{w}_h)_{U^{\mathrm{ad}}_{\mathcal{T}}} = b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{w}_h) \text{ for all } \mathbf{w}_h \in U^{\mathrm{ad}}_{\mathcal{T},h}\Big\}\,.$$

Theorem

For $\mathbf{f} \in \mathrm{L}_2(Q)^J$, a unique solution $(\hat{\mathbf{u}}_h, \mathbf{u}_h) \in \hat{U}_h \times H_h$ exists solving

$$b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_Q\,, \qquad \mathbf{v}_h \in U^{\mathrm{opt}}_{\mathcal{T},h}\,.$$

For the discretization error holds

$$\|(\hat{\mathbf{u}}, \mathbf{u}) - (\hat{\mathbf{u}}_h, \mathbf{u}_h)\|_{\hat{U}\times H} \leq C \inf_{(\hat{\mathbf{w}}_h, \mathbf{w}_h) \in \hat{U}_h \times H_h} \|(\hat{\mathbf{u}}, \mathbf{u}) - (\hat{\mathbf{w}}_h, \mathbf{w}_h)\|_{\hat{U}\times H}\,.$$

W/Wohlmuth: Substructuring methods for first order systems, submitted 2013

Why Does It Work? Conforming or Non-conforming?

Karlsruhe Institute of Technology

Set $U_{\mathcal{T}} = \prod_{\tau \in \mathcal{T}} U_\tau$ and $\hat{U}_{\mathcal{T}} = \prod_{\tau \in \mathcal{T}} \hat{U}_\tau$ with $\hat{U}_\tau = \gamma_\tau(U_\tau)$. Then, we have

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & \longrightarrow & \gamma_{\mathcal{T}}(U) = \hat{U} & & \\
 & & & \nearrow & \downarrow & & \downarrow & \searrow & \\
0 & \longrightarrow & \mathcal{N}(\gamma_{\mathcal{T}}) & \longrightarrow & U_{\mathcal{T}} & \longrightarrow & \gamma_{\mathcal{T}}(U_{\mathcal{T}}) = \hat{U}_{\mathcal{T}} & \longrightarrow & 0 \\
 & \searrow & & & \downarrow & & \downarrow & \nearrow & \\
 & & \mathcal{N}([\gamma_{\mathcal{T}}]) & \longrightarrow & [U_{\mathcal{T}}] = U_{\mathcal{T}}/U & \longrightarrow & [\hat{U}_{\mathcal{T}}] = \hat{U}_{\mathcal{T}}/\hat{U} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

introducing the face jumps $[\gamma_{\mathcal{T}}]$, i.e. $U = \mathcal{N}([\gamma_{\mathcal{T}}])$.

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\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & \longrightarrow & \gamma_\mathcal{T}(U) = \hat U & & \\
 & & & \nearrow & \downarrow & & \downarrow & \searrow & \\
0 & \longrightarrow & \mathcal{N}(\gamma_\mathcal{T}) & \longrightarrow & U_\mathcal{T} & \longrightarrow & \gamma_\mathcal{T}(U_\mathcal{T}) = \hat U_\mathcal{T} & \longrightarrow & 0 \\
 & \searrow & & & \downarrow & & \downarrow & \nearrow & \\
 & & \mathcal{N}([\gamma_\mathcal{T}]) & \longrightarrow & [U_\mathcal{T}] = U_\mathcal{T}/U & \longrightarrow & [\hat U_\mathcal{T}] = \hat U_\mathcal{T}/\hat U & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

introducing the face jumps $[\gamma_\mathcal{T}]$, i.e. $U = \mathcal{N}([\gamma_\mathcal{T}])$.

Conforming spaces

$U_h \subset \big\{\mathbf{u} \in U \colon \langle[\gamma_\mathcal{T}(\mathbf{u})], [\hat{\mathbf{v}}]\rangle = 0 \text{ for all } [\hat{\mathbf{v}}] \in [\hat U_\mathcal{T}^{\mathrm{ad}}]\big\}, \quad \hat U_h \subset \hat U$

Non-conforming spaces Select $\Lambda_h \subset [\hat U_\mathcal{T}^{\mathrm{ad}}]$ and set

$U_h = \big\{\mathbf{u} \in U \colon \langle[\gamma_\mathcal{T}(\mathbf{u})], [\hat{\mathbf{v}}_h]\rangle = 0 \text{ for all } [\hat{\mathbf{v}}_h] \in \Lambda_h\big\}, \quad \hat U_h = \Lambda_h' \subset [\hat U_\mathcal{T}^{\mathrm{ad}}]'$

The Hybrid Discontinuous Petrov-Galerkin Method

Consider the extension

$$b_{\text{hdPG}} : [\hat{U}^{\text{ad}}_{\mathcal{T}}]' \times H \times U^{\text{ad}}_{\mathcal{T}} \longrightarrow \mathbb{R}\,, \qquad b_{\text{hdPG}}(\hat{\mathbf{u}}, \mathbf{u}, \mathbf{v}) = (\mathbf{u}, L^{\text{ad}}\mathbf{v})_{\mathcal{T}} + \langle \hat{\mathbf{u}}, \gamma^{\text{ad}}_{\mathcal{T}}\mathbf{v}\rangle\,.$$

The inf-sup condition and the optimal test space are defined as above.

Lemma

For $\mathbf{f} \in \mathrm{L}_2(Q)^J$, a unique solution $(\hat{\mathbf{u}}_h, \mathbf{u}_h) \in \hat{U}_h \times H_h$ exists solving

$$b_{\mathrm{dPG}}(\hat{\mathbf{u}}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_Q\,, \qquad \mathbf{v}_h \in U^{\mathrm{opt}}_{\mathcal{T},h}\,.$$

The non-conforming trace approximation $\hat{\mathbf{u}}_h \in \hat{U}_h$ is determined by a symmetric positive definite Schur complement problem

$$\hat{S}_h \hat{\mathbf{u}}_h = \hat{\mathbf{f}}_h\,,$$

and the local solutions $\mathbf{u}_{\tau,h} \in H_{\tau,h}$ can be reconstructed from $\hat{\mathbf{u}}_h$.

On nested decompositions $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ of the space-time cylinder Q, the corresponding sequence of hybrid spaces allow for multilevel preconditioning.

The Double-Greedy Concept

Successive adaptive selection of ansatz and test spaces:

Input: $\varepsilon > 0$, coarse decomposition $\mathcal{T}_0$,
coarse nonconforming trace space $\hat{U}_0$, local ansatz and test spaces $H_{\tau,0}$, $U^{\text{ad}}_{\tau,0}$

solve the coarse problem $\hat{\mathbf{u}}_0 \in \hat{U}_0$
for $m = 1, \ldots,$ MaxIter **do**
  estimate the error by an indicator η_{m-1}; STOP if $\eta_{m-1} \le \varepsilon$.
  mark for refinement depending on $\eta_{\tau,m}$
  refinement of the decomposition $\mathcal{T}_m \subset \mathcal{T}_{m-1}$
  select a finer discretization for traces $\hat{U}_m$ and local ansatz spaces $H_{\tau,m}$
  select locally finer test spaces $U^{\text{ad}}_{\tau,m}$ to ensure inf-sup stability
  compute $\hat{\mathbf{u}}_m \in \hat{U}_m$ with multigrid on $\hat{U}_0$, ..., $\hat{U}_m$
end for

A simple error is given by

$$\eta_m^2 = \sum_\tau \eta_{\tau,m}^2 \,, \qquad \eta_\tau \approx \|\hat{\mathbf{u}}_{\tau,m} - \gamma_\tau \mathbf{w}_\tau\|_{\hat{U}_\tau} \,,$$

where $\mathbf{w}_\tau \in U_\tau$ is a local reconstruction of the weak solution $\mathbf{u}_{\tau,m} \in H_{\tau,m}$.